PACIFIC JOURNAL OF MATHEMATICS
Vol. 41, No. 2, 1972

SUPER-REFLEXIVE SPACES WITH BASES

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Super-reflexivity is defined in such a way that all super-reflexive Banach spaces are reflexive and a Banach space is super-reflexive if it is isomorphic to a Banach space that is either uniformly convex or uniformly non-square. It is shown that, if $0 < 2\phi < \varepsilon \leqq 1 < \Phi$ and B is super-reflexive, then there are numbers r and s for which $1 < r < \infty$, $1 < s < \infty$ and, if $\{e_i\}$ is any normalized basic sequence in B with characteristic not less than ε, then

$$\phi\,[\Sigma\mid a_i\mid^r]^{1/r} \leqq \|\,\Sigma a_i e_i\,\| \leqq \Phi\,[\Sigma\mid a_i\mid^s]^{1/s}\,,$$

for all numbers $\{a_i\}$ such that $\Sigma a_i e_i$ is convergent. This also is true for unconditional basic subsets in nonseparable super-reflexive Banach spaces. Gurariĭ and Gurariĭ recently established the existence of ϕ and r for uniformly smooth spaces, and the existence of Φ and s for uniformly convex spaces [Izv. Akad. Nauk SSSR Ser. Mat., 35 (1971), 210-215].

A *basis* for a Banach space B is a sequence $\{e_i\}$ such that, for each x in B, there is a unique sequence of numbers $\{a_i\}$ such that $\sum_1^\infty a_i e_i$ converges strongly to x, i.e.,

$$\lim_{n\to\infty}\left\| x - \sum_1^n a_i e_i \right\| = 0\,.$$

A *normalized basis* is a basis $\{e_i\}$ such that $\|e_i\| = 1$ for all i.

A *basic sequence* is any sequence that is a basis for its closed linear span.

It apparently was known to Banach (see [1, pg. 111] and [3]) that a sequence $\{e_i\}$ whose linear span is dense in a Banach space B is a basis for B if and only if there is a number $\varepsilon > 0$ such that

$$\left\| \sum_1^n a_i e_i \right\| \geqq \varepsilon \left\| \sum_1^k a_i e_i \right\|$$

if $k < n$ and $\{a_i\}$ is any sequence of numbers. The largest such number ε is the *characteristic* of the basis. It follows directly from the triangle inequality that, if $1 \leqq p \leqq q \leqq n$, then

$$\left\| \sum_1^n a_i e_i \right\| \geqq \frac{1}{2}\,\varepsilon \left\| \sum_p^q a_i e_i \right\|.$$

An *unconditional basis* for a Banach space B is a subset $\{e_\alpha\}$ of B such that for each x in B there is a unique sequence of ordered pairs $(a_i, e_{\alpha(i)})$ such that $\sum_1^\infty a_i e_{\alpha(i)}$ converges strongly and unconditionally to x. By arguments similar to those used in [1] and [3] for a basis, it can be shown that a subset $\{e_\alpha\}$ whose linear span is dense in B is an unconditional basis for B if and only if there is a *characteristic* ε for which

$$\|\textstyle\sum_A a_\alpha e_\alpha\| \geqq \varepsilon \|\textstyle\sum_B a_\alpha e_\alpha\| ,$$

if $B \subset A$ and A is a finite subset of the index set.

A *uniformly non-square* Banach space is a Banach space B for which there is a positive number δ such that there do not exist members x and y of B for which $\|x\| \leqq 1$, $\|y\| \leqq 1$,

$$\left\|\frac{1}{2}(x+y)\right\| > 1-\delta \quad \text{and} \quad \left\|\frac{1}{2}(x-y)\right\| > 1-\delta .$$

A uniformly convex space is uniformly non-square and a uniformly non-square space is reflexive [6, Theorem 1.1].

