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THE SPECTRUM OF CERTAIN LOWER TRIANGULAR MATRICES AS OPERATORS ON THE l_p SPACES

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In this paper we compute the spectrum of the lower triangular matrices $A_a = (c_{m,n})$, where $c_{m,n} = (n+1)^a/(m+1)^{a+1}$, $m \geqq n \geqq 0$, a is real and the corresponding operator on l_p is bounded (see 4.1). This result and other lemmas are used to determine the spectrum of lower triangular matrices $p(n)/q(m)$, $m \geqq n \geqq 0$ as operators on l_p where p is a monic polynomial of degree a, q is a monic polynomial of degree $a+1$ and $q(m) \neq 0$ for $m = 0, 1, \cdots$. The spectrum is the diagonal together with the set $C_{a-p} - 1_{+1}$ when $a - p^{-1} + 1 > 0$, where $C_b = \{\lambda: |\lambda - (2b)^{-1}| \leqq (2b)^{-1}\}$ (see 4.3).

Our initial interest in this problem stems from conversations with Prof. Charles J. A. Halberg, Jr., who had conjectured and partially proved the conclusions of Theorem 4.1 for an operator equivalent to the special case $a = 1$.

1. Preliminaries.

In this section we set down the general notation and prove some preliminary lemmas.

General notation 1.1. Let X be a complex normed linear space. The norm is denoted $\| \ \|$ or $\| \ \|_X$ if it can be confused with another norm. The normed algebra of bounded linear operators on X is denoted $O(X)$. For $T \in O(X)$, $\mathrm{sp}(T)$ or $\mathrm{sp}(T, X)$ denotes the spectrum; that is, all complex numbers λ such that $\lambda - T$ does not have an inverse in $O(X)$. If $T \in O(X)$ and K is a subspace of X such that $TK \subset K$, we let $T|K$ denote the operator in $O(K)$ obtained by restricting T to K.

For $1 \leqq p \leqq \infty$ (p will always denote a number in this range) l_p is the usual normed linear space of complex p-summable sequences $x = (x_0, x_1, \cdots)$. We will be concerned with complex matrices $A = (a_{m,n})$, $0 \leqq m, n < \infty$. It is well known that there is a one-to-one correspondence between $O(l_p)$ and a class of matrices and that this correspondence is an algebra isomorphism. In this paper we will not disinguish between an operator in $O(l_p)$ and its corresponding matrix; in particular, we will speak of matrices as elements in $O(l_p)$. A lower triangular matrix $A = (a_{m,n})$ is a matrix such that $a_{m,n} = 0$ if $m < n$. $\mathscr{L}_p$ will denote the lower triangular matrices in $O(l_p)$. The set $\{a_{n,n}: n = 0, 1, \cdots\}$ is denoted $d(A)$. A sequence A_k of matrices is said to converge to A entrywise if the m, nth entry of A_k converges to the m, nth entry of A as $k \to \infty$ for each m, n. For $m = 0, 1, \cdots$ J_m will denote the sequences x in l_p such that $x(k) = 0$ if $k \geqq m$ and K_m will denote the sequences x in l_p such that $x(k) = 0$ if $k < m$. For $T \in \mathscr{L}_p$, the final spectrum of T, denoted $\mathrm{sp}_f(T)$, is defined as the set $\cap \{\mathrm{sp}(T \mid K_m): m = 0, 1, \cdots\}$. Finally, for each complex number z, A_z will denote the lower triangular matrix with m, nth entry

$$(n+1)^z/(m+1)^{z+1}$$

for $m \geqq n$ (real for real z).

The following three lemmas are easily established.

LEMMA 1.2. *A matrix $A = (a_{m,n})$ determines a bounded linear operator on l_1 if and only if $\sup_n \sum_m |a_{m,n}| < \infty$. The matrix A determines a bounded linear operator on l_∞ if and only if $\sup_m \sum_n |a_{m,n}| < \infty$. When one of these suprema is finite it is equal to the corresponding operator norm.*

LEMMA 1.3. *If $A = (a_{m,n})$ and $B = (b_{m,n})$ are matrices, $b_{m,n} \geqq 0$, $B \in O(l_p)$ and $|a_{m,n}| \leqq b_{m,n}$ for $0 \leqq m, n$, then $A \in O(l_p)$ and $\|A\| \leqq \|B\|$.*

