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GENERALIZED RAMSEY THEORY FOR GRAPHS, III. SMALL OFF-DIAGONAL NUMBERS

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The classical Ramsey theory for graphs studies the Ramsey numbers $r(m, n)$. This is the smallest p such that every 2-coloring of the lines of the complete graph K_p contains a green K_m or a red K_n. In the preceding papers in this series, we developed the theory and calculation of the diagonal numbers $r(F)$ for a graph F with no isolated points, as the smallest p for which every 2-coloring of K_p contains a monochromatic F. Here we introduce the off-diagonal numbers: $r(F_1, F_2)$ with $F_1 \neq F_2$ is the minimum p such that every 2-coloring of K_p contains a green F_1 or a red F_2. With the help of a general lower bound, the exact values of $r(F_1, F_2)$ are determined for all graphs F_i with less than five points having no isolates.

1. Introduction. The small ($p \leqq 4$ points) graphs F_i having no isolated points are shown in Figure 1, together with their symbolic names, following the notation for operations on graphs in the book [3, p. 21]. In fact, we follow the terminology and notation of this book throughout.

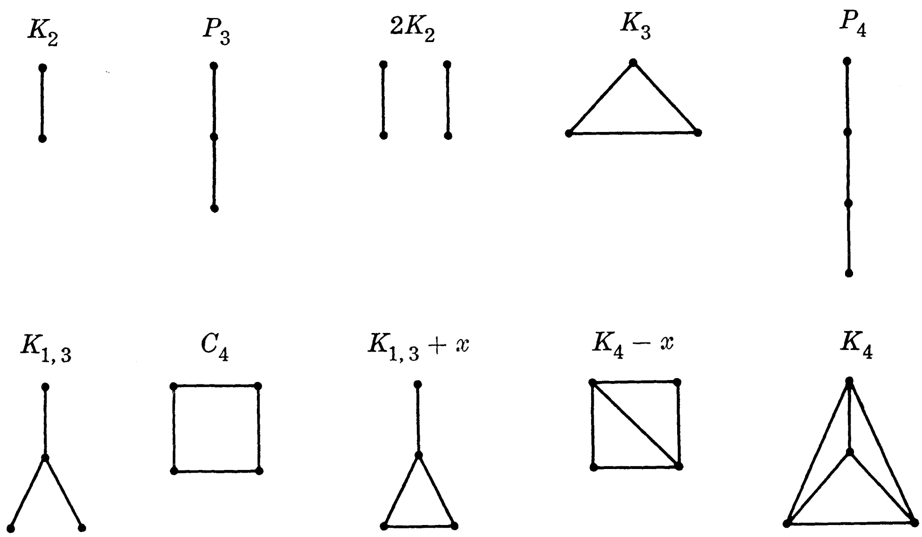

FIGURE 1

In [1, 2], we defined the number $r(F)$ as the minimum p for which every 2-coloring (of the lines) of K_p contains a monochromatic subgraph F. The number $r(F_1, F_2)$ is the corresponding smallest p

such that every 2-coloring of K_p contains a green F_1 or a red F_2. Obviously $r(F) = r(F, F)$, so that the numbers $r(F)$ are diagonal within the $r(F_1, F_2)$.

There is an equivalent formulation of the definition of $r(F_1, F_2)$ in terms of graphical complementation rather than 2-colorings of a complete graph. Namely, $r(F_1, F_2)$ is the minimum p such that whenever a p-point graph G does not have F_1 as a subgraph, then its complement $\bar{G}$ contains F_2. It is convenient to assign numbers to the following immediate consequences of the definition: symmetry, monotonicity, and a crude lower bound,

$$(1) \qquad r(F_1, F_2) = r(F_2, F_1)$$

$$(2) \qquad F_1' \subset F_1 \text{ and } F_2' \subset F_2 \text{ imply } r(F_1', F_2') \leqq (F_1, F_2)$$

$$(3) \qquad r(F_1, F_2) \geqq \max\,(p(F_1), p(F_2))\ .$$

When F_1 and F_2 are both complete graphs, we have specialized to $r(K_m, K_n) = r(m, n)$, the classical Ramsey numbers for graphs. As all the numbers $r(m, n)$ are known for $m, n = 2, 3, 4$, we begin with some information about off-diagonal Ramsey numbers for small F_1 and F_2. The existence of the diagonal numbers $r(n, n)$ was established by Ramsey [4] himself; that of all the other numbers $r(F_1, F_2)$ follows from (2).

