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ALGEBRAS OF ANALYTIC FUNCTIONS IN THE PLANE

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Let X be a compact subset of the complex plane and let A be an algebra of functions analytic near X which contains the polynomials and is complete in its natural topology. This paper is concerned with determining the spectrum of A and describing A in terms of its spectrum. It is shown that the spectrum of A is formed from the disjoint union of certain compact subsets of C (suitably topologized) by making certain identifications. A is closed under differentiation exactly when no identifications need be performed, and then A admits a simple, complete description. In particular, if X is connected, then the completion of A is merely the restriction to X of the algebra of all functions analytic near the union of X with some of the bounded components of $C - X$.

Our principal tool in these investigations is the theory of analytic structure in the spectrum of a function algebra developed by Bishop in [2] and extended by Bjork in [4, 5]. We view the algebra A as the inductive limit of function algebras and induce analytic structure in the spectrum of A. When A is closed under differentiation, topological considerations lead quickly to the desired results. In the general case, we pass to the smallest algebra B containing A which is closed under differentiation. By introducing differentiation in the spectrum of A, we show that every continuous complex-valued homomorphism of A may be extended to B. It follows that the spectrum of A is obtained from the spectrum of B by making certain identifications. When no identifications need be performed, $A = B$.

2. Preliminaries. If U is an open set, we let $\mathcal{O}(U)$ denote the algebra of functions analytic on U, endowed with the topology of uniform convergence on compact sets. If V is an open subset of U, we let $r_{UV}: \mathcal{O}(U) \to \mathcal{O}(V)$ be the restriction. If X is a compact set, $\mathcal{O}(X)$ denotes the algebra of functions on X which have analytic extensions to a neighborhood of X. We view $\mathcal{O}(X)$ as the inductive limit (in the sense of functions) of the system $\{\mathcal{O}(U); r_{UV}\}$ and equip $\mathcal{O}(X)$ with the inductive limit topology; i.e., the finest topology rendering the restriction maps $r_U: \mathcal{O}(U) \to \mathcal{O}(X)$ continuous.

If A is a subalgebra of $\mathcal{O}(X)$ and U is an open set containing X, we let $A(U) = \{f \in \mathcal{O}(X): f \mid X \in A\}$. Similarly, if K is a compact set containing X, we let $A(K) = \{f \in \mathcal{O}(K): f \mid X \in A\}$. For compact sets K, L with $K \supset L$, we let $r_{KL}: \mathcal{O} \to \mathcal{O}(L)$ be the restriction. Then it is easy to see that:

$$\begin{aligned} A &= \text{inductive limit } \{A(U); r_{UV}\} \\ &= \text{inductive limit } \{A(K); r_{KL}\} \end{aligned}$$

and that the inductive limit topologies thus induced on A coincide with the relative topology from $\mathcal{O}(X)$. A is complete in this topology if and only if $A(U)$ is closed in $\mathcal{O}(U)$ for each open set U containing X. Details of the above may be found in §2 of [11].

We also regard a subalgebra A of $\mathcal{O}(X)$ as a normed algebra with the norm:

$$\|f\|_X = \sup\{|f(x)| : x \in X\} .$$

We relate these two topologies in the following proposition.

PROPOSITION. *Let A be a subalgebra of $\mathcal{O}(X)$ containing the constants. Then the norm topology and the inductive limit topology on A admit the same continuous complex-valued homomorphisms.*

Proof. If this were not so, there would be a homomorphism ϕ of A, continuous relative to the inductive limit topology, and a function f in A such that

$$\phi(f) = 1 > \|f\|_X .$$

We could then find an open set U containing X and a function F in $A(U)$ such that $F|X = f$ and $|F| < 1$ on U. Then $(1 - F)$ would be invertible in the closure of $A(U)$ in $\mathcal{O}(U)$. Moreover, $\phi \cdot r_U$ would be a continuous homomorphism of $A(U)$, and would thus extend to its closure. Since $\phi \cdot r_U(1 - F) = \phi(1 - f) = 0$, we would then have the following contradictory chain of equalities:

$$1 = \phi \cdot r_U[(1 - F)(1 - F)^{-1}] = \{\phi \cdot r_U(1 - F)\}\{\phi \cdot r_U[(1 - F)^{-1}]\} = 0 .$$

This contradiction establishes the proposition.