LEMMA 1.4. *Suppose that S_n is a bounded sequence of matrices in $O(l_p)$, $1 \leqq p \leqq \infty$, such that S_n converges entrywise to the matrix S as $n \to \infty$. Then S is in $O(l_p)$ and $\|S\| \leqq \sup_n \|S_n\|$.*

LEMMA 1.5. *Let $T \in \mathscr{L}_p$. Suppose that λ is a complex number $\neq 0$, that the sequence $\sum_{n=0}^m (\lambda^{-1}T)^n$ canverges entrywise to the matrix U as $m \to \infty$ and that U is a bounded operator on l_p. Then $\lambda^{-1}U$ is the inverse of $\lambda - T$ in the algebra of bounded linear operators on l_p.*

Proof. Let $U_m = \lambda^{-1} \sum_{n=0}^m (\lambda^{-1}T)^n$. $(\lambda - T) U_m = U_m(\lambda - T) = I - (\lambda^{-1}T)^{m+1}$ and $(\lambda^{-1}T)^{m+1} \to 0$ entrywise as $m \to \infty$. Also, $U_m \to \lambda^{-1}U$ entrywise as $m \to \infty$. From these observations and the fact that the matrices involved are all lower triangular matrices, it is clear that $(\lambda - T)\lambda^{-1}U = \lambda^{-1}U(\lambda - T) = I$.

A decomposition 1.6. For a lower triangular matrix S and $m = 0, 1, \cdots$ we define the corresponding matrices E_m, N_m and B_m by: E_m has the same j, kth entry as S if $0 \leqq j, k < m$ and has all other entries $= 0$; N_m has the same j, kth entry as S if $k < m$, $j \geqq m$ and has all other entries $= 0$; $B_m = S - E_m - N_m$.

LEMMA 1.7. *If $S \in \mathscr{L}_p$, $m = 1, 2, \cdots$ and E_m, N_m, B_m correspond to S as in* 1.6, *then*

$$\mathrm{sp}\,(S) = \mathrm{sp}\,(E_m | J_m) \cup \mathrm{sp}\,(B_m | K_m) ,$$

(J_m, K_m defined in paragraph preceding 1.2).

Proof. For convenience, let $E = E_m$, $N = N_m$, $B = B_m$. It is obvious that $\mathrm{sp}\,(E + B) = \mathrm{sp}\,(E | J_m) \cup \mathrm{ps}\,(B | K_m)$. Thus, we need only show that

$$\text{(1)} \qquad \mathrm{sp}\,(S) = \mathrm{sp}\,(E + B) .$$

If $\lambda - S$ has an inverse, then

$$\lambda - E - B = \lambda - S + N = (I + N(\lambda - S)^{-1})(\lambda - S) ;$$

and, if $\lambda - E - B$ has an inverse, then

$$\lambda - S = \lambda - E - B - N = (I - N(\lambda - E - B)^{-1})(\lambda - E - B) .$$

From these two observations it is clear that in order to establish (1) we need only prove that NA is a nilpotent operator for each $A \in \mathscr{L}_p$. A simple computation shows that the matrix $A(NA)^m = (a_{j,k})$ has the property that $a_{j,k} = 0$, if $j < k + m$. Thus, $(NA)^{m+1} = 0$. This completes the proof.

LEMMA 1.8. *Suppose that $S \in \mathscr{L}_p$ and let S_m denote the operator in $O(K_m)$ obtained by restricting S to K_m, $m = 0, 1, \cdots$. If $\lambda - S_m$ has an inverse in $O(K_m)$, then $\lambda - S_k$ has an inverse in $O(K_k)$ for $k \geqq m$.*

Proof. Let T_m be the inverse of $\lambda - S_m$ and $k \geqq m$. If $x \in J_k$, then $T_m x = u + v$ where $u, v \in J_m$, $u(j) = 0$ for $j \geqq k$ and $v(j) = 0$ for $j < k$. It is clear from Lemma 1.7 (since l_p looks like K_m) that $\lambda \neq a_{k,k}$ for $k \geqq m$, where $S = (a_{m,n})$. If $u \neq 0$ and r is the smallest integer such that $u(r) \neq 0$, then $(\lambda - S_m)u(r) \neq 0$. Thus, $(\lambda - S_m)u \notin J_k$; and, since $(\lambda - S_m)v \in J_k$, it follows that $x = (\lambda - S_m)T_m x \notin J_k$. This contradiction shows that $u = 0$. Thus, T_m leaves J_k invariant; and, consequently, $\lambda - S_k$ has an inverse in $O(J_k)$.