From [3, p. 17], we have the following values of $r(m, n)$:

$m \backslash n$	2	3	4
2	2	3	4
3		6	9
4			18

In [2], the numbers $r(F)$ are determined for the 10 graphs of Fig. 1:

F	K_2	P_3	$2K_2$	K_3	P_4	$K_{1,3}$	C_4	$K_{1,3}+x$	K_4-x	K_4
$r(F)$	2	3	5	6	5	6	6	7	10	18

It is obvious that $r(K_2, F) = p(F)$, the number of points in F.

2. The simplest Ramsey numbers.

We now obtain two equations which give the next two rows in Table 1.1, the first for Ramsey numbers involving $2K_2$ and the second for P_3.

LEMMA 1. *For any graph F with no isolates,*

$$r(2K_2, F) = \begin{cases} p(F) + 2 & \textit{if } F \textit{ is complete} \\ p(F) + 1 & \textit{otherwise.} \end{cases}$$

Proof. First, when F is complete, we have $r(2K_2, F) > p(F) + 1$ because a 2-coloring of K_{p+1} in which the green lines form just one triangle cannot have a red K_p. On the other hand, if a 2-coloring of K_{p+2} has no green $2K_2$, then the green lines form either a star or a triangle, so there must be a red K_p.

Secondly when F is not complete, it is a subgraph of $K_p - x$. In an arbitrary 2-coloring of K_{p+1} which does not contain a green $2K_2$, the green lines again form a star or a triangle. When there is a green star, there must be a red K_p. And when we have a green triangle, there must appear a green $K_p - x$. Thus $r(2K_2, F) \leqq p(F) + 1$. The equality follows from the 2-coloring of K_p with red K_{p-1} and a green star $K_{1,p-1}$.

The next question is a bit more subtle.

LEMMA 2. *For any graph F with no isolates,*

$$r(P_3, F) = \begin{cases} p(F) \textit{ if } F \textit{ has a 1-factor} \\ 2p(F) - 2\beta_1(F) - 1 \textit{ otherwise.} \end{cases}$$

Proof. In each 2-coloring of K_m without a green P_3, all the green lines are independent. In other words, the green graph is a subgraph of $[m/2]K_2$ or, equivalently, the red graph contains $K_m - [m/2]K_2$. (For m even, this graph has been called a "party graph" by A. J. Hoffman because everyone talks to everyone else with the exception that nobody talks to his own spouse.) Thus, $r(P_3, F)$ is the smallest m such that F is a subgraph of $K_m - [m/2]K_2$.

For any graph F with p points, we have the maximum number of independent lines in the complement of F, $\beta_1(\bar{F}) = n$ if and only if $F \subset K_p - nK_2$. Thus, if $\bar{F}$ has a 1-factor, i.e., $\beta_1(\bar{F}) = p/2$, then we have $F \subset K_p - (p/2)K_2$ or $r(P_3, F) \leqq p$. The equality follows trivially from (2).

Now, let $\bar{F}$ have no 1-factor, so that $\beta_1(\bar{F}) = n < p/2$. If $m = 2p - 2n - 1$, then any 2-coloring of K_m having no green P_3 has a red $K_m - [m/2]K_2 = K_m - (p - n - 1)K_2$. We will show that such a coloring has a red F. Starting with the simple inclusion $(p - n - 1)K_2 \cup K_1 \subset nK_2 \cup (p - 2n)K_1$, and taking complements by merely removing the indicated number of independent lines from a complete graph of the proper size, we obtain $K_p - nK_2 \subset K_m - (p - n - 1)K_2$. Thus, we have $r(P_3, F) \leqq 2p - 2n - 1$. On the other hand, the 2-coloring of K_{m-1} which has just $(m - 1)/2 = p - n - 1$ green independent lines and leaves as the remaining red graph $K_{m-1} - ((m-1)/2)K_2$ already has no green P_3. It contains no red F either, for otherwise $((m-1)/2)K_2 \subset \bar{F}$ or equivalently $n = \beta_1(\bar{F}) > (m-1)/2 = p - n - 1$, contradicting $n < p/2$ and proving Lemma 2.

3. A useful lower bound. For our last lemma, we easily derive a simple lower bound which is not at all sharp in general, but luckily happens to be rather useful in establishing the values of $r(F_1, F_2)$ for the 10 small graphs of Fig. 1.