If B is a topological algebra, we denote the spectrum of B (the space of nonzero continuous complex-valued homomorphisms, with the weak* topology) by M_B. We may regard an element b of B as a function of M_B via the Gelfand transform $\hat{b}(\phi) = \phi(b)$, for each ϕ in M_B. If B is a normed algebra with identity, then M_B is compact. Then if A is a subalgebra of $\mathcal{O}(X)$ containing the constants, M_A is compact and a standard argument may be used to show that (see [11]):

$$\begin{aligned} M_A &= \text{projective limit } \{M_{A(U)}; r^*_{UV}\} \\ &= \text{projective limit } \{M_{A(K)}; r^*_{KL}\} \end{aligned}$$

where r^*_{UV} and r^*_{KL} are the adjoints of the restrictions.

We refer to [6] for standard material concerning function algebras. If B is a function algebra with spectrum M_B we denote its Silov boundary by S_B. We make use of the techniques developed by Bishop and Bjork in [2, 4, 5] and assume some familiarity with these papers. In particular, if f is an element of B we say that a component W of $C - \hat{f}(S_B)$ is f-regular of multiplicity n if for each w in W there are at most n homomorphisms ζ in M_B for which $\hat{f}(\zeta) = w$; and that for some w there are exactly n such homomorphisms. In that case, there is a discrete subset E of W such that for each ζ in M_B such that $\hat{f}(\zeta) \in (W - E)$, there is a neighborhood Q of ζ in M_B mapped homeomorphically by $\hat{f}$ onto a disk, and such that $\hat{g} \circ (\hat{f} \mid Q)^{-1}$ is analytic for each g in B. The neighborhood Q is called an analytic disk about ζ, relative to the function $\hat{f}$.

We conclude this section with a topological lemma.

LEMMA. *Let M be a compact connected real 2-manifold with boundary and let p be a continuous map of M into the 2-sphere S^2. If p is locally one-to-one and is one-to-one on the boundary of M, then p is one-to-one.*

Proof. We will reduce to the case of a 2-manifold without boundary. To this end, suppose that M has k boundary components $J_1, \cdots, J_k$. Each J_i is a 1-sphere, so that $p(J_i)$ is a 1-sphere in S^2 for each i. Hence $S^2 - p(J_i)$ consists of two disjoint connected open sets. A compactness argument, using the fact that p is locally-one-to-one, may be used to show that there is a connected neighborhood of J_i in M on which p is one-to-one. It follows that we may choose a neighborhood W_i of J_i such that p is one-to-one on W_i and $p(W_i)$ does not intersect one of the components of $S^2 - p(J_i)$. It is easy to see that we may attach a disk to M along J_i and extend p to this disk; since $p(W_i)$ lies in only one component of $S^2 - p(J_i)$ this may be effected in such a way that the extension remains locally one-to-one. If we perform this surgery for each boundary component J_i we arrive at a compact connected real 2-manifold N without boundary and a continuous map q of N into S^2 which is locally one-to-one. If q is one-to-one then p must certainly be.

For each x in S^2, the fiber $q^{-1}(x)$ is compact and discrete (since q is locally one-to-one) and hence finite. Then, using the invariance of domain, we may choose an open set U about x such that $q^{-1}(U)$ consists of open, connected components, each mapped homeomorphically onto U by q; thus q is a covering map. Since S^2 is its own universal covering space, it follows that q, and hence p, must be one-to-one, as desired.

3. Main results. If A is a subalgebra of $\mathcal{O}(X)$ that contains the constants and the coordinate function Z, we say that A is *stable* if it is complete in the inductive limit topology and each of the algebras $A(U)$ is closed under differentiation.

In order to see how stable algebras may arise, consider the following construction. Let X be a compact subset of $\boldsymbol{C}$ and let $\{X_\alpha\}$ be a partitioning of X into disjoint closed sets. For each α let Y_α be the union of X_α with some of the bounded components of $\boldsymbol{C} - X_\alpha$. Then let

$$A = \{f \in \mathcal{O}(X): f \mid X_\alpha \in \mathcal{O}(Y_\alpha) \mid X_\alpha \text{ for each } \alpha\} .$$

It is easy to see that A is a stable algebra and that the spectrum of A is the disjoint union of the Y_α, suitably topologized. The following theorem shows that this is the only way in which stable algebras may arise.