Recall that for a matrix $A = (a_{m,n})$, $d(A)$ denotes the set $\{a_{n,n}: n = 0, 1, \cdots\}$.

THEOREM 1.9. *If $T \in \mathscr{L}_p$, then*

$$\text{(i)} \qquad \mathrm{sp}\,(T) = d(T) \cup \mathrm{sp}_f(T) .$$

Proof. It follows immediately from Lemma 1.7 and Lemma 1.8 that the left-hand set in (i) contains the right-hand set. If $\lambda \in \mathrm{sp}\,(T)$, then again by Lemma 1.7 it follows that either $\lambda \in \mathrm{sp}\,(B_m | K_m)$ for $m = 1, 2, \cdots$ or $\lambda \in \mathrm{sp}\,(E_m | J_m)$ for some $m = 1, 2 \cdots$. This completes the proof.

LEMMA 1.10. *Suppose that $S, T \in \mathscr{L}_p$ and let S_m, T_m denote the elements of $O(K_m)$ obtained by restricting S, T to K_m, respectively. If $\|T_m - S_m\| \to 0$ as $m \to \infty$, then*

$$\mathrm{sp}_f(S) = \mathrm{sp}_f(T) .$$

Proof. Suppose λ is not in $\mathrm{sp}_f(T)$. From the definition of $\mathrm{sp}_f(T)$ and Lemma 1.8 we conclude that there is an m_0 such that for $m \geqq m_0$

$$\lambda - S_m = (\lambda - T_m)(I + (\lambda - T_m)^{-1}(T_m - S_m))$$

in the algebra $O(J_m)$. Clearly, $\|(\lambda - T_m)^{-1}\|$ is bounded for $m \geqq m_0$ since $(\lambda - T_m)^{-1}$ is just the restriction of $(\lambda - T_{m_0})^{-1}$ to K_m, $m \geqq m_0$ (see proof of Lemma 1.8); and, by hypothesis, $\|T_m - S_m\| \to 0$ as $m \to \infty$. Thus, for some $m \geqq m_0$, $\lambda - S_m$ has an inverse in $O(J_m)$. This shows that λ is not in $\mathrm{sp}_f(S)$. By interchanging S and T in this argument we complete the proof.

LEMMA 1.11. *If $T \in \mathscr{L}_p$, then each isolated point in $\mathrm{sp}(T)$ is in $d(T)$.*

Proof. Suppose λ is an isolated point of $\mathrm{sp}(T)$. Let J be the image of l_p under the projection that corresponds to the spectral set $\{\lambda\}$ (see [2, p. 573]). Then J is an invariant subspace of T and $T_1 = \lambda + Q$ where $T_1 = T|J$ and $Q = (T - \lambda)|J$. Furthermore, $\mathrm{sp}(Q) = \{0\}$. Let r be the smallest integer $\geqq 0$ for which there corresponds an x in J such that $x(r) \neq 0$. From the definition of r and the fact that $T \in \mathscr{L}_p$ it follows that $Tx(r) = a_{r,r}x(r)$ for each $x \in J$ where $T = (a_{m,n})$. This together with the fact that $T_1 = \lambda + Q$ yields

$$Qx(r) = (a_{r,r} - \lambda)x(r),\ x \in J . \tag{2}$$

Since each $Q^n x$ is in J, (2) yields

$$Q^n x(r) = (a_{r,r} - \lambda)^n x(r),\ x \in J .$$

Choose $x \in J$ such that $x(r) \neq 0$. Then

$$|(a_{r,r} - \lambda)^n x(r)| \leqq \|Q^n\|\,\|x\| \quad n = 1, 2, \cdots$$

and the fact that $\mathrm{sp}(Q) = \{0\}$ shows that $a_{r,r} = \lambda$. This completes the proof.

2. **An Inequality.** Let V be the vector space of all complex matrices with the topology of entrywise convergence. It is clear that $O(l_p)$, $1 \leqq p \leqq \infty$, is continuously embedded in V. It is with respect to this embedding that we define the interpolation spaces

$$Q_s = [O(l_\infty), O(l_1)]_s\,, \quad 0 \leqq s \leqq 1\,,$$

as in [1, p. 114]. Let N_s denote the norm of the space Q_s.

LEMMA 2.1. *If $T \in Q_s$, then $T \in O(l_p)$ where $p^{-1} = s$ and $\|T\|_{O(l_p)} \leqq N_s(T)$.*

This lemma is an immediate consequence of [1, Sec. 10.2].

