PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 1, 1972

STRUCTURE OF RIGHT SUBDIRECTLY IRREDUCIBLE RINGS II

M. G. DESHPANDE

The object of this paper is to determine the structure and properties of right subdirectly irreducible rings which are either local or self-injective. The rings in the latter class form a special case of the so-called right *PF* rings. By employing the notion of Feller's *X*-rings, it is proved that right *PF X*-rings are finite direct sums of full matrix rings over self-injective right subdirectly irreducible rings. Thus, whether or not right *PF X*-rings are left *PF* depends on the answer to the same question for the more elementary case of self-injective right subdirectly irreducible rings.

For a discussion of artinian and noetherian RSI rings, see [2].

1. Notation and preliminaries.

All rings considered have an identity and all modules are unitary. A module M_R is R-subdirectly irreducible if the intersection of all nonzero submodules of M is nonzero, which will then be called the heart of M_R. A ring R is RSI (right subdirectly irreducible) if R_R is R-subdirectly irreducible. The heart H of a RSI ring R is a two sided ideal. These and some of the following definitions and observations are given in [2] and we rewrite them for completeness. We will always use the following notation is connection with a RSI ring R. $H =$ heart, $N = H^l = \{x \in R: xH = 0\}$, $D = \mathrm{Hom}_R(H_R, H_R)$, $\hat{R} =$ injective hull of R_R, $K = \mathrm{Hom}_R(\hat{R}, \hat{R})$ and $L = \{f \in K: \ker f \neq 0\}$. In addition, for a local ring R, J will always denote the unique maximal right ideal. A ring R will be termed self-injective if R_R is injective.

We state the following theorem showing the relationship between R, N, H, D, K, L which has been proved in [2, p. 319].

THEOREM 1.1. *If R is RSI, then R/N is isomorphic to a subring of the division ring D and $D \cong K/L$.*

In connection with $QF-1$ algebras, faithful indecomposable modules play an important role. In the following proposition we prove that a RSI ring has a unique faithful indecomposable injective module. In this respect, it may be remarked that an artinian semisimple ring which is not simple is an example of a ring for which faithful indecomposable injectives don't exist; while over the ring of integers, for each prime p, by using [12, p. 145, Th. 7] or otherwise, one can verify that Z_{p^∞} is a faithful indecomposable injective module, and obviously they are all nonisomorphic.

PROPOSITION 1.2. *A RSI ring R has, up to isomorphism, a unique faithful indecomposable injective module.*

Proof. $\hat{R}_R$ is certainly faithful and injective. It is also indecomposable because every nonzero submodule contains H. Let M_R be any other such module. If $h \in H$ is a nonzero element, then $Mh \neq 0$ because M is faithful. Thus, for some $m \in M$, $mh \neq 0$. The mapping $x \rightarrow mx$ is then an isomorphism on H to mH which can be extended to an isomorphism of $\hat{R}$ into M. If N be the image of $\hat{R}$ under this isomorphism, N is injective and hence a direct summand of M. By indecomposability of M, we have $N = M$ and therefore $\hat{R} \cong M$.

2. Local RSI rings. We recall that a ring R is a left S-ring in the sense of F. Kasch [5, p. 455] if each proper right ideal has a nonzero left annihilator. It is known that [11, p. 412, Th. 2.9] a (right) self-injective ring is local iff it is right uniform. In the following an analogue of this is considered for RSI rings.

PROPOSITION 2.1. *A self-injective ring R is RSI iff it is a local left S-ring.*

Proof. If R is self-injective and RSI, then it is right uniform and thus local by the above. If $h \in H$, $h \neq 0$; then h^r is a maximal right ideal and so must be the unique maximal right ideal J. Thus $J^l \neq 0$. If A is any proper right ideal, then $A \subseteqq J$ implies that $0 \neq J^l \subseteqq A^l$ and hence R is a left S-ring. Conversely, R is self-injective and local implies that it is right uniform. Since R is a left S-ring, by [6, p. 237, 2.1] R contains a copy of the simple R-module R/J. Clearly, a right uniform ring containing a minimal right ideal must be RSI.

