PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 2, 1972

RINGS ON ANALYTIC FUNCTIONS ON ANY SUBSET OF THE COMPLEX PLANE

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We prove that for any two subsets X, Y of C, the complex plane, X and Y are conformally homeomorphic if there is an isomorphism between $\mathfrak{A}(X)$ and $\mathfrak{A}(Y)$ which is the identity on constant functions.

It has been known for some time that the conformal structure of a domain in the complex plane or a Riemann surface is determined by the algebraic structure of certain rings of analytic functions on it. (See [3], [11], [12], [10], [9] and [8].) Iss'sa [5] shows this is also true for a Stein variety of positive dimension.

All functions considered here are complex single-valued.

DEFINITION 1. Let X be an arbitrary subset of C. A function f on X is said to be analytic at a point $p \in X$ if there is a power series $\sum_{n=0}^{\infty} \alpha_n (z-p)^n$ which converges for $|z-p| < R$, and $f(z) = \sum_{n=0}^{\infty} \alpha_n (z-p)^n$ for all $z \in X$ and $|z-p| < R$, where $R > 0$, and α_n is a complex number for each $n = 0, \cdots$, and f is said to be analytic on X if it is analytic at each point of X.

DEFINITION 2. Let X and Y be two arbitrary subspaces of C. A mapping τ from X to Y is said to be analytic mapping if τ is an analytic function on X and has values in Y. τ is said to be a conformal mapping if τ is analytic, one-to-one, and onto. (See [2, Ch. II. §2].) For any two subsets X, Y of C, X, Y are said to be conformally homeomorphic if there is a one-to-one conformal mapping from X onto Y.

Let X be an arbitrary subset of C, and $\mathfrak{A}(X) = \{f : f \text{ is analytic on } X\}$. We can then easily show that $\mathfrak{A}(X)$ forms a ring with the constant function of value 1 as the identity u. By [1, p. 145], if $f \in \mathfrak{A}(X)$ and $Z(f) = \{x \in X : f(x) = 0\} = \varnothing$, then $1/f \in \mathfrak{A}(X)$.

LEMMA 3. *For $p \in X$, there is an $f \in M_p = \{f \in \mathfrak{A}(X) : f(p) = 0\}$ such that $Z(f) = \{p\}$ and f belongs to no maximal ideal other than M_p.*

Proof. Let $f(z) = z - p$. Then that $f \in M_p$ and f belongs to no other fixed maximal ideal [4, 4.4] is clear. Now, suppose that M is a free maximal ideal [4, 4.1] such that $f \in M$. Since M is free, there is $g \in M$ such that $g(p) \neq 0$. Thus, we have $g(z) = \alpha + \sum_{j=0}^{\infty} \alpha_{k+j} (z-p)^{k+j}$ for $z \in X$ and $|z-p| < R$, for some $R > 0$, $\alpha_0 \neq 0$, $\alpha_k \neq 0$ and $k \geqq 1$.

Hence $\underline{\alpha_0}^* = g(z) - (z-p)^{k-1}\cdot f(z)\cdot h(z)$ for some $h \in \mathfrak{A}(X)$. Now f, $g \in M$ which is an ideal, $\underline{\alpha_0} \in M$. This is impossible as $\alpha_0 \neq 0$. Hence, the assertion holds.

LEMMA 4. *If Φ is an isomorphism from $\mathfrak{A}(X)$ onto $\mathfrak{A}(Y)$, then $\Phi(M_p)$ is a fixed maximal ideal.*

Proof. That $\Phi(M_p)$ is a maximal ideal is clear. From Lemma 3, there is an $f_0 \in M_p$ such that $Z(f_0) = \{p\}$, and f_0 belongs to no other maximal ideal. Consider $Z(\Phi(f_0))$. If $Z(\Phi(f_0)) = \varnothing$, then $\Phi(f_0)$ is a unit so that $\Phi(M_p)$ is the whole ring, $\mathfrak{A}(X)$. This is impossible. Hence, $Z(\Phi(f_0)) \neq \varnothing$. But if $Z(\Phi(f_0))$ contains more than one point, say q_1 and q_2, then $\Phi(f_0) \in M_{q_1}$ and M_{q_2} so that f_0 would belong to at least two maximal ideals which is again impossible. Hence $Z(\Phi(f_0)) = \{q\}$ for some $q \in Y$. Hence $\Phi(M_p) = M_q$ is fixed ideal.

THEOREM 5. *Let X and Y be two subsets of C, and Φ be an isomorphism from $\mathfrak{A}(Y)$ onto $\mathfrak{A}(X)$ such that it is the identity on the constant functions. Then Φ induces a mapping $\tau: X \to Y$, defined by $\Phi(g) = g \circ \tau$, and τ is a conformal mapping of X onto Y.*