Recall that A_z is defined in 1.1.

LEMMA 2.2. *There is an absolute constant C such that if $u > 0$, $1 \leqq p \leqq \infty$ and $\operatorname{Re} z - p^{-1} + 1 = u$, then $A_z \in O(l_p)$ and*

$$\text{(a)} \qquad \|A_z\|_{O(l_p)} \leqq Cu^{-1}\,.$$

Proof. For $\operatorname{Re} z > -1$ and $1 > r > 0$ let $B_{r,z}$ denote the matrix obtained by multiplying the m, nth entry of A_z by $(n+1)^{-r}$. Fix $u > 0$ and for $r > 0$ and $0 \leqq \operatorname{Re} z \leqq 1$ let $f(r, \xi) = B_{r,\xi-1+u}$. In order to prove the lemma we must show that for each $r > 0$ the following hold:

(1) $f(r, \cdot)$ is a continuous function from $0 \leqq \operatorname{Re} \xi \leqq 1$ into $O(l_\infty)$;

(2) $f(r, \cdot)$ is an analytic function from $0 < \operatorname{Re} \xi < 1$ into $O(l_\infty)$;

(3) $f(r, \cdot)$ is continuous function from $\operatorname{Re} \xi = 1$ into $O(l_1)$;

(4) there is an absolute constant C such that

$$\|f(r, \xi)\|_{O(l_p)} \leqq Cu^{-1} \quad \text{if} \quad \operatorname{Re} \xi = j,\ p^{-1} = j,\ j = 0, 1\,.$$

We will now establish (1)-(4). Fix $r > 0$. For $k = 1, 2, \cdots$, let $E_{k,\xi}$ denote the matrix that has the same m, nth entry as $f(r, \xi)$ when $0 \leqq m, n < k$ and that has m, nth entry $= 0$ otherwise. It is obvious that (1) and (2) hold if $f(r, \xi)$ is replaced by $E_{k,\xi}$ for any $k = 1, 2, \cdots$. Furthermore, a simple computation using Lemma 1.2 shows that

$$\text{(5)} \qquad \|f(r, \xi) - E_{k,\xi}\|_{O(l_\infty)} \leqq C(k+2)^{-r}$$

where C is a constant which depends only on u. Thus, (1) and (2) hold. Again using Lemma 1.2, we see that

$$\text{(6)} \quad \|f(r, 1+it) - E_{k,1+it}\|_{O(l_1)} \leqq u^{-1}k^{-r} \quad \text{for} \quad k = 1, 2, \cdots, t \quad \text{real}\,.$$

(3) follows from (6) and the fact that $E_{k,1+it}$ is a continuous function of t with values in $O(l_1)$. (4) also follows from a simple computation using Lemma 1.2. From (1)-(4) we see that we can apply [1, p. 114] to $f(r, \xi)e^{\varepsilon(z-\xi)^2}$ and deduce $f(r, \xi) \in Q_s$ where $\operatorname{Re} \xi = p^{-1} = s$ and

$$N_s(f(r, \xi)) \leqq Cu^{-1} \quad \text{for} \quad 1 > r > 0; \operatorname{Re} \xi = s\,; \quad 0 \leqq s \leqq 1\,. \tag{7}$$

If $\operatorname{Re} z - p^{-1} + 1 = u$ and $\xi = z + 1 - u$, then $\operatorname{Re} \xi = p^{-1}$; therefore by Lemma 2.1, (7) and the fact that $B_{r,z} = f(r,\xi)$, we see that

$$\begin{aligned} &\|B_{r,z}\|_{O(l_p)} \leqq Cu^{-1} \quad \text{if} \quad \operatorname{Re} z - p^{-1} + 1 = u; 1 \leqq p \leqq \infty\,; \\ &u > 0; 1 > r > 0\,. \end{aligned} \tag{8}$$

From (8), the fact that $B_{1/n,z}$ converges to A_z entrywise as $n \to \infty$ and Lemma 1.4 we obtain the conclusion of the Lemma.

3. The transform.

In Lemmas 3.1 and 3.3 we will establish two properties of the series

$$\sum_{n=0}^{\infty} (-1)^n (n+1)^a \binom{w}{n} x_n$$

which will be used to obtain a lower bound for the spectrum of operators A_z on l_p spaces. Recall that A_z is defined in 1.5. We use $\sum$ and $\sum_m$ to denote sums taken over all nonnegative integers.