LEMMA 4. *Let F_1 and F_2 be two graphs (not necessarily different) with no isolated points. Let c be the number of points in a largest connected component of F_1, and let χ be the chromatic number of F_2. Then the following lower bound holds:*

$$r(F_1, F_2) \geqq (c-1)(\chi-1) + 1 .$$

Proof. Consider the graph $G = (\chi - 1)\, K_{c-1}$. Since G has no component with at least c points, it cannot possibly contain F_1. On the other hand, the complement $\bar{G}$ is $(\chi - 1)$-chromatic and hence cannot contain the χ-chromatic graph F_2. The inequality follows at once, as G has $(c-1)(\chi-1)$ points.

Remarkably, we shall find that in all but the two instances $r(K_{1,3}, C_4) \geqq 4$ and $r(K_4 - x, K_4) \geqq 10$, this lower bound turns out to yield the exact number for $r(F_1, F_2)$.

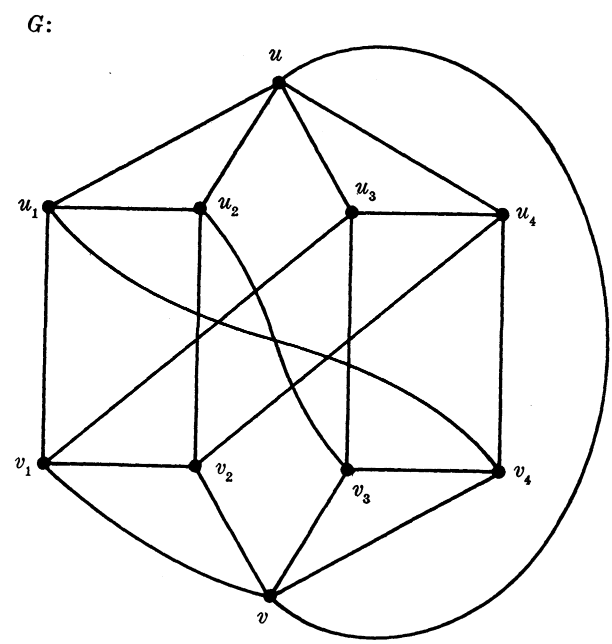

FIGURE 2

Referring to Table 2 below, we next show that better lower bounds than 4 and 10 respectively are given by

$$(4) \qquad r(K_{1,3}, C_4) \geqq 6$$

$$(5) \qquad r(K_4 - x, K_4) \geqq 11 .$$

Later we will see that (4) and (5) give the correct values of these two Ramsey numbers.

To prove (4) we need only exhibit a graph G with 5 points such that G has no $K_{1,3}$ (i.e., no point of degree exceeding 2) and $\bar{G}$ has no 4-cycle. Clearly $G = C_5$ works.

Similarly (5) can be verified by producing G with 10 points not containing $K_4 - x$ such that $\beta_0(G) < 4$. This example is a bit trickier, but we finally found it.

The graph G of Fig. 2 has just four triangles, no two having a common line. Hence G does not contain $K_4 - x$. It is also easily seen that G has no set of 4 independent points.

4. Forcing forbidden subgraphs. For each pair F_1, F_2 of forbidden graphs, we must argue that when the number r of points is right, every graph G with r points not containing F_1 must have F_2 in its complement. In particular, we will prove the next 8 upper bounds which establish the remaining off-diagonal Ramsey numbers.

$$(6) \qquad r(P_4, K_{1,3}) \leqq 5$$

$$(7) \qquad r(P_4, C_4) \leqq 5$$

$$(8) \qquad r(K_{1,3}, C_4) \leqq 6$$

$$(9) \qquad r(K_{1,3} + x, K_4 - x) \leqq 7$$

$$(10) \qquad r(C_4, K_4 - x) \leqq 7$$

$$(11) \qquad r(K_{1,3} + x, K_4) \leqq 10$$

$$(12) \qquad r(C_4, K_4) \leqq 10$$

$$(13) \qquad r(K_4 - x, K_4) \leqq 11 .$$

Proof of (6) *and* (7). By coincidence, both (6) and (7) may be shown at one fell swoop. Let G have no 4-point path P_4 on its 5 points. There are only two possibilities for such a graph: either $G \subset K_2 \cup K_3$ or $G \subset K_{1,4}$. Taking complements, $K_{2,3} \subset \bar{G}$ or $K_4 \subset \bar{G}$, so that necessarily both $K_{1,3}$ and C_4 are subgraphs of $\bar{G}$.