THEOREM 1. *The following properties are equivalent for normed linear spaces X, each of them is implied by nonreflexivity of the completion of X, and each is self-dual. If a normed linear space X has any one of these properties, then X is not isomorphic to any space that is uniformly non-square.*

(i) *There exists a positive number θ such that, for every positive integer n, there are subsets $\{z_1, \cdots, z_n\}$ and $\{g_1, \cdots, g_n\}$ of the unit balls of X and X^*, respectively, such that*

$$g_i(z_j) = \theta \quad \text{if} \quad i \leqq j,\ g_i(z_j) = 0 \quad \text{if} \quad i > j .$$

(ii) *There exist positive numbers α and β such that, for every positive integer n, there is a subset $\{x_1, \cdots, x_n\}$ of the unit ball of X for which $\|x\| > \alpha$ if $x \in \operatorname{conv}\{x_1, \cdots, x_n\}$ and, for every positive integer $k < n$ and all numbers $\{a_1, \cdots, a_n\}$,*

$$\left\|\sum_1^n a_i x_i\right\| \geqq \beta \left\|\sum_1^k a_i x_i\right\| .$$

(iii) *There exist positive numbers α' and β' such that, for every positive integer n, there is a subset $\{x_1, \cdots, x_n\}$ of X which has the property that, for every positive integer $k < n$ and all numbers $\{a_i\}$,*

$$\left\|\sum_1^n a_i x_i\right\| \geqq \alpha' \sup |a_i| \quad \text{and} \quad \left\|\sum_1^k x_i\right\| < \beta'.$$

Proof. It is known that Theorem 1 is valid for properties (i) and (ii) [8, Theorem 6]. We shall show that (i) and (iii) are equivalent.

If (i) is satisfied, let $x_1 = z_1$ and $x_i = z_i - z_{i-1}$ if $1 < i \leqq n$. Then $g_i(x_j) = \delta_i^j \theta$, so that

$$\left\| \sum_1^n a_i x_i \right\| \geqq \left| g_k\left(\sum_1^n a_i x_i \right) \right| = \theta \left| a_k \right| ,$$

and $\| \sum_1^k x_i \| = \| z_k \| \leqq 1$. Thus (iii) is satisfied.

If (iii) is satisfied, let $z_k = \sum_1^k x_i / \beta'$. Define g_j on $\mathrm{lin}\{z_1, \cdots, z_n\}$ by letting $g_i(x_j) = \delta_i^j \alpha'$. Then $\| z_k \| < 1$ and

$$\left| g_j\left(\sum_1^n a_i x_i \right) \right| = \alpha' \mid a_j \mid \leqq \left\| \sum_1^n a_i x_i \right\| ,$$

so that g_j can be extended to all of the space with $\| g_j \| \leqq 1$. Also, $g_i(z_j) = \alpha'/\beta'$ if $i \leqq j$ and $g_i(z_j) = 0$ if $i > j$, so that (i) is satisfied.

DEFINITION. A *super-reflexive Banach space* is a Banach space that does not have any of the equivalent properties (i), (ii) and (iii) described in the statement of Theorem 1.

This is a natural definition, since a Banach space is non-reflexive if and only if (i) of Theorem 1 is satisfied by infinite sequences $\{z_i\}$ and $\{g_i\}$. Moreover, there are several other finitely stated properties that are equivalent to (i), but which become equivalent to non-reflexivity when stated for infinite sequences [8, Theorem 3].

THEOREM 2. *Let B be a super-reflexive Banach space. If $\Phi > 1$ and $0 < \varepsilon \leqq 1$, then there is a number s for which $1 < s < \infty$ and, if $\{e_i\}$ is any normalized basic sequence in B with characteristic not less than ε, then*

$$\| \sum a_i e_i \| \leqq \Phi \left[\sum \mid a_i \mid^s \right]^{1/s} \tag{1}$$

for all numbers $\{a_i\}$ such that $\sum a_i e_i$ is convergent.