LEMMA 3.1. *To each real number $a, 1 \leqq p \leqq \infty$ and K, a compact subset of $\operatorname{Re} w > a - p^{-1}$, there corresponds a constant C depending only on a, p and K such that*

$$\sum_{n=0}^{\infty} \left| (-1)^n (n+1)^a \binom{w}{n} x_n \right| \leqq C \|x\|_p$$

for each $x \in l_p$ and $z \in K$.

We need the following inequality for the proof of this lemma and Lemma 3.3.

PROPOSITION 3.2. *For $n = 2, 3, \cdots$ and w a complex number,*

$$\left|\binom{w}{n}\right| \leqq |w/n| (n+1)^{-\operatorname{Re} w} \exp(|w| + |w|^2)$$

where C is an absolute constant.

Proof.

$$\begin{aligned} \left|\binom{w}{n}\right| &\leqq \left|\frac{w}{n}\right| \exp\left\{\frac{1}{2}\sum_{k=1}^{n-1} \log\left(1 - \frac{2}{k}\operatorname{Re} w + \frac{|w|^2}{k^2}\right)\right\} \\ &\leqq \left|\frac{w}{n}\right| \left\{\exp\left(-\operatorname{Re} w \sum_{k=1}^{n-1} \frac{1}{k} + \frac{1}{2}\sum_{k=1}^{n-1} \frac{|w|^2}{k^2}\right)\right\} \end{aligned}$$

$$\leqq \left|\frac{w}{n}\right|(n+1)^{-\operatorname{Re} w} \exp\left\{|w|\left|\log(n+1) - \sum_{k=1}^{n-1}\frac{1}{k}\right| + |w|^2\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}\right\}$$

$$\leqq \left|\frac{w}{n}\right|(n+1)^{-\operatorname{Re} w} \exp\{|w| + |w|^2\} .$$

This proves 3.2.

We now prove Lemma 3.1. By using Hölder's inequality and the estimate in Proposition 3.2 we see that

$$\sum_{n=0}^{\infty}\left|(-1)^n(n+1)^a\binom{w}{n}x_n\right| \leqq C_1\left(\sum_{n=0}^{\infty}(n+1)^{(a-1-\operatorname{Re} w)q}\right)^{1/q}||x||_{l_p}$$

where C_1 depends on K and $p^{-1} + q^{-1} = 1$. To obtain a constant C we need only require that $(a - 1 - \operatorname{Re} w) < -1/q$, even in the case $q = \infty$; or, equivalently, $\operatorname{Re} w > a - p^{-1}$.

LEMMA 3.3. *If* $1 \leqq p \leqq \infty$, $\operatorname{Re} w > a - p^{-1}$, $a - p^{-1} + 1 > 0$ *and* $x \in l_p$, *then*

$$\sum_m (-1)^m(m+1)^a\binom{w}{m}A_a x(m) = (1+w)^{-1}\sum_m(-1)^m(m+1)^a\binom{w}{m}x(m) , \tag{3.3.1}$$

both series converging absolutely.

Proof. It follows from Lemma 2.2 and Lemma 3.1 that both sides of (3.3.1) converge absolutely and determine continuous linear functionals on l_p.

In order to establish (3.3.1) we first assume that $a = 0$ and $1 < p \leqq \infty$. Consider a sequence $x(m) = t^m$ where $0 < t < 1$. Then

$$A_0 x(m) = (m+1)^{-1}\sum_{0\leqq k\leqq m} t^k = (1-t)^{-1}\int_t^1 s^m ds . \tag{1}$$

From Proposition 3.1 and Taylor's theorem it follows that

(2) $\sum(-1)^m\binom{w}{m}s^m = (1-s)^w$, $-1 < s < 1$, z complex and the series converges uniformly for s in a compact subset of $(-1, 1)$ and w in any compact set in the plane.

We also note that

$$(1-t)^{-1}\int_t^1(1-s)^w ds = (1+w)^{-1}(1-t)^w , \quad w \neq -1 . \tag{3}$$

From (1), (2) and (3) we see that

$$\sum (-1)^m \binom{w}{m} A_0 x(m) = (1 + w)^{-1}(1 - t)^w . \tag{4}$$

Again, from (2) we see that the right side of (4) is

$$(1 + w)^{-1} \sum_m (-1)^m \binom{w}{m} t^m .$$

This establishes (3.3.1) for the special sequences $x(m) = t^m$, $0 < t < 1$ and for any w. If $y \in l_q$, $(p^{-1} + q^{-1} = 1)$ and $y \neq 0$, then $\sum t^m y(m)$ must be $\neq 0$ for some t in $(0, 1)$ since the series converges to an analytic function of t. This shows that the linear span of the sequences (t^m), $0 < t < 1$, is a dense subspace of l_p, $1 < p \leqq \infty$. Since both sides of (3.3.1) determine continuous linear functionals on l_p, we conclude that (3.3.1) holds for the case $a = 0$ and $1 < p \leqq \infty$.