The above proof shows that a RSI local ring is necessarily a left S-ring. We now prove the following theorem[1] which will imply that a LSI (left subdirectly irreducible) left S-ring is local.

THEOREM 2.2. *Let R be a ring. For the three statements*

(i) *R is local,*

(ii) *there exists a bimodule* ${}_TM_R$ *such that* A^l *in* ${}_TM$ *is nonzero for any proper right ideal A of R, and* ${}_TM$ *is subdirectly irreducible,*

[1] The author is obliged to the referee for this version of the theorem and other helpful suggestions.

(iii) ${}_K\hat{R}$ *is subdirectly irreducible and* R *is a left* S*-ring; we have* (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). *If, further* R *is RSI, then we also have* (i) $\Rightarrow$ (iii).

Proof. (iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i). If M_0 is the heart of ${}_TM$, then $M_0 \subseteqq A^l$ for any proper right ideal A of R. Thus $A \subseteqq A^{lr} \subseteqq M_0^r$ which proves that R is local with $J = M_0^r$.

Now we assume that R is RSI and prove (i) $\Rightarrow$ (iii). That R is then a left S-ring is already noted above. Now, let h and a be any two nonzero elements respectively from H and $\hat{R}$. Since R is local and $h^r = J$, we have $a^r \subseteqq h^r$. Thus $ax \to hx$ is a homomorphism of aR into H which can be extended to some element f of K. Then $h = f(a) \in Ka$ which shows that each nonzero submodule of ${}_K\hat{R}$ contains h and so ${}_K\hat{R}$ is subdirectly irreducible. In fact, it can be easily seen that ${}_K\hat{R}$ and R_R have the same heart H.

Since a LSI, left S-ring R satisfies condition (ii) of the above theorem, we have in particular,

COROLLARY 2.3. *For a RSI and LSI ring, the following are equivalent.*

(i) R *is local*

(ii) R *is a left (right)* S*-ring.*

3. Self-injective RSI **rings.** In a local RSI ring R, the left annihilator N of H, and the right annihilator J of H (which will be the unique maximal right ideal) need not coincide, though obviously we must have $N \subseteqq J$. We show by an example[2] that this inclusion can be proper and then prove that for self-injective rings, $N = J$.

EXAMPLE 3.1. Let $F = k(x_1, \cdots, x_n, \cdots)$ be the field of rational functions in $x_1, x_2, \cdots$ over the field k of real numbers and L the subring of fractions with denominators prime to x_1. Let α and β denote an epimorphism and a monomorphism respectively on L to F given by $f(x_1, \cdots, x_n)^\alpha = f(0, x_1, \cdots, x_{n-1})$ and $\beta =$ inclusion. Let R be the ring defined by $(R, +) = L \oplus F$ and $(a, b)(c, d) = (ac, bc^\alpha + a^\beta d)$. If h denotes the element $(0, 1)$ of R, then every element of R can be written as $a + hb$ and $h^2 = 0$. It can be verified that R is RSI with heart $H = hR$ and that $a + hb$ is a unit in R iff a is a unit in L. Since L is a local ring, so is R. For this ring R, $N = H^l = \{(0, b): b \in F\}$ and $J = \{(a, b): a^\alpha = 0\}$. Thus $N \subsetneqq J$.

[2] Professor P. M. Cohn has kindly communicated this example to the author.

PROPOSITION 3.2. *If R is self-injective and RSI, $R/N \cong D$.*

Proof. It was shown in [2] that $f\colon R/N \to D$ defined by $f(a+N) = f_a$, where $f_a\colon H \to H$ is the left multiplication by a, is a monomorphism. If R is self-injective and $d\colon H \to H$ is any element of D, then there exists an element $a \in R$ such that $d = f_a = f(a+N)$ and $R/N \cong D$.