Proof. It will be shown that, if there are numbers Φ and ε for which $\Phi > 1$, $0 < \varepsilon \leqq 1$, and there does not exist such a number s, then B has property (ii) of Theorem 1 with $\alpha = 1/2$ and $\beta = \varepsilon$. Let n be an arbitrary positive integer greater than 1. Let θ be a number for which

$$1 - \frac{1}{2n} < \theta < 1 .$$

Then choose λ such that $\theta^{1/4} < \lambda < 1$, $\lambda^2 \Phi > 1$, and

$$\frac{(\Phi + 1)(1 - \lambda^2)}{\lambda^2 \Phi - 1} < \frac{1}{n} (1 - \theta^{1/4}) . \tag{2}$$

Choose $s > 1$ and close enough to 1 that $\lambda n < n^{1/s}$. Then

$$(3) \qquad (\alpha+\beta)^{1/s} \geqq \lambda\,(\alpha^{1/s}+\beta^{1/s}) \quad \text{if} \quad \alpha \geqq 0 \quad \text{and} \quad \beta \geqq 0\,,$$

$$(4) \qquad \lambda n(\inf \beta_i)^{1/s} \leqq \left(\sum_1^n \beta_i\right)^{1/s} \quad \text{if } \beta_i \geqq 0 \quad \text{for each } i\,.$$

Since there is a basic sequence $\{e_i\}$ with characteristic not less than ε and a sequence $\{a_i\}$ for which (1) is false, there also is a least positive integer m for which

$$(5) \qquad \sup \frac{\left\| \sum_1^m a_i e_i \right\|}{\left[\sum_1^m |a_i|^s\right]^{1/s}} = M > \Phi\,,$$

where the sup is over all m-tuples of numbers $(a_1, \cdots, a_m)$. Since

$$\frac{\left\| \sum_1^{m-1} a_i e_i + a_m e_m \right\|}{\left[\sum_1^{m-1} |a_i|^s + |a_m|^s\right]^{1/s}} \leqq \frac{\left\| \sum_1^{m-1} a_i e_i \right\|}{\left[\sum_1^{m-1} |a_i|^s\right]^{1/s}} + \frac{\| a_m e_m \|}{[|a_m|^s]^{1/s}} \leqq \Phi + 1\,,$$

we have $\Phi < M \leqq \Phi + 1$ and it follows from (2) that

$$(6) \qquad \left[\frac{M(1-\lambda^2)}{\lambda^2 M - 1}\right]^s < \frac{M(1-\lambda^2)}{\lambda^2 M - 1} < \frac{1}{n}(1-\theta^{1/4})\,.$$

Let $(\alpha_1, \cdots, \alpha_m)$ be an m-tuple such that $\| \sum_1^m \alpha_i e_i \| = 1$ and

$$(7) \qquad \frac{1}{\left[\sum_1^m |\alpha_i|^s\right]^{1/s}} = \frac{\left\| \sum_1^m \alpha_i e_i \right\|}{\left[\sum_1^m |\alpha_i|^s\right]^{1/s}} > \lambda M\,.$$

We shall show first that, for each k,

$$(8) \qquad |\alpha_k|^s < \frac{1}{n}(1-\theta^{1/4}) \sum_1^m |\alpha_i|^s\,.$$

It follows from (3), (7) and (5) that, for each k,

$$\left[\sum_1^m |\alpha_i|^s\right]^{1/s} \geqq \lambda \left\{ |\alpha_k| + \left[\sum_{i\neq k} |\alpha_i|^s\right]^{1/s} \right\},$$

and

$$(9) \qquad \lambda^2 M < \frac{|\alpha_k| + \left\| \sum_{i\neq k} \alpha_i e_i \right\|}{|\alpha_k| + \left[\sum_{i\neq k} |\alpha_i|^s\right]^{1/s}} \leqq \frac{|\alpha_k| + M\left[\sum_{i\neq k} |\alpha_i|^s\right]^{1/s}}{|\alpha_k| + \left[\sum_{i\neq k} |\alpha_i|^s\right]^{1/s}}$$