Now consider the general case. Let x be an element in l_p with finite support, that is, for some n_0, $x(n) = 0$ if $n \geqq n_0$. Let $y(n) = (n + 1)^a x(n)$, $n = 1, 2, \cdots$. Then by the case already established we have

$$\begin{aligned}
&\sum_m (-1)^m (m + 1)^a \binom{w}{m} A_a x(m) \\
&\quad = \sum_m (-1)^m \binom{w}{m} A_0 y(m) = (1 + w)^{-1} \sum_m (-1)^m \binom{w}{m} y(m) \\
&\quad = (1 + w)^{-1} \sum_m (-1)^m (m + 1)^a \binom{w}{m} x(m) .
\end{aligned}$$

Again, since both sides of (3.3.1) determine continuous linear functionals on l_p and since these linear functionals agree on a dense subspace of l_p, the space of sequences with finite support, it follows that (3.3.1) holds. This completes the proof.

4. The two theorems.

We will now obtain the two main results of the paper, Theorems 4.1 and 4.3. It is convenient to let C_b, $0 < b$, denote the set of all complex numbers z such that $|z - (2b)^{-1}| \leqq (2b)^{-1}$. An easy computation shows that this set is equal to the set $\{\lambda\colon \operatorname{Re} \lambda^{-1} \geqq b\}$.

THEOREM 4.1. *If* $1 \leqq p \leqq \infty$ *and* $a - p^{-1} + 1 > 0$, *then* $A_a \in O(l_p)$ *and*

$$\operatorname{sp}(A_a, l_p) = C_{a-p^{-1}+1} \cup \{(m + 1)^{-1}\colon m = 0, 1, \cdots\} . \tag{1}$$

Proof. The fact that $A_a \in O(l_p)$ follows from Lemma 2.2. We will first show that the left-hand-set of (1) contains the right-hand-set.

From Lemma 3.1, Lemma 3.3 and the fact A_a is a bounded linear operator on l_p it follows that $A_a x - (1+w)^{-1}x$ is in the null space of a nonzero continuous linear functional on l_p for each x in l_p, if $\operatorname{Re} w > a - p^{-1}$. Since $\operatorname{Re} w > a - p^{-1}$ is equivalent to $\operatorname{Re}((1+w)^{-1})^{-1} > a - p^{-1} + 1$, which is equivalent to $(1+w)^{-1}$ being in the interior of the set $C_{a-p^{-1}+1}$, we see that $\operatorname{sp}(A_a, l_p) \supset C_{a-p^{-1}+1}$. This, together with Lemma 1.9, shows that the left-hand-set of (1) contains the right-hand-set. In order to prove that the right-hand set of (1) contains the left-hand set we need the following estimate.

PROPOSITION 4.1A. *To each complex number* $\lambda \neq 0$ *and* $\varepsilon > 0$ *there corresponds an* m *such that if* $i \geqq k \geqq m$, *then*

$$\left|\prod_{i \geqq j \geqq k}(1 - \lambda^{-1}(j+1)^{-1})\right|^{-1} \leqq (1+\varepsilon)((i+1)/(k+1))^{\operatorname{Re}\lambda^{-1}}.$$

Proof. By using the power series for $\log(1+t)$ one can show that

$$\log(1+t) \geqq t - t^2 \quad \text{for} \quad t \geqq -3/5 .$$

Thus, if $-2|\lambda^{-1}|(j+1)^{-1} \geqq -3/5$, then

$$\begin{aligned}
&\log|1 - \lambda^{-1}(j+1)^{-1}| \\
&\quad \geqq \frac{1}{2}\log(1 - 2\operatorname{Re}\lambda^{-1}(j+1)^{-1} + |\lambda|^{-2}(j+1)^{-2}) \\
&\quad \geqq -(j+1)^{-1}\operatorname{Re}\lambda^{-1} + \frac{1}{2}|\lambda|^{-2}(j+1)^{-2} \\
&\quad\quad - \frac{1}{2}(2|\lambda|^{-1}(j+1)^{-1} + |\lambda|^{-2}(j+1)^{-2})^2 .
\end{aligned}$$