COROLLARY 3.3. *If R is self-injective and RSI, $N = J$.*

Proof. Self-injective and RSI implies local. Since R/N is a division ring, N must be a maximal right ideal and hence $N = J$.

We now give a characterization of RSI rings in the class of self-injective rings which is analogous to McCoy's Theorem [9, p. 382, Th. 1] for the commutative subdirectly irreducible rings.

THEOREM 3.4. *Let R be self-injective. Then R is RSI iff there exists a nonzero principal right ideal $X = xR$ and an ideal Y in R such that*

(i) *$Y^l = X$, so that X is a two sided ideal,*

(ii) *$X^l = Y$,*

(iii) *R/Y is a division ring, and*

(iv) *If a is an element of Y not in X, there exists an element b of Y not in X, such that $ab = x$.*

Proof. ($\Rightarrow$). If R is RSI, we choose X to be $H = hR$ and $Y = N$. Then (ii) holds by definition of N and (iii) is a consequence of 3.2 above. Since R is in this case local with $N =$ unique maximal right ideal, $aN = 0$ if and only if aR is a minimal right ideal. Thus $N^l = H$ which proves (i). Now let $a \in N$ such that $a \notin N$. Since $H \subseteqq aR$ we have $h = ab$ for some $b \in R$. (ii) implies that $b \notin H$. Also if $b \notin N$ then b is a unit which implies $a = hb^{-1} \in H$ contradicting the hypothesis on a.

($\Leftarrow$). By assuming (i), (ii), (iii), and (iv) we will show that every nonzero right ideal of R contains the fixed element x of R. Accordingly, let a be a nonzero element in a right ideal A of R. If $a \notin Y$, by (iii) we have $1 - ay \in Y$ for a suitable $y \in R$. Then by (ii), $(1-ay)x = 0$ which implies $x = ayx \in aR \subseteqq A$. If $a \in Y$ and $a \notin X$ then by (iv) we have $x \in aR \subseteqq A$. Lastly, if $a \in X$, then $a = xc$ for some $c \in R$. c cannot be in Y because then we would have $a = xc \in XY = 0$. Thus, again by (iii) $1 - cd = u \in Y$ for some $d \in R$. Then $ad = xcd = x(1-u) = x$ which proves that $x \in A$. Thus R is RSI with heart $X = xR$.

4. **Right *PF* *X*-rings.** Utumi [14, p. 56] defined a ring R to be right PF if every faithful right R-module is completely faithful.

These rings afford a nice generalization of QF rings and have been discussed by Azumaya [1], Osofsky [10], Kato [6, 7], Utumi [13, 14] and others. A ring R is right PF if and only if [1, p. 701, Th. 6] it is a finite direct sum of indecomposable injective right ideals each of which contains a unique minimal right ideal. It is not known whether a right PF ring is also left PF. Clearly, a self-injective RSI ring is right PF. Either by using the theory of right PF rings developed by the authors mentioned above, or directly as a consequence of our Theorem 2.2 it can be seen that a self-injective RSI ring is left subdirectly irreducible. (In this case, R is local, $\hat{R} = R$ and $K \cong R$). We state this as a

PROPOSITION 4.1. *A self-injective RSI ring is LSI.*

The following definition is due to Feller [3, p. 20, 2.2].

DEFINITION 4.2. *A ring R is called an X-ring if for every pair e, f of primitive idempotents such that $eR \not\cong fR$; $a \in eR$ and $a^r \cap fR \neq 0$ implies $afR = 0$.*

We state the following useful lemma whose proof is straightforward.