Proof. Define τ to be a mapping from X to Y as follows: $\tau(p) = \cap Z[\Phi^{-1}(M_p)]$. By hypothesis Φ^{-1} is an isomorphism of $\mathfrak{A}(X)$ onto $\mathfrak{A}(Y)$. By Lemma 4, $\Phi^{-1}(M_p)$ is a fixed maximal ideal in $\mathfrak{A}(Y)$. Thus, τ is a single-valued mapping. Evidently, $M_{\tau(p)} = \Phi^{-1}(M_p)$, and τ is one-to-one and onto. Now, for each $g \in \mathfrak{A}(Y)$, and $p \in X$, let $\Phi(g)(p) = \alpha$. Then $\Phi(g) - \underline{\alpha} \in M_p$, $g - \Phi^{-1}(\underline{\alpha}) \in M_{\tau(p)}$, so that $g(\tau(p)) = \Phi^{-1}(\underline{\alpha})(\tau(p)) = \alpha = \Phi(g)(p)$. Hence $\Phi(g) = g \circ \tau$. Similarly, $\Phi^{-1}(f) = f \circ \tau^{-1}$, where τ^{-1}; $Y \to X$ with $\tau^{-1}(q) = \cap Z[\Phi(M_q)]$. If we choose $g(w) = w$ on Y, and $f(z) = z$ on X, then $\tau(p) = g \circ \tau(p)$, and $\tau^{-1}(q) = f \circ \tau^{-1}(q)$ are analytic. Hence, τ is a conformal mapping.

COROLLARY 6. *Let X and Y be two subsets of C, and Φ be an isomorphism of $\mathfrak{A}(X)$ onto $\mathfrak{A}(Y)$ which is the identity on real constant functions. Then X and Y can be decomposed respectively into $X_1 \cup X_2$ and $Y_1 \cup Y_2$ such that the sets X_1, X_2 are open and disjoint in X and similarly for Y_1 and Y_2, in such a way that X_1 is conformal with Y_1, and X_2 is anti-conformal with Y_2, where some of X_1, X_2, Y_1 and Y_2 could be empty.*

Note that a set is anti-conformal with another set if it is conformal with its complex conjugate.

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* $\underline{\alpha_0}$ stands for the constant function of value α_0.

Proof. As in Theorem 5, the mapping τ defined by $\tau(p) = \cap Z[\Phi^{-1}(M_p)]$ is one-to-one and onto. We know that $(\Phi(\underline{i}))^2 = \Phi(-\underline{1}) = -\underline{1}$, hence $\Phi(\underline{i}) = \underline{i}$, $-\underline{i}$ or $\underline{i}$ on one clopen subset of X, say X_1, and $-i$ on $X_2 = X - X_1$, (which is then a clopen subset). We will set $X_1 = X$ and $X_2 = X$, respectively, according as $\Phi(\underline{i}) = i$ and $\Phi(\underline{i}) = -i$. Therefore, $\Phi(\underline{\alpha}) = \underline{\alpha}$ on X_1, and $\bar{\alpha}$ on X_2 for any constant α. Then, by an argument similar to that used in Theorem 5, we can show that $\Phi(g) = g \circ \tau$ on X_1, and $\overline{g \circ \tau}$ on X_2; and $\Phi^{-1}(f) = f \circ \tau^{-1}$ on X_1 and $\overline{f \circ \tau^{-1}}$ on X_2, for any $g \in \mathfrak{A}(Y)$ and $f \in \mathfrak{A}(X)$. Hence the assertion holds.

REMARK. In Theorem 5, the condition that Φ is the identity on the constant functions can not be omitted. Consider $X = \{p\}$, $Y = \{q\}$. Then $\mathfrak{A}(X) = \boldsymbol{C} = \mathfrak{A}(Y)$. We know that there is an isomorphism of $\boldsymbol{C}$ to $\boldsymbol{C}$ other than $z \to z$ and $z \to \bar{z}$ (see [7, Remark on p. 119]). Define $\Phi: \mathfrak{A}(X) \to \mathfrak{A}(Y)$ in the obvious way. Then $\Phi(\alpha) \neq \alpha$ for some $\alpha \in \mathfrak{A}(Y)$. On the other hand, $\alpha \circ \tau(p) = \alpha$. Hence, $\Phi(\alpha) \neq \alpha \circ \tau$.

However, L. Bers shows that if X and Y are domains with boundary points, then every isomorphism of $\mathfrak{A}(Y)$ onto $\mathfrak{A}(X)$ induces a mapping which is either conformal or anti-conformal. (See [3].) Nevertheless, Royden [10], and Ozawa and Mizumoto [9] assumed that the given isomorphism preserves the constant functions. Recently, Nakai [8]** shows that if X and Y are open Riemann surfaces and Φ is such that $\Phi(i) = i$ (or $-i$), then Φ induces a conformal (or conjugate-conformal, resp.) mapping. Iss'sa [5]** shows that if X and Y are Stein varieties of positive dimensions, then Φ induces a unique conformal or a unique conjugate-conformal mapping.

THEOREM 7. *Let X and Y be two subsets of $\boldsymbol{C}$, and τ be a conformal mapping of X onto Y. Then the induced mapping τ', defined by $\tau'(g) = g \circ \tau$, is an isomorphism of $\mathfrak{A}(Y)$ onto $\mathfrak{A}(X)$ leaving the constant function unchanged.*

Proof. Use the Weirstrass' double-series theorem in [6] to show the composition of $g \circ \tau \in \mathfrak{A}(X)$ for any $g \in \mathfrak{A}(Y)$. The others are obvious.

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** The author wishes to express her thanks to the referee for bringing her attention to these two articles.

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Received April 14, 1971.

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