Consequently, if $-2|\lambda^{-1}|(k+1)^{-1} \geqq -3/5$, then

$$\begin{aligned}
&\left|\prod_{i \geqq j \geqq k}(1 - \lambda^{-1}(j+1)^{-1})\right|^{-1} \\
&\quad \leqq \exp\left\{\operatorname{Re}\lambda^{-1}\cdot\sum_{j=k}^{i}\frac{1}{j+1} + \frac{3}{2}|\lambda|^{-2}\sum_{j=k}^{i}(j+1)^{-2}\right. \\
&\quad\quad \left. + 2|\lambda|^{-3}\sum_{j=k}^{i}(j+1)^{-3} + \frac{1}{2}|\lambda|^{-4}\sum_{j=k}^{i}(j+1)^{-4}\right\} \\
&\quad \leqq \exp\left\{\operatorname{Re}\lambda^{-1}\log\frac{i+1}{k+1}\right\} .
\end{aligned}$$

$$\exp\left\{|\lambda|^{-1}\left|\sum_{j=k}^{i}\frac{1}{j+1} - \int_{k+1}^{i+1}\frac{dt}{t}\right| + \frac{3}{2}|\lambda|^{-2}k^{-1} + |\lambda|^{-3}k^{-2} + \frac{1}{6}|\lambda|^{-4}k^{-3}\right\}$$

$$(*) \quad \leqq \left(\frac{i+1}{k+1}\right)^{\operatorname{Re}\lambda^{-1}}\cdot\exp\left\{2|\lambda^{-1}|k^{-1} + \frac{3}{2}|\lambda|^{-2}k^{-1} + |\lambda|^{-3}k^{-2} + \frac{1}{6}|\lambda|^{-4}k^{-3}\right\}.$$

The conclusion of 4.1A now follows by choosing m such that

$$-2|\lambda|^{-1}(m+1)^{-1} \geqq -3/5$$

and the second factor in (*) is $\leqq (1+\varepsilon)$ when $k \geqq m$.

Now suppose that λ is not in the right-hand set of (1). In particular, $\mathrm{Re}\,\lambda^{-1} < a - p^{-1} + 1$ or

$$a - \mathrm{Re}\,\lambda^{-1} - p^{-1} + 1 > 0 . \tag{2}$$

By 4.1A we can choose an m_1 corresponding to λ and $\varepsilon = 1$ such that

$$\left|\prod_{i \geqq j \geqq k} (1 - \gamma_j/\lambda)\right|^{-1} \leqq 2((i+1)/(k+1))^{\mathrm{Re}\lambda^{-1}} \quad \text{if} \quad k \geqq m_1 . \tag{3}$$

Choose $m \geqq m_1$ so that $|\lambda| > (m+1)^{-1}$. Let E_m and B_m correspond to A_a as in 1.6. Since $\mathrm{sp}\,(A_a) \cup \{0\} = \mathrm{sp}\,(E_m) \cup \mathrm{sp}\,(B_m)$ and $\lambda \notin \mathrm{sp}\,(E_m)$, to show that $\lambda \notin \mathrm{sp}\,(A_a)$ it suffices to show that $\lambda \notin \mathrm{sp}\,(B_m)$. Because of Lemma 1.5 in order to show that $\lambda \notin \mathrm{sp}\,(B_m)$ it suffices to show that

$$\sum_{n=1}^{\infty} \lambda^{-n} B_m^{n+1} \tag{4}$$

converges entrywise to a matrix which is a bounded operator on l_p. Recall that the i, jth entry of B_m, which we will denote by a_{ij}, is $\alpha_i \gamma_j / \alpha_j$ where $\alpha_i = (i+1)^{-a-1}$ and $\gamma_j = (j+1)^{-1}$ for $i \geqq j \geqq m$ and the i, jth entry is 0 otherwise. The i, kth entry of B_m^{n+1} is

$$\sum_{i \geqq j_1 \geqq \cdots \geqq j_n \geqq k} a_{ij_1} a_{j_1 j_2} \cdots a_{j_n k} = \sum_{i \geqq j_1 \geqq \cdots \geqq j_n \geqq k} a_{ik} \gamma_{j_1} \cdots \gamma_{j_n} .$$