Proof of (8). Taking G as a 6-point graph with all degrees $\leqq 2$

forces $\bar{G}$ to have each degree $\geqq 3$. Thus, in $\bar{G}$, the neighborhoods of any two nonadjacent points have at least two common points, so that $\bar{G}$ must contain C_4.

The next assertion (9) will automatically have several consequences by the monotonicity condition (2).

Proof of (9). Let G be an arbitrary graph of 7 points not containing $K_{1,3} + x$. We assume $\bar{G}$ does not contain $K_4 - x$ and proceed to derive a contradiction. There are two possibilities, depending on whether $G \supset K_3$. If G does have a triangle $u_1u_2u_3$, with the remaining points labeled v_j, then there can be no line u_iv_j in G. Now each pair of the points v_j is forced to be adjacent in G, for otherwise $\bar{G}$ would contain $K_4 - x$. Hence the points v_j induce K_4 in G, a contradiction.

Next, if G has no triangle, then it has 3 independent points u_1, u_2, u_3 since $r(K_3, K_3) = r(K_3) = 6$. Again, we denote the remaining four points by v_j. Each v_j must be adjacent in G to at least two of the points u_i, for otherwise $G \supset K_4 - x$. If there is even one line v_iv_j, then G contains $K_{1,3} + x$, contrary to the hypothesis. Thus $\bar{G}$ is forced to contain K_4, and *a fortiori* $K_4 - x$.

We now apply (2) and the inclusions

$$K_{1,3} + x \supset K_{1,3},\ P_4,\ K_3$$

to (9) to obtain at once the lower bounds

$$r(K_3, K_4 - x) \leqq 7 \tag{14}$$

$$r(P_4, K_4 - x) \leqq 7 \tag{15}$$

$$r(K_{1,3}, K_4 - x) \leqq 7\ . \tag{16}$$

Similarly $K_4 - x \supset K_{1,3} + x$, C_4, $K_{1,3}$, P_4 and (2) applied to (14) give

$$r(K_3, P_4) \leqq 7 \tag{17}$$

$$r(K_3, K_{1,3}) \leqq 7 \tag{18}$$

$$r(K_3, C_4) \leqq 7 \tag{19}$$

$$r(K_3, K_{1,3} + x) \leqq 7\ . \tag{20}$$

Similarly by (15),

$$r(P_4, K_{1,3} + x) \leqq 7\ , \tag{21}$$

and by (16),

(22) $$r(K_{1,3},\ K_{1,3}+x) \leqq 7\ .$$

Proof of (10). Let G be an arbitrary graph with 7 points and no C_4. We will assume $\bar{G} \not\supset K_4 - x$ and deduce a contradiction.

In the proof, we distinguish two cases according to whether there is or is not a point u of degree smaller than three. In the first case, we delete the point u together with its neighbors and are left with a subgraph H of G having at least four points. Clearly, H has no C_4 because G has none. Thus, as $r(P_3, C_4) = 4$ by Lemma 2, $\bar{H}$ is forced to contain P_3. By definition of H, u is adjacent to no point in H. Therefore, $\bar{G}$ contains $K_4 - x$, contradicting the assumption.

Next, we consider the second case where each point in G has degree at least three. Now the inequality (9), $r(K_{1,3} + x, K_4 - x) \leqq 7$, proved above, implies $K_{1,3} + x \subset G$. *A fortiori*, G contains a triangle $u_1u_2u_3$. Now, since each point of G has degree at least three and G contains no C_4, we conclude that there are three other points v_1, v_2, v_3 such that u_iv_i is a line of G for each $i = 1, 2, 3$. In other words, G contains the subgraph shown in Figure 3. Actually, it is easy to check that the graph in Fig. 3 is the subgraph of G *induced* by u_1, u_2, u_3, v_1, v_2, v_3, for the addition of any line to this graph produces C_4. But then $\bar{G}$ contains $K_4 - x$ with points u_1, v_1, v_2, v_3 again contradicting the assumption.