Since $\lambda^2 M - 1 > \lambda^2 \Phi - 1 > 0$, direct computation shows that (9) implies

$$|\alpha_k| < \frac{M\left[\sum_{i\neq k}|\alpha_i|^s\right]^{1/s}(1-\lambda^2)}{\lambda^2 M - 1} \leqq \left[\sum_1^m |\alpha_i|^s\right]^{1/s}\frac{M(1-\lambda^2)}{\lambda^2 M - 1},$$

which with (6) implies (8). Now that (8) has been established, we know there is a sequence of n integers $\{m(1), \cdots, m(n) = m\}$ such that, for each j,

$$\left|\left[\sum_{i=1}^{m(j)}|\alpha_i|^s - \frac{j}{n}\sum_1^m |\alpha_i|^s\right]\right| < \frac{1}{2n}(1-\theta^{1/4})\sum_1^m |\alpha_i|^s .$$

Let us write

$$\begin{aligned}\sum_1^m \alpha_i e_i &= \sum_1^{m(1)}\alpha_i e_i + \sum_{m(1)+1}^{m(2)}\alpha_i e_i + \cdots + \sum_{m(n-1)+1}^{m}\alpha_i e_i \\ &= \sum_1^n u_j ,\end{aligned}$$

where $u_j = \sum_{m(j-1)+1}^{m(j)}\alpha_i e_i$ with $m(0) = 0$. Then we have, for each j,

$$\left|\left[\sum_{m(j-1)+1}^{m(j)}|\alpha_i|^s - \frac{1}{n}\sum_1^m |\alpha_i|^s\right]\right| < \frac{1}{n}(1-\theta^{1/4})\sum_1^m |\alpha_i|^s .$$

This implies that

$$\frac{1}{n}\theta^{1/4}\sum_1^m |\alpha_i|^s < \sum_{m(j-1)+1}^{m(j)}|\alpha_i|^s < \frac{1}{n}(2-\theta^{1/4})\sum_1^m |\alpha_i|^s < \frac{1}{n}\theta^{-1/4}\sum_1^m |\alpha_i|^s$$

and

$$\sum_{m(j-1)+1}^{m(j)}|\alpha_i|^s < \theta^{-1/2}\inf\left\{\sum_{m(k-1)+1}^{m(k)}|\alpha_i|^s : 1 \leqq k \leqq n\right\} \tag{10}$$

for each j. It follows from (7), (5), (10), (4) and $\lambda^2 > \theta^{1/2}$ that

$$\begin{aligned}\frac{1}{\left[\sum_1^m |\alpha_i|^s\right]^{1/s}} > \lambda M \geqq \frac{\lambda \| u_j \|}{\left[\sum_{m(j-1)+1}^{m(j)}|\alpha_i|^s\right]^{1/s}} &> \frac{(\theta^{1/2})^{1/s}\lambda \| u_j \|}{\left[\inf_k \sum_{m(k-1)+1}^{m(k)}|\alpha_i|^s\right]^{1/s}} \\ &> \frac{n(\theta^{1/2})^{1/s}\lambda^2 \| u_j \|}{\left[\sum_1^m |\alpha_i|^s\right]^{1/s}} > \frac{n\theta \| u_j \|}{\left[\sum_1^m |\alpha_i|^s\right]^{1/s}},\end{aligned}$$

so that $\| u_j \| < 1/(n\theta)$. We are now prepared to show that $\{x_1, \cdots, x_n\}$ satisfies (ii) of Theorem 1 if $x_j = n\theta u_j$ for each i, $\alpha = 1/2$ and $\beta = \varepsilon$. Note first that if $\Sigma\beta_j = 1$ and $\beta_j \geqq 0$ for each j, then

$$\begin{aligned}\| \Sigma\beta_j x_j \| &\geqq \| \Sigma x_j \| - \| \Sigma(1-\beta_j)\, x_j \| \\ &\geqq n\theta \| \Sigma u_j \| - \Sigma(1-\beta_j) .\end{aligned}$$

Since $\| \Sigma u_j \| = \| \Sigma\alpha_i e_i \| = 1$ and $\theta > 1 - 1/(2n)$, we have

$$\| \Sigma\beta_j x_j \| \geqq \left(n - \frac{1}{2}\right) - (n-1) = \frac{1}{2} = \alpha .$$

Since the characteristic of the basic sequence $\{e_i\}$ is not less than $\varepsilon = \beta$, we also have

$$\left\| \sum_1^n a_i x_i \right\| \geqq \beta \left\| \sum_1^k a_i x_i \right\| \quad \text{if} \quad k < n .$$

The duality argument used by Gurariĭ and Gurariĭ [4] in a similar situation does not seem easily adaptible to give a proof of Theorem 3 that makes explicit use of Theorem 2. Therefore a direct proof of Theorem 3 will be given.