Thus, the i, kth entry of $D_m = B_m + \sum_{n=1}^{\infty} \lambda^{-n} B_m^{n+1}$ is

$$a_{ik}\Big(1 + \sum_{n+1}^{\infty} \lambda^{-n} \sum_{i \geqq j_i \geqq \cdots \geqq j_n \geqq k} \gamma_{j_1} \cdots \gamma_{j_n}\Big) \tag{5}$$

if $i \geqq k \geqq m$. Since $|\lambda| > (m+1)^{-1} \geqq \gamma_j$ for $j \geqq m$, it follows from the general theory of power series that the series in (5) conveges and that the value of (5) is

$$a_{jk} \prod_{i \geqq j \geqq k} (1 - \gamma_j/\lambda)^{-1} . \tag{6}$$

Since $k \geqq m \geqq m_1$, it follows from (3) and the definition of B_m that the modulus of the quantity in (6) is dominated by

$$2(k+1)^{a - \mathrm{Re}\lambda^{-1}} (i+1)^{-(a - \mathrm{Re}\lambda^{-1}) - 1} .$$

From this, Lemma 1.3, (2) and Lemma 2.2 (let $z = a - \mathrm{Re}\,\lambda^{-1}$) we conclude that the matrix $D_m \in O(l_p)$. This shows that $\lambda \notin \mathrm{sp}(B_m)$. Hence, we have shown that the right-hand set of (1) contains the left-hand set. This completes the proof of the theorem.

LEMMA 4.2. *If* $1 \leqq p \leqq \infty$ *and* $a - p^{-1} + 1 > 0$, *then*

$$\mathrm{sp}_f (A_a, l_p) = C_{a-p^{-1}+1} \,. \tag{i}$$

Proof. From Theorem 4.1, Lemma 1.9 and the fact that the final spectrum is always closed, it follows that the left-hand-side of (i) contains the right-hand-side. Now let B_m correspond to A_a as in 1.6 where $(m + 1)^{-1} \in C_{a-p^{-1}+1}$. By Lemma 1.8

$$\mathrm{sp}\,(B_m) \supset \mathrm{sp}\,(B_m \,|\, K_m) \;; \tag{7}$$

and, clearly,

$$\mathrm{sp}\,(B_m \,|\, K_m) = \mathrm{sp}\,(A_a \,|\, K_m) \supset \mathrm{sp}_f\,(A_a, l_p) \,. \tag{8}$$

Also by Lemma 1.8,

$$\mathrm{sp}\,(B_m) \subset \mathrm{sp}\,(A_a) \cup \{0\} = \mathrm{sp}\,(A_a) \,. \tag{9}$$

By Lemma 1.11, the only isolated points in $\mathrm{sp}\,(B_m)$ are points on the diagonal of B_m; from this, (9), Theorem 4.1 and the choice of m we see that

$$\mathrm{sp}\,(B_m) \subset C_{a-p^{-1}+1} \,. \tag{10}$$

From (10), (7) and (8) we deduce that the right-hand-side of (i) contains the left-hand-side. This completes the proof.

THEOREM 4.3. *Let* $p(x)$ *be a monic polynomial of degree* a *and* $q(x)$ *a monic polynomial of degree* $a+1$ *such that* $q(m) \neq 0$, $m = 0, 1 \cdots$. *Let* S *be the lower triangular matrix with* m, nth *entry* $p(n)/q(m)$, $m \geqq n$. *If* $a > 0$ *and* $1 \leqq p \leqq \infty$, *or* $a = 0$ *and* $1 < p \leqq \infty$, *then* $S \in O(l_p)$ *and*

$$\mathrm{sp}\,(S, l_p) = C_{a-p^{-1}+1} \cup \{p(m)/q(m);\, m = 0, 1, \cdots\} \,.$$

Proof. Let D_m be the operator obtained by restricting $S - A_a$ to the subspace K_m of l_p and let B_m correspond to $S - A_a$ as in 1.6. Clearly

$$\| D_m \|_{O(K_m)} \leqq \| B_m \|_{O(l_p)} \,. \tag{11}$$

A simple computation using Lemma 1.3 shows that the right-hand-side of (11) is $O(m^{-1})$ if $p = 1$ or $p = \infty$. From this and the Riesz convexity theorem we see that $\| D_m \| \to 0$ as $m \to \infty$. From this and Lemma 1.10 we conclude that $\mathrm{sp}_f\,(S) = \mathrm{sp}_f\,(A_a)$. Consequently, by Lemma 4.2, $\mathrm{sp}_f\,(S) = C_{a-p^{-1}+1}$. From this and Lemma 1.9 we obtain the conclusion of the theorem.

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