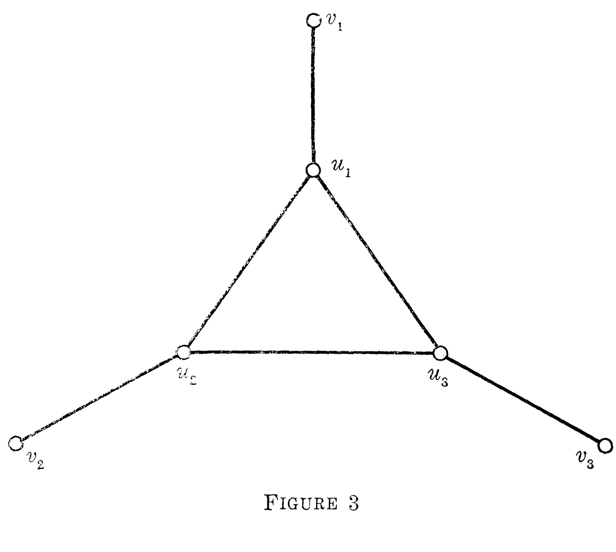

FIGURE 3

Proof of (11). Assume there is a graph G with 10 points such that G contains no $K_{1,3} + x$ and $\beta_0(G) < 4$. As $r(K_3, K_4) = r(3, 4) = 9$, G contains a triangle $u_1u_2u_3$. Let the other points in G be $v_j (j = 1, 2, \cdots, 7)$. There cannot be any line u_iv_j for otherwise G would contain a $K_{1,3} + x$. Now, let us consider the subgraph H of G spanned by the

v_j's. H has 7 points and no $K_{1,3} + x$ because G has none. Thus, the inequality (20) written in the form $r(K_{1,3} + x, K_3) \leqq 7$ implies the existence of three independent v_j's. Since u_1 is adjacent to none of these, we then have $\beta_0(G) \geqq 4$, contrary to the initial assumptions, completing the proof of (11).

Now we can apply (2) and the inclusions $K_{1,3} + x \supset K_{1,3}, P_4$ to (11) to obtain two more upper bounds,

$$r(K_{1,3}, K_4) \leqq 10 \tag{23}$$

$$R(P_4, K_4) \leqq 10 . \tag{24}$$

It is quite convenient to have another lemma for the proof of (12).

LEMMA 3. *If a graph G with p points has minimum degree d and $d(d-1) > p-1$, then G contains C_4.*

Proof. Let n be the total number of paths P_3 contained in G. There are exactly p choices for the midpoint of P_3, and for each fixed midpoint at least $\binom{d}{2}$ choices of the endpoints. Therefore $n \geqq p\binom{d}{2} > \binom{p}{2}$ so there must be two distinct paths P_3 in G with the same pair of endpoints, and hence a cycle C_4.

Proof of (12). Let G be a graph with 10 points such that the point independence number $\beta_0(G) < 4$. Then necessarily the chromatic number $\chi(G) \geqq 4$. Hence by Brooks' Theorem, see [3, p. 128], either K_4 (and hence C_4) is contained in G, or the degree of each point of G is at least four in which case the conclusion follows from Lemma 3.

Proof of (13). We have to show that there is no graph G with 11 points such that $K_4 - x \not\subset G$ and $\beta_0(G) < 4$, so again we assume the contrary. Our first aim is to show that G must be regular of degree 4. This will be done by degrees, considered as possible separate cases.

Case 1. *G has a point u of degree* $\geqq 7$. Then the neighborhood subgraph H of u (induced by the neighborhood of u) has at least 7 points and clearly contains no set of four independent points. By Lemma 2, $r(P_3, K_4) = 7$, so H must contain P_3, which on joining u implies $K_4 - x \subset G$. This contradiction proves the impossibility of Case 1.

Case 2. *G has a point u of degree* 6. Then the neighborhood subgraph H of u has exactly six points, no four of them being independent. As G contains no $K_4 - x$, H cannot contain P_3. It is easy to see that these conditions imply $H = 3K_2$; let the three independent lines of H be v_1w_1, v_2w_2 and v_3w_3. There are four other points in G; call one of them u_0. This point cannot be adjacent to both v_i and w_i for some $i \in \{1, 2, 3\}$ since otherwise G would contain $K_4 - x$. Thus, we may assume u_0 not adjacent to v_1, v_2, v_3. But then the points u_0, v_1, v_2, v_3 are independent contradicting $\beta_0(G) < 4$. Hence the asumption of Case 2 is false.