THEOREM 1. *Let A be a stable subalgebra of $\mathcal{O}(X)$ and let Y'_α be a component of M_A. Then $\hat{Z} \mid Y'_\alpha$ is a homeomorphism. The set $Y_\alpha = \hat{Z}(Y'_\alpha)$ is the union of $X_\alpha = X \cap Y_\alpha$ with some of the bounded components of $\boldsymbol{C} - X_\alpha$. Finally, the collection $\{X_\alpha$: Y'_α is a component of $M_A\}$ is a partitioning of X into disjoint closed sets and $A = \{f \in \mathcal{O}(X)$: $f \mid X_\alpha \in \mathcal{O}(Y_\alpha) \mid X_\alpha$ for each component Y'_α of $M_A\}$.*

Proof. Let K be a compact set whose interior contains X and whose boundary is the disjoint union of a finite number of smooth, simple closed curves. Let $A(K)^*$ denote the completion of the algebra $A(K)$ in the norm $\|\cdot\|_K$. We proceed by examining the algebra $A(K)^*$ and its spectrum and then passing to the projective limit.

We identify K with a subset of $M_{A(K)^*}$. Clearly, $S_{A(K)^*}$ is contained in the boundary (relative to $\boldsymbol{C}$) of K. Let Δ denote the set of points in $M_{A(K)^*}$ having a neighborhood which is an analytic disk (relative to the function $\hat{Z}$). We show that $M_{A(K)^*} - S_{A(K)^*} - \Delta$ is at most countable. First, a standard argument shows that the unbounded component of $\boldsymbol{C} - \hat{Z}(S_{A(K)^*})$ is Z-regular of multiplicity 0. If T is the boundary of this component, then it follows from [5] that there are no points ζ of $M_{A(K)^*} - S_{A(K)^*}$ for which $\hat{Z}(\zeta)$ belongs to T. We conclude from [5] that each component of $\boldsymbol{C} - \hat{Z}(S_{A(K)^*})$ that adjoins the unbounded component is Z-regular of multiplicity at most 1. Similarly, if T' denotes the boundary of one of these components, then there is at most one point ζ in $M_{A(K)^*} - S_{A(K)^*}$ for which $\hat{Z}(\xi) \in T'$. Then each component of $\boldsymbol{C} - \hat{Z}(S_{A(K)^*})$ that adjoins one of these components is Z-regular of multiplicity at most 2. Proceeding inward in this way, we see that each component of $\boldsymbol{C} - \hat{Z}(S_{A(K)^*})$ is Z-regular of some multiplicity. Again from [5], it follows that there is a discrete subset

E of $\boldsymbol{C} - \hat{Z}(S_{A(K)^*})$ such that each ξ in $M_{A(K)^*} - S_{A(K)^*}$ for which $\hat{Z}(\xi)$ does not lie in $E \cup \hat{Z}(S_{A(K)^*})$, is a point of Λ. Moreover, if x is in E, then there are only finitely many homomorphisms ψ for which $\hat{Z}(\psi) = x$.

Now let us turn to the points ξ of $M_{A(K)^*} - S_{A(K)^*}$ for which $\hat{Z}(\xi) \in \hat{Z}(S_{A(K)^*})$. Since the boundary of K is the finite union of smooth curves, it follows that each boundary point is a triangle point in the sense of Bishop [2]. Hence for each ξ in $M_{A(K)^*} - S_{A(K)^*}$ for which $\hat{Z}(\xi) \in \hat{Z}(S_{A(K)^*})$, there is a deleted neighborhood W_ξ lying in Λ. From the compactness of the boundary of K and the fact that for each point y of the boundary there are only finitely many homomorphisms ζ for which $\hat{Z}(\zeta) = y$, it follows that all but finitely many of the points ξ of $M_{A(K)^*} - S_{A(K)^*}$ for which $\hat{Z}(\xi) \in \hat{Z}(S_{A(K)^*})$ actually lie in Λ. Then $M_{A(K)^*} - S_{A(K)^*} - \Lambda$ is a countable set, as was asserted.

