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A NOTE ON SECOND ORDER DIFFERENTIAL INEQUALITIES AND FUNCTIONAL DIFFERENTIAL EQUATIONS

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Criteria are established for the nonexistence of eventually positive solutions of a second order differential inequality. The oscillation of all solutions of large classes of functional differential equations follows as corollaries.

1. Introduction. Study of the behavior of solutions of equations like

$$x'' + F(t, x, x') = 0 ,$$

where $xF \geqq 0$, often entails study of the behavior of solutions of an inequality system like

$$(1) \qquad x'' + H(t, x) \leqq 0 , \quad x \geqq 0 ,$$

where $xF \geqq xH \geqq 0$ and H is selected for its tractability to analysis [6].

In this note it is shown that oscillation properties of large classes of equations

$$(2) \qquad x'' + F(t, x(t), x(t - \tau(t))) = 0$$

can be established by use of inequalities like (1).

Thus, whenever feasible, inequalities like (1) should be primary objects of investigation.

2. Preliminaries. The inequality system discussed in this note is

$$(3) \qquad x'' + a(t)N(x) \leqq 0 , \quad x \geqq 0 ,$$

where $a(t)$ is nonnegative and continuous on $[0, \infty)$, and $N(x)$ is positive on $(0, \infty)$ and continuous and nondecreasing on $[0, \infty)$. Note that $N(0) > 0$ is permitted.

Three theorems are given on the nonexistence of eventually positive solutions of (3). Each theorem has a corollary concerning (2) where $F(t, u, v)$ is continuous on $[0, \infty) \times R_2$, and nondecreasing in u and v for $uv > 0$, $\tau(t)$ is continuous on $[0, \infty)$, and

$$(4) \qquad \begin{aligned} F(t, u, u) &\geqq a(t)N(u) , \quad u \geqq 0 , \\ -F(t, u, u) &\geqq a(t)N(-u) , \quad u \leqq 0 . \end{aligned}$$

The term "solution" refers only to those solutions of equation (2) or inequality (3) which are defined and have a continuous second derivative on some interval $[T, \infty)$, $T \geqq 0$. Inequality (3) does not restrict a solution at those t-values where it is negative.

P. K. Wong [7] has discussed an inequality system like

$$x'' - a(t)N(x) > 0\,, \quad x > 0\,.$$

3. The results. Theorem 1 is suggested by an oscillation criterion for a special case of the equation (1) due to F. V. Atkinson [1].

THEOREM 1. *Suppose*

$$\int^{\infty} ta(t)dt = \infty \tag{7}$$

and, if $\alpha > 0$,

$$\int_{\alpha}^{\infty} N^{-1}(u)du < \infty\,. \tag{8}$$

If $x(t)$ *is a solution of* (3) *and* $x(T_0) > 0$ *at some* T_0 *in* $[0, \infty)$, *then* $x(t)$ *has a zero in* (T_0, ∞).

Proof. Suppose $x(t) > 0$ on $[T_0, \infty)$. Then $x(t)$ satisfies

$$x''(t) + a(t)N(x(t)) \leqq 0\,. \tag{9}$$

If $x'(t)$ has a zero in $[T_0, \infty)$ then, by the conditions on a, N, (9) implies $x''(t) \leqq 0$, $x''(t) \not\equiv 0$, on $[T_0, \infty)$. Thus, it is readily seen that $x(t)$ must have a zero in $[T_0, \infty)$.

Therefore, suppose $x'(t) > 0$ on $[T_0, \infty)$. An integration of (9) over $[s, t]$, $T_0 \leqq s < t$, gives, by neglect of positive $x'(t)$,

$$-x'(s) \leqq -\int_s^t a(r)N(x(r))dr \leqq -N(x(s))\int_s^t a(r)dr\,.$$

Division by $N(x(s))$ and an integration over $[T_0, t]$ gives

$$-\int_{x(T_0)}^{x(t)} N^{-1}(u)du \leqq -\int_{T_0}^{t} (s - T_0)a(s)ds\,.$$

Clearly, if t is sufficiently large (7) and (8) are contradicted. This proves the theorem.

COROLLARY 1. *Given* (2), *suppose there exist functions* a, N, *which satisfy the conditions of the theorem and condition* (4). *Suppose* $\tau = \sup_t \tau(t) < \infty$. *Then, each solution of* (2) *has a zero in each interval* $[T_0, \infty)$ (*is oscillatory*).

Proof. If $x(t)$ is a solution of (2), but is not oscillatory, then $x(t) > 0$ on some interval $[T_0 - \tau, \infty)$. And, necessarily, $x'(t) > 0$ on $[T_0, \infty)$, whereby, $x(t - \tau(t)) \geqq x(t - \tau)$ on $[T_0, \infty)$.