Case 3. *G has a point u of degree* 5. Similarly as above, we can prove that the neighborhood graph H of u must be $2K_2 \cup K_1$. Let its two lines be u_1v_1 and u_2v_2, and let its fifth point be w. There are five other points in G. If all of them are adjacent to w, then the degree of w equals six. As we saw, this assumption led to a contradiction in Case 2. Thus there is a point w_0 adjacent neither to u nor to v. Clearly, w_0 cannot be adjacent to both u_1 and v_1 (nor to both u_2 and v_2) as otherwise G would contain $K_4 - x$. Thus, we may assume w_0 not adjacent to u_1, u_2. But then w_0, w, u_1 and u_2 form a set of four independent points, contradicting $\beta_0(G) < 4$.

Finally, to rule out any degree other than 4, we consider

Case 4. *G contains a point u of degree* $\leqq 3$. Then there is a set S of seven points in G which are distinct from u and not adjacent to u. The subgraph $\langle S \rangle$ of G induced by S contains no $K_4 - x$. Since by (14), $r(K_4 - x, K_3) \leqq 7$, $\langle S \rangle$ necessarily contains three independent points u_1, u_2, u_3 and hence G contains four independent points, namely u, u_1, u_2, u_3 contradicting $\beta_0(G) < 4$.

We have shown that each of the Cases 1–4 leads to a contradiction. Therefore, G must be regular of degree 4. Clearly, every line of G is contained in *at most* one triangle, for otherwise G would contain $K_4 - x$. On the other hand, if every line of G is in *exactly* one triangle, then the number of lines of G would be divisible by three. However, G has 22 edges and so it has a line, say uv, contained in no triangle. Let the other three neighbors of u be u_1, u_2, u_3 and let the other three neighbors of v be v_1, v_2, v_3. As uv is contained in no triangle, all these are distinct. Now, we show that the subgraph of G spanned by u_1, u_2, u_3 must contain exactly one line. For if it has none, then the points u_1, u_2, u_3, v would be independent; if it has more than one, then G would contain $K_4 - x$ with points u, u_1, u_2, u_3. Similarly, the subgraph of G spanned by v_1, v_2, v_3 also contains exactly one line. Let these two lines be u_1u_2 and v_1v_2. Next, let w be one of the remaining three points w_1, w_2, w_3 in G. This point cannot be adjacent to both u_1 and u_2 for G would then contain $K_4 - x$.

Thus, we may assume w not adjacent to u_1. If w is not adjacent to u_3, then u_1, u_3, w, v are four independent points, contradicting $\beta_0(G) < 4$. So w must be adjacent to u_3. As w is arbitrary, we conclude that each of the points w_1, w_2, w_3 is adjacent to u_3. By a symmetry argument, each of w_1, w_2, w_3 is adjacent to v_3. Then there can be no line w_iw_j in G, for otherwise F would contain $K_4 - x$ with points u_3, v_3, w_i, w_j. Thus the points w_1, w_2, w_3 are independent. But then the points u, w_1, w_2, w_3 are independent, contradicting $\beta_0 < 4$.

5. Conclusions. The following table summarizes the results obtained (for both diagonal and off-diagonal) generalized Ramsey numbers.

TABLE 2. Small generalized Ramsey numbers

	K_2	P_3	$2K_2$	K_3	P_4	$K_{1,3}$	C_4	$K_{1,3}+x$	K_4-x	K_4
K_2	2	3	4	3	4	4	4	4	4	4
P_3		3	4	5	4	5	4	5	5	7
$2K_2$			5	5	5	5	5	5	5	6
K_3				6	7	7	7	7	7	9
P_4					5	5	5	7	7	10
$K_{1,3}$						6	6	7	7	10
C_4							6	7	7	10
$K_{1,3}+x$								7	7	10
K_4-x									10	11
K_4										18

Notice the irregularity of the behavior of $r(F_1, F_2)$:

$$r(P_4, K_3) > r(P_4, P_4), r(K_3, K_3) .$$

On the other hand,

$$r(P_3, P_3) < r(P_3, K_3) < r(K_3, K_3)$$

(inequalities which continue to hold when all subscripts are increased to 4). These suggest the following

Conjecture. For any graphs F_1, F_2 with no isolates,

$$r(F_1, F_2) \geqq \min (r(F_1), r(F_2)) .$$

It would be a formidable task indeed to extend this table to *all* 23 of the 5-point graphs with no isolates. In particular this would include the determination (exact, of course) of $r(5, 5)$ which appears not intractable, but extremely complicated. Our experience show that some of these 5-point graphs will be more delicate to handle than others. Unless and until some more analytic, powerful, and automatic method is found for calculating the numbers $r(F_1, F_2)$, it is highly unlikely that these will be found for all the 6-point graphs and larger ones.

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