Now let L be a connected component of $M_{A(K)^*}$. We assert that $\hat{Z} | L$ is a homeomorphism. If this were not so, we could find homomorphisms ϕ and λ in L such that $\hat{Z}(\phi) = \hat{Z}(\lambda)$. Since $M_{A(K)^*} = M_{A(K)}$, we could then find an open set U containing K and a function f in $A(U)$ such that $\phi(f | K) \neq \lambda(f | K)$. Since $A(U)$ is complete, closed under differentiation and contains the polynomials, it follows from a theorem of Bishop [3] that $M_{A(U)}$ is a 1-dimensional complex analytic manifold and that $\hat{Z}$ on $M_{A(U)}$ is a local analytic isomorphism. Let $\rho: A(U) \rightarrow A(K)$ be the restriction and $\rho^*: M_{A(K)} \rightarrow M_{A(U)}$ be its adjoint. Then $\rho^*(L)$ is a compact connected subset of $M_{A(U)}$. By the invariance of domain theorem, $\rho^*(\Lambda)$ lies in the interior of $\rho^*(L)$. Hence $\hat{Z}$ is one-to-one on the boundary of $\rho^*(L)$. Since $\hat{Z}$ is a local homeomorphism on $M_{A(U)}$, we may find a compact connected set L' containing $\rho^*(L)$ in its interior such that $\hat{Z}$ is one-to-one on the boundary of L' and L' is a 2-manifold with boundary. Regarding $\boldsymbol{C}$ as a subset of S^2, we may then apply the lemma to conclude that $\hat{Z}$ is one-to-one on L' and hence on $\rho^*(L)$. But ϕ and λ restrict to different homomorphisms of $A(U)$ so that $\rho^*(\phi) \neq \rho^*(\lambda)$, while $\hat{Z}[\rho^*(\phi)] = \hat{Z}[\rho^*(\lambda)]$, which is a contradiction. It must be therefore, that $\hat{Z} | L$ is a homeomorphism.

From the Silov idempotent theorem, it follows that each component of $M_{A(K)^*}$ contains a component of the boundary of K. It follows that for each component L of $M_{A(K)^*}$, the boundary of $\hat{Z}(L)$ coincides with $\hat{Z}(S_{A(K)^*} \cap L)$, so that $\hat{Z}(L)$ is formed from $K \cap \hat{Z}(L)$ by the addition of certain components of $\boldsymbol{C} - K \cap \hat{Z}(L)$.

Now let us return to M_A. For a component Y'_α of M_A, and a compact set K with smooth boundary, containing X in its interior, let $r_K: A(K)^* \rightarrow A$ be the restriction and let $r_K^*: M_A \rightarrow M_{A(K)^*}$ be its adjoint. Let K'_α be the component of $M_{A(K)^*}$ that contains $r_K(Y_\alpha)$. It is clear that

$$Y'_\alpha = \text{projective limit } \{K'_\alpha; r^*_{KL}\} .$$

From the description of K'_α derived above, it follows that $\hat{Z}|Y'_\alpha$ is a homeomorphism and that $\hat{Z}(Y'_\alpha)$ is the union of $X \cap \hat{Z}(Y'_\alpha) = X_\alpha$ with some of the bounded components of $C - X_\alpha$.

If f belongs to A, then it is in $A(K)^*$ for some compact K with smooth boundary containing X in its interior. Since $\hat{Z}$ is a homeomorphism on each component of $M_{A(K)^*}$, it follows that Δ, the set of points in $M_{A(K)^*}$ having neighborhoods which are analytic disks, is all of $M_{A(K)^*} - S_{A(K)^*}$. Now we may see that $f|(K \cap \hat{Z}(L'))$ belongs to $\mathcal{O}(\hat{Z}(L'))|(K \cap \hat{Z}(L'))$ for each component L' of $M_{A(K)^*}$. It follows that $f|(X \cap \hat{Z}(Y'_\alpha))$ belongs to $\mathcal{O}(\hat{Z}(Y'_\alpha))|(X \cap \hat{Z}(Y'_\alpha))$ for each component Y'_α of M_A.

Finally, suppose that U is an open set containing X and that f is a function in $\mathcal{O}(U)$ such that $f|(X \cap \hat{Z}(Y'_\alpha))$ belongs to

$$\mathcal{O}(\hat{Z}(Y'_\alpha))|(X \cap \hat{Z}(Y'_\alpha))$$

for each component Y'_α of M_A. For each such Y'_α, choose a compact set K_α with smooth boundary containing X in its interior and such that $Z(L'_\alpha) \subset (U \cup \hat{Z}(Y'_\alpha))$ where L'_α is the component of $M_{A(K)^*}$ that contains $r^*_K(Y'_\alpha)$. If Y'_β is sufficiently close to Y'_α, we may choose K_β to be K_α. Then the compactness of M_A enables us to choose a single compact set K' with smooth boundary, containing X in its interior, and such that $\hat{Z}(L''_\alpha) \subset (U \cup \hat{Z}(Y'_\alpha))$ for each α, where L''_α is the component of $M_{A(K')}{}^*$ that contains $r^*_{K'}(Y'_\alpha)$. Without loss, we may assume that every component of K' contains a point of X. Then for each component L' of $M_{A(K')^*}$ we see that $f|(K' \cap \hat{Z}(L'))$ belongs to $\mathcal{O}(\hat{Z}(L'))|(K' \cap \hat{Z}(L'))$. The Silov idempotent theorem and the Arens-Calderon theorem then imply that $f|K'$ belongs to $A(K')^*$. Since A is complete and K' contains X in its interior, it follows that $f|X$ belongs to A, which completes the proof.