THEOREM 3. *Let B be a super-reflexive Banach space. If ϕ and ε are numbers for which $0 < 2\phi < \varepsilon \leqq 1$, then there is a number r for which $1 < r < \infty$ and, if $\{e_i\}$ is any normalized basic sequence in B with characteristic not less than ε, then*

$$\phi\, [\Sigma \mid a_i \mid^r]^{1/r} \leqq \| \Sigma\, a_i e_i \| , \tag{11}$$

for all numbers $\{a_i\}$ such that $\Sigma a_i e_i$ is convergent.

Proof. Suppose that $0 < 2\phi < \varepsilon \leqq 1$. It will be shown that if no such number r exists, then B has property (iii) of Theorem 1 with $\alpha' = 2\phi^2/\varepsilon$ and $\beta' > 1/\varepsilon$.
Let n be an arbitrary positive integer greater than 1. Let λ be a positive number for which

$$2\phi < \lambda^2\varepsilon \quad \text{and} \quad \lambda < 1 .$$

Then choose $r > 1$ and large enough that

$$n^{1/r} < \lambda^{-1}(1-\lambda)^{1/r} . \tag{12}$$

If $\beta_i \geqq 0$ for each i, then it follows from (12) that

$$\left(\sum_1^n \beta_i\right)^{1/r} < \lambda^{-1}\,(\sup \beta_i)^{1/r} . \tag{13}$$

Since there is a basic sequence $\{e_i\}$ with characteristic not less than ε and a sequence $\{a_i\}$ for which (11) is false, there also is an m for which

$$\inf \frac{\left\| \sum_1^m a_i e_i \right\|}{\left[\sum_1^m \mid a_i \mid^r \right]^{1/r}} = M < \phi , \tag{14}$$

where the inf is over all m-tuples of numbers $(a_1, \cdots, a_m)$. Let

$(\alpha_1, \cdots, \alpha_m)$ be an m-tuple such that $\| \sum_1^m \alpha_i e_i \| = 1$ and

$$(15) \qquad \frac{1}{\left[\sum_1^m |\alpha_i|^r\right]^{1/r}} = \frac{\left\|\sum_1^m \alpha_i e_i\right\|}{\left[\sum_1^m |\alpha_i|^r\right]^{1/r}} < M\lambda^{-1} .$$

As is true for all basic sequences with characteristic not less than ε, $\| \sum_1^m \alpha_i e_i \| \geqq (1/2)\varepsilon |\alpha_k|$ for each k. Thus it follows from (15) that

$$(16) \qquad |\alpha_k| \leqq \frac{2}{\varepsilon}\left\|\sum_1^m \alpha_i e_i\right\| < \frac{2M}{\varepsilon\lambda}\left[\sum_1^m |\alpha_i|^r\right]^{1/r} .$$

Since $M < \phi$ and $2\phi < \lambda^2\varepsilon$, it follows from (16) and (12) that

$$|\alpha_k|^r < \lambda^r \sum_1^m |\alpha_i|^r < \frac{1}{n}(1-\lambda)\sum_1^m |\alpha_i|^r .$$

Therefore, there is a sequence of n integers $\{m(1), \cdots, m(n) = m\}$ such that, for each j,

$$\left|\left[\sum_{i=1}^{m(j)} |\alpha_i|^r - \frac{j}{n}\sum_1^m |\alpha_i|^r\right]\right| < \frac{1}{2n}(1-\lambda)\sum_1^m |\alpha_i|^r .$$