LEMMA 4.3. *If A and B are rings and $R = A \oplus B$ is the ring theoretic direct sum, then R is a self-injective ring iff each of A and B are self-injective rings.*

We are now in a position to prove the following

THEOREM 4.4. *A ring R is a right PF X-ring if and only if R is isomorphic to a finite direct sum of full matrix rings over self-injective RSI rings. Further, this decomposition is unique.*

Proof. If S is a self-injective RSI ring with heart H, then the $n \times n$ matrix ring S_n is self-injective [13, p. 172, Th. 8.3] and is the direct sum of indecomposable right ideals $e_{ii}S_n$, $i = 1, 2, \cdots, n$ each of which contains a unique minimal right ideal $e_{ii}H_n$. Consequently S_n is a right PF ring which is trivially an X-ring. Now, by using 4.3 we can see that any finite direct sum of such rings is again right PF. In order to show that it is also an X-ring, it is enough to remark that eR and fR are nonisomorphic only if they belong to different matrix rings, in which case $eRfR = 0$.

Conversely, Let R be a right PF X-ring. Then

4.5. $$R = e_1R \oplus \cdots \oplus e_nR$$

where $e_1, \cdots, e_n$ are primitive and let us assume that $e_1R, \cdots, e_kR$ $(k \leqq n)$ denote a complete set of nonisomorphic right ideals among the n summands in 4.5. Since each e_iR is indecomposable and injective, it is right uniform. By the same argument as in [3, p. 20, Th. 2.3] we conclude that $R = A_1 \oplus \cdots \oplus A_k$ where each A_i is the sum of all summands in 4.5 which are isomorphic to e_iR and $A_1, \cdots, A_k$ are all two sided ideals. Further, each A_i is isomorphic to a full matrix ring over e_iRe_i. That this decomposition is unique follows from [4, p. 42, Th. 1]. Also, by 4.3, each of the rings $A_1, \cdots, A_k$ is self-injective and hence by [13, p. 172, Th. 8.3] so are the rings e_iRe_i. Finally, if H_i is the unique minimal right ideal of R contained in e_iR, it can be verified that e_iRe_i is RSI with heart $e_iH_ie_i$. This prove the Theorem.

As a consequence of this theorem, a right PF X-ring will be left PF (if and) only if the self-injective RSI rings over which matrix rings appear in the above decomposition are also left self-injective. The author does not know if such is always the case.

REFERENCES

1. Goro Azumaya, *Completely faithful modules and selfinjective rings*, Nagoya Math. J., **27**, (1966), 697-708.
2. M. G. Deshpande, *Structure of right subdirectly irreducible ring I*, J. Alg., **17** (1971), 317-325.
3. E. H. Feller, *A type of quasi-Frobenius ring*, Canad. Math. Bull., **10** (1967), 19-27.
4. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. publ. **37** (1964).
5. F. Kasch, *Grundlagen Einer Theorie der Frobeniuserweiterungen*, Math. Ann., **127** (1954), 453-474.
6. T. Kato, *Torsionless modules*, Tohoku Math. J., **20** (1968), 234-243.
7. ———, *Some generalisations of QF-Rings*, Proc. Japan Acad., **44** (1968), 114-119.
8. J. Lambek, *Lectures on rings and modules*, Blaisdell, Waltham, Mass. (1966).
9. N. H. McCoy, *Subdirectly irreducible commutative rings*, Duke Math. J., **12** (1945), 381-387.
10. B. L. Osofsky, *A generalization of quasi-Frobenius rings*, J. Alg., **4** (1966), 373-387.
11. M. Satyanarayana, *Characterisation of local rings*, Tohoku Math. J., **19** (1967), 411-416.
12. Chi Te Tsai, *Report on Injective modules*, Queen's papers in pure and appl. Math., No. 6, Queen's Univ., Kingston, Ontario, (1966).
13. Yuzo Utumi, *On continuous rings and self-injective rings*, Trans. Amer. Math. Soc., **118** (1965), 158-173.
14. ———, *Self-injective rings*, J. Alg., **6** (1967), 56-64.

Received January 12, 1971 and in revised form June 16, 1971. A portion of this paper is included in the author's doctoral dissertation under the supervision of Professor Edmund H. Feller, submitted to University of Wisconsin-Milwaukee, in June, 1969.

MARQUETTEE UNIVERSITY