The above theorem gives a complete description of stable algebras. In what follows, we use stable algebras to describe the structure of more general subalgebras of $\mathcal{O}(X)$. We let A be a complete subalgebra of $\mathcal{O}(X)$ containing the polynomials and let A_0 be the smallest stable algebra containing A; A_0 is the completion of the algebra generated by the functions in A together with all their derivatives. We let $i: A \to A_0$ be the inclusion and $i^*: M_{A_0} \to M_A$ be its adjoint (the restriction map).

THEOREM 2. *The map $i^*: M_{A_0} \to M_A$ is onto. If Y and Y' are components of M_{A_0} then $i^*|Y$ and $i^*|Y'$ are one-to-one and there are at most finitely many pairs (μ, ν) in $Y \times Y'$ such that $i^*(\mu) = i^*(\nu)$. If f^* is a homeomorphism, then $A = A_0$.*

Proof. We show first that i^* is onto. Choose a compact set K with smooth boundary, whose interior contains X and is dense in K, and each component of which meets X. Let A_1 be the (non-complete) subalgebra of $\mathcal{O}(X)$ generated by the functions in A and all their derivatives. Let $i_K: A(K) \to A_1(K)$ be the inclusion and $i_K^*: M_{A_1(K)^*} \to M_{A(K)^*}$ be its adjoint. If we show that i_K^* is onto for each K belonging to a fundamental system of neighborhoods of X, then by passage to the projective limit, it will follow that i^* is onto. So suppose that for a particular choice of K, i_K^* is not onto.

Let Λ be the set of points of $M_{A(K)^*}$ which have a neighborhood which is an analytic disk relative to $\hat{Z}$. As in the proof of Theorem 1, we see that $M_{A(K)^*} - S_{A(K)^*} - \Lambda = E$ is at most countable. In view of the Silov idempotent theorem, no point of $M_{A(K)^*}$ is isolated, so that there is an open subset of Λ disjoint from $i_K^*(M_{A_1(K)^*})$. Let W be a component of Λ containing such an open set. We distinguish two cases.

Regard K as a subset of $M_{A(K)^*}$ and consider first the case in which W contains a point of K. Then W is a Riemann surface with the local coordinate $\hat{Z}$. If f is a function in $A(K)$, then $\hat{f}$ is analytic on W. Denote the derivative of $\hat{f}$ with respect to the coordinate $\hat{Z}$ by $D\hat{f}$ and the derivative of f with respect to Z by f'. If f and f' both belong to $A(K)$, then the connectedness of W, together with the fact that W contains a point of K and hence an open subset of K, implies that $D\hat{f} = \hat{f}'$ on W.

Let h belong to $A(K)$ and let $g = h'$. Define a function $\tilde{g}$ on W by $\tilde{g}(\zeta) = D\hat{h}(\zeta)$. The analysis of the previous paragraph implies that $\tilde{g} = \hat{g}$ if $g = h'$ belongs to $A(K)$. Thus the functions in $A(K)$ together with their first derivatives, extend to be analytic on W. By iteration of this process, we may extend each of the functions g in $A_1(K)$ to an analytic function $\tilde{g}$ on W; since W is connected, this extension is unique.

Thus if δ is a homomorphism in W, δ extends to a homomorphism of $A_1(K)$ by defining $\bar{\delta}(g) = \tilde{g}(\delta)$. If we show that $\bar{\delta}$ is a continuous homomorphism of $A_1(K)$, and thus extends to $A_1(K)^*$, then we will have that $i_K^*(\bar{\delta}) = \delta$ and this contradiction will complete the analysis of this case.

To this end, let us consider the boundary of W in $M_{A(K)^*}$. Since W is a connected component of Λ, no point of Λ is a boundary point of W. Thus the boundary points of W belong either to K or to E. If p is a boundary point of W that belongs to K, the fact that the interior of K is dense in K and that the boundary of K is smooth implies that there is a half-disk about p belonging to K. By enlarging K slightly we may effect a modification of K so that some half-disk around p belongs to $K \cap W$. An argument using the compactness of the part of the boundary of W that lies in K shows that all the boundary points of W that belong to K may be assumed to have half-disks in $K \cap W$ about them (modifying K as necessary).