Let us write

$$\sum_1^m \alpha_i e_i = \sum_1^{m(1)} \alpha_i e_i + \sum_{m(1)+1}^{m(2)} \alpha_i e_i + \cdots + \sum_{m(n-1)+1}^{m} \alpha_i e_i$$
$$= \sum_1^n u_j ,$$

where $u_j = \sum_{m(j-1)+1}^{m(j)}$ with $m(0) = 0$. Then we have, for each j,

$$\left|\left[\sum_{m(j-1)+1}^{m(j)} |\alpha_i|^r - \frac{1}{n}\sum_1^m |\alpha_i|^r\right]\right| < \frac{1}{n}(1-\lambda)\sum_1^m |\alpha_i|^r .$$

This implies that

$$\frac{1}{n}\lambda\sum_1^m |\alpha_i|^r < \sum_{m(j-1)+1}^{m(j)} |\alpha_i|^r < \frac{1}{n}(2-\lambda)\sum_1^m |\alpha_i|^r < \frac{1}{n}\lambda^{-1}\sum_1^m |\alpha_i|^r$$

and

$$(17) \qquad \sum_{m(j-1)+1}^{m(j)} |\alpha_i|^r > \lambda^2 \sup\left\{\sum_{m(k-1)+1}^{m(k)} |\alpha_i|^r \colon 1 \leqq k \leqq n\right\} .$$

It follows from (15), (14), (17), and (13) that, for each j,

$$\frac{\lambda}{\left[\sum_1^m |\alpha_i|^r\right]^{1/r}} < M \leqq \frac{||u_j||}{\left[\sum_{m(j-1)+1}^{m(j)} |\alpha_i|^r\right]^{1/r}} < \frac{||u_j||}{\lambda^{2/r}\left[\sup_k \sum_{m(k-1)+1}^{m(k)} |\alpha_i|^r\right]^{1/r}}$$
$$< \frac{||u_j||}{\lambda^3\left[\sum_1^m |\alpha_i|^r\right]^{1/r}},$$

so that $||u_j|| > \lambda^4$. Since $\{e_i\}$ is a basis with constant not less than ε and $\lambda^4 > 4\phi^2/\varepsilon^2$, this implies

$$\left\|\sum_1^n a_j u_j\right\| \geqq \frac{1}{2}\varepsilon\, ||a_k u_k|| \geqq \frac{1}{2}\varepsilon\lambda^4\, |a_k| \geqq \frac{2\phi^2}{\varepsilon}\, |a_k| = \alpha'\, |a_k|$$

for all numbers $\{a_i\}$ and each $k \leqq n$. Now we can use

$$1 = \left\|\sum_1^m \alpha_i e_i\right\| = \left\|\sum_1^n u_j\right\| \geqq \varepsilon\left\|\sum_1^k u_j\right\|$$

to obtain $||\sum_1^k u_j|| \leqq 1/\varepsilon < \beta'$.

THEOREM 4. *Let B be a Banach space that is super-reflexive. If $0 < 2\phi < \varepsilon \leqq 1 < \Phi$, then there are numbers r and s for which $1 < r < \infty$, $1 < s < \infty$ and, if $\{e_i\}$ is any normalized basic sequence in B with characteristic not less than ε, then*

$$\phi\, [\sum |a_i|^r]^{1/r} \leqq ||\sum a_i e_i|| \leqq \Phi\, [\sum |a_i|^s]^{1/s}$$

for all numbers $\{a_i\}$ such that $\sum a_i e_i$ is convergent.

An examination of the proofs of Theorems 2 and 3 will show that essentially the same arguments can be used for nonseparable Banach spaces and unconditional basic subsets. Therefore:

THEOREM 5. *Let B be Banach space that is super-reflexive. If $0 < 2\phi < \varepsilon \leqq 1 < \Phi$, then there numbers r and s for which $1 < r < \infty$, $1 < s < \infty$ and, if $\{e_\alpha\}$ is any normalized unconditional basic subset of B with characteristic not less than ε, then*

$$\phi\, [\sum |a_\alpha|^r]^{1/r} \leqq ||\sum a_\alpha e_\alpha|| \leqq \Phi\, [\sum |a_\alpha|^s]^{1/s},$$

for all numbers $\{a_\alpha\}$ such that $\sum a_\alpha e_\alpha$ is convergent.