If $\tau \leqq 0$, $x(t - \tau) \geqq x(t)$ and $x(t)$ satisfies (9); thus, by Theorem 1, there is a contradiction.

If $\tau > 0$, then $x''(t) \leqq 0$ on $[T_0, \infty)$ implies

$$x(t) - x(t - \tau) \leqq x'(T_0)\tau, \; t \geqq T_0 + \tau \; . \tag{10}$$

Hence, there is a β, $0 < \beta < 1$, such that $\beta x(t) \leqq x(t - \tau)$ and $x(t)$ is a solution of (3) on $[T_0 + \tau, \infty)$ with $N(\beta x)$ in place of $N(x)$. The resulting contradiction proves the corollary.

Theorem 2 corresponds to the equation (1) where $F \equiv a(t)|x|^{\gamma} \operatorname{sgn} x$, $0 < \gamma < 1$, whereas Theorem 1 holds for this F with $\gamma > 1$. The proof is based on a proof given by J. W. Heidel [2].

Notice that if on $[t_0, \infty)$ $f(t) > 0$, $f'(t) > 0$ continuous and nonincreasing, an integration shows that for each ν, $0 < \nu < 1$,

$$f(t) \geqq \nu t f'(t), \; t \geqq (1 - \nu)^{-1} t_0 \; . \tag{11}$$

THEOREM 2. *Suppose that in* (3) $N(x)$ *satisfies*

$$N(uv) \geqq \eta(u) N(v) \; , \tag{12}$$

for $u, v \geqq 0$, u *bounded,* v *large, and for some continuous function* $\eta(u)$. *Suppose, also,*

$$\int^{\infty} N(t) a(t) dt = \infty \; , \tag{13}$$

and the possibly improper integral,

$$\int_0^v \eta^{-1}(u) du \; , \tag{14}$$

exists for each finite v. *If* $x(t)$ *is a solution of* (3) *and* $x(T_0) > 0$, *then* $x(t)$ *must have a zero in* (T_0, ∞).

Proof. If $x(t) > 0$ on $[T_0, \infty)$, then $x'(t) > 0$ and $x(t) \geqq 1/2(tx'(t))$ on $[T_1, \infty)$, $T_1 = 2T_0$. Therefore, $N(x(t)) \geqq \eta(1/2(x'(t)))N(t)$ and (9) leads to

$$x''(t)\eta^{-1}(x'(t)) + N(t)a(t) \leqq 0 \; , \quad t \geqq T_1 \; .$$

An integration produces

$$\int_{x'(T_1)}^{x'(t)} \eta^{-1}\Big(\frac{1}{2}u\Big) du + \int_T^t N(s)a(s) ds \leqq 0 \; .$$

Since $x'(t)$ is nonincreasing this inequality with (13) contradicts the boundedness of (14). Hence, $x'(t)$ must have a zero in (T_0, ∞) and the theorem follows.

COROLLARY 2. *If there exist functions* $a(t)$, $N(u)$, *satisfying the conditions of the theorem and also satisfying* (4), *and if* $\sup_t \tau(t) < \infty$, *then all solutions of* (2) *oscillate.*

P. Waltman gave an example for an equation like (2) in which $t - \tau(t)$ increases more slowly than t, and the equation (2) has a nonoscillatory solution, while (2) with $\tau(t) \equiv 0$ has only oscillatory solutions. Corollary 3 slightly generalizes a sufficient condition for oscillation given in [5] for such cases. It holds for the linear equation

$$x''(t) + a(t)x(t - \tau(t)) = 0 .$$

THEOREM 3. *In* (3) *let* $a(t)$ *and* $N(x)$ *have the properties mentioned, and in addition let*

$$\int^{\infty} a(t)dt = \infty . \tag{15}$$

Then; if $x(t)$ *is a solution of* (3), $x(T_0) > 0$ *implies* $x(t)$ *has a zero in* (T_0, ∞).

Proof. If the theorem is false $x(t) > x(T_0)$ on (T_0, ∞) and $N(x(t))$ in (9) is greater than $N(x(T_0))$. An integration produces a contradiction to $x'(t) > 0$ on $[T_0, \infty)$. This proves the theorem.

COROLLARY 3. *Given* (3) *suppose there exist funtions* $a(t)$, $N(u)$, *satisfying* (4). *If* $a(t)$ *satisfies* (15) *and* $t - \tau(t) \to \infty$ *as* $t \to \infty$, *then all solutions of* (2) *are oscillatory.*

REMARK. The corollaries could have been given for equations more general than (2) where $F \equiv F(t, x(t - \tau_1(t)), \cdots, x(t-\tau_n(t)))$ as in [3]. Also the $\tau(t)$ could have been expressed as explicitly dependent on $x(t)$ in certain ways. In [4] equations (2) which are nearly linear are discussed.

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