Now consider the boundary points of W that belong to E. If q is one of these points, then the results of [2] imply that there is a neighborhood Q of q with the property that $Q - q$ lies in Δ and consists of finitely many components, each of which is mapped by $\hat{Z}$ homeomorphically onto a disk minus its center. Thus we may cover $W \cup \{q\}$ with a Riemann surface W_q in such a way that the functions in $A(K)$ extend to be analytic on W_q. We may certainly do this for each of the boundary points of W lying in E. Thus, passing to a covering Riemann surface when necessary, and modifying W as necessary (by enlargement of K), we arrive at a Riemann surface W' which has a subset of the interior of K as a neighborhood of its boundary, and to which the functions in $A(K)$ extend naturally. As before, we see that the functions in $A_1(K)$ extend to W'. Hence no function in $A_1(K)$ assumes a larger value on W than on K. It follows that δ is indeed a continuous homomorphism of $A_1(K)$.

We have shown that each compact set K whose interior contains X and is dense in K, and all of whose components meet X, can be modified slightly to produce another such compact set K' with the property that $i_{K'}^*: M_{A_1(K')^*} \to M_{A(K')^*}$ is onto. Since the collection of such sets K' forms a fundamental system of neighborhoods of X, it follows from a passage to the projective limit that $i^*: M_{A_0} \to M_A$ is onto.

Now let Y and Y' be distinct components of M_{A_0}. Considered as a map on M_{A_0}, $\hat{Z} | Y$ is one-to-one and $\hat{Z} = \hat{Z} \circ i^*$ so that i^* is certainly one-to-one. If there are infinitely many pairs (μ, ν) in $Y \times Y'$ such that $i^*(\mu) = i^*(\nu)$ then some point (λ, ξ) is a limit point of such pairs. We may choose a compact set K with smooth boundary, whose interior contains X and is dense in K, and such that $i_K^*(Y)$ and $i_K^*(Y')$ belong to different components of $M_{A(K)^*}$; say T and T' respectively. If f is in $A(K)$, then $\hat{f}$ is analytic on $T - (T \cap S_{A(K)^*})$ and $T' - (T' \cap S_{A(K)^*})$, and the derivative of $\hat{f}$ may be obtained, as in the first part of the proof, by differentiating with respect to the local coordinate $\hat{Z}$. Then the functions $\hat{f} \circ (\hat{Z} | T)^{-1}$ and $\hat{f} \circ (\hat{Z} | T')^{-1}$ are analytic in a neighborhood of $\hat{Z}(\xi) = \hat{Z}(\lambda)$ and agree to infinite order there. Now it follows that $\hat{g}(i_K^*(\xi)) = \hat{g}(i_K^*(\lambda))$ for each g in $A_1(K)$, since $A_1(K)$ is generated by functions in $A(K)$ and their derivatives. It follows that $i_K^*(\lambda) = i_K^*(\xi)$ which is a contradiction. It follows that only finitely many pairs in $Y \times Y'$ are not separated by i^*, as desired.

Finally, suppose that i^* is a homeomorphism, and let f be in $A_0(U)$ for some open set U containing X. As in the proof of Theorem 1, we may choose a compact set containing X in its interior such that $f | (K \cap \hat{Z}(L))$ belongs to $\mathcal{O}(\hat{Z}(L)) | (K \cap \hat{Z}(L))$ for each component L of $M_{A(K)^*}$. As before, we may then conclude that f belongs to $A(K)^*$

and hence that $f|X$ belongs to A. Since U is arbitrary, this completes the proof.

COROLLARY 1. *Let X be compact and connected. If A is a complete subalgebra of $\mathcal{O}(X)$ containing the polynomials, then there is a compact connected set X' containing X and such that $X' - X$ is open and $A = \mathcal{O}(X')$.*

Proof. Let A_0 be the stable algebra generated by A. The Silov idempotent theorem implies that M_A is connected and the corollary now follows quickly from Theorems 1 and 2.

The above results have easy applications to questions of approximation on open sets as well. We mention one result that seems particularly striking.

COROLLARY 2. *Let U be a connected open set and let g be an analytic function on U that admits no analytic extension to the union of U with any of the bounded components of $\mathbf{C} - U$. Then every analytic function on U is the limit, uniformly on compact subsets of U, of polynomials in g and Z.*

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