It is stated in [4] that it is not known whether B is isomorphic to a space that is uniformly convex and uniformly smooth if, for each normalized basic sequence $\{e_i\}$ in B, there are positive numbers ϕ, Φ, r and s such that $1 < r < \infty, 1 < s < \infty$, and

$$\phi\, [\sum |a_i|^r]^{1/r} \leqq ||\sum a_i e_i|| \leqq \Phi\, [\sum |a_i|^s]^{1/s}.$$

This conjecture would be strongly suggested by the next theorem, if it should be true that every super-reflexive space is isomorphic to a uniformly convex space. It would then also follow that uniform convexity, uniform smoothness, and super-reflexivity are equivalent within isomorphism and that the existence of numbers ϕ, Φ, r and s that satisfy the inequalities of Theorem 4 could be deduced from the results of Gurariǐ and Gurariǐ [4].

THEOREM 6. *Each of the following is a necessary and sufficient condition for a Banach space B to be super-reflexive.*

(a) *If $0 < 2\phi < \varepsilon \leqq 1 < \Phi$, then there are numbers r and s for which $1 < r < \infty$, $1 < s < \infty$, and, if $\{e_i\}$ is any normalized basic sequence in B with characteristic not less than ε, then*

$$\phi\,[\sum |a_i|^r]^{1/r} \leqq \|\sum a_i e_i\| \leqq \Phi\,[\sum |a_i|^s]^{1/s}\,,$$

for all number $\{a_i\}$ such that $\sum a_i e_i$ is convergent.

(b) *If $0 < \varepsilon \leqq 1 < \Phi$, then there is a number s for which $1 < s < \infty$, and, if $\{e_i\}$ is any normalized basic sequence in B with characteristic not less than ε, then*

$$\|\sum a_i e_i\| \leqq \Phi\,[\sum |a_i|^s]^{1/s}\,,$$

for all numbers $\{a_i\}$ such that $\sum a_i e_i$ is convergent.

(c) *There exist numbers ε, Φ and s such that $0 < \varepsilon < 1/2$, $1 < s < \infty$, and, if $\{e_i\}$ is any normalized basic sequence in B with characteristic not less than ε, then*

$$\|\sum a_i e_i\| \leqq \Phi\,[\sum |a_i|^s]^{1/s}\,, \tag{18}$$

for all numbers $\{a_i\}$ such that $\sum a_i e_i$ is convergent.

Proof. It follows from Theorem 4 that super-reflexivity implies (a). The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are purely formal. To prove that (c) implies that B is super-reflexive, let us suppose that B is not super-reflexive and that there exist numbers ε, Φ and s as described in (c). Choose a positive integer n such that

$$n^{1-1/s} > \frac{\Phi}{\varepsilon}\,. \tag{19}$$

It is known that in (ii) of Theorem 1 we can require that $\varepsilon < \alpha = \beta$ (see the definition of P_3 and Theorem 6, both in [8]). Therefore there is a subset $\{x_1, \cdots, x_n\}$ of the unit ball for which $\|x\| > \varepsilon$ if $x \in \operatorname{conv}\{x_1, \cdots, x_n\}$ and $\|\sum_1^n a_i x_i\| \geqq \beta \|\sum_1^k a_i x_i\|$ for all $k < n$ and all numbers $\{a_1, \cdots, a_n\}$. Then $\{x_i\}$ can be the initial segment of a basic sequence with characteristic not less than ε and it follows from

(18) that

$$\left\| \sum_1^n x_i \right\| \leqq \Phi n^{1/s} .$$

Since $\| \sum_1^n x_i \| > n\varepsilon$, we have a contradiction of (19).

Recall that, relative to a basis $\{e_i\}$, a *block basic sequence* is a sequence $\{e_i'\}$ for which there is an increasing sequence of positive integers $\{n(i)\}$ such that $n(1) = 1$ and

$$e_k' = \sum_{n(k)}^{n(k+1)-1} a_i e_i , \qquad k = 1, 2, \cdots .$$

THEOREM 7. *A Banach space B is reflexive if B has a basis $\{e_i\}$ and, for each normalized block basic sequence $\{e_i'\}$ of $\{e_i\}$, there are positive numbers ϕ, Φ, r and s such that $1 < r < \infty$, $1 < s < \infty$, and*

$$\phi\, [\sum | a_i |^r]^{1/r} \leqq \| \sum a_i e_i' \| \leqq \Phi\, [\sum | a_i |^s]^{1/s} , \tag{20}$$

for all numbers $\{a_i\}$ such that $\sum a_i e_i'$ is convergent.

Proof. If $\{e_i\}$ is not boundedly complete, there is a sequence $\{u_i\}$ and a positive number Δ such that $\| \sum_1^n u_i \|$ is bounded, $\| u_i \| > \Delta$, and

$$u_i = \sum_{n(k)}^{n(k+1)-1} a_i e_i , \qquad k = 1, 2, \cdots ,$$

where $\{n(i)\}$ is an increasing sequence of positive integers. Let $e_i' = u_i/\| u_i \|$. Then $\| (\sum_1^n \| u_i \| e_i') \|$ is bounded, but there do not exist $\phi > 0$ and $1 < r < \infty$ such that $\phi \sum_1^n \| u_i \|^r > \phi n \Delta^n$ is bounded. If $\{e_i\}$ is not shrinking, there is a normalized block basic sequence $\{e_i''\}$ such that $\| \sum_1^n e_i'' \| > (1/2)n$ for all n. But there do not exist Φ and $s > 1$ such that $\Phi n^{1/s} > (1/2)n$ for all n. Thus $\{e_i\}$ is boundedly complete and shrinking, which implies B is reflexive [2, Theorem 3, p. 71].

The next example shows that Theorem 7 can not be strengthened by assuming that (20) is satisfied only for a basis for B, even if $\phi = \Phi = 1$, $s = 2$, and r is close to 2.

EXAMPLE. Choose $r > 2$ and positive integers $\{n_i\}$ so that $(n_i)^{(1/2)r-1} > 2^i$ for each i. For each k, let v^k be the sequence that has zeros except for k initial blocks, the ith block having n_i components each equal to $(n_i)^{-1/2}$. Let B be the completion of the space of all sequences of real numbers with only a finite number of nonzero components and, if $x = \{x_i\}$,

$$\| x \| = \inf \{(\sum u_i^2)^{1/2} + \sum | a_k |: x = u + \sum a_k v^k\} . \tag{21}$$

If $\| \{y_i\} \|_r$ denotes $[\sum | y_i |^r]^{1/r}$, then $(\sum u_i^2)^{1/2} \geqq \| u \|_r$ and

$$\| v^k \|_r = [n_1^{1-1/2r} + n_2^{1-1/2r} \cdots + (n_{p(k)})^{1-1/2r}]^{1/r} < 1 .$$

Therefore

$$\| x \| \geqq \| u \|_r + \sum \| a_k v^k \|_r \geqq \| x \|_r .$$

It follows directly from (21) that $\| x \| \leqq (\sum x_i^2)^{1/2}$. It follows from the facts that $\| v^k \| \leqq 1$ for all k and that a sequence has norm 1 if it contains all zeros except for one block of n_i terms each equal to $n_i^{-1/2}$, that the natural basis for B is not boundedly complete and B is not reflexive.

It was shown by N. I. Gurariĭ [5, Theorem 7] that, for any r and s with $1 < r < \infty$ and $1 < s < \infty$, there is a basis $\{e_i\}$ for Hilbert space such that for any positive numbers ϕ and Φ there are finite sequences $\{a_i\}$ and $\{b_i\}$ for which

$$\phi [\sum | a_i |^r]^{1/r} > \| \sum a_i e_i \| \quad \text{and} \quad \| \sum b_i e_i \| > \Phi [\sum a_i |^s]^{1/s} .$$

Thus for Hilbert space there can be neither an upper bound $\rho < \infty$ for r nor a lower bound $\sigma < 1$ for s in Theorems 2-5, even if ϕ and Φ are allowed to depend on the basic sequence.

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Received July 20, 1971. This work was supported in part by NSF Grant GP 20838.

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