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GENERALIZED CONTINUATION

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In this paper the operation of analytic continuation is generalized by relaxing the condition that a direct continuation of a function must have the same values as the original on the intersection of their domains of definition. Thus the generalized continuations of a function can have some other property in common with the original function such as being preimages of a single function under a local integral operator. This generalization is accomplished by developing $\mathscr{A}$-continuation of $\mathscr{F} = \{(f_\alpha, S_\alpha) \mid f_\alpha \in \Phi$ and S_α a ball in $\mathscr{C}^n\}$ with respect to a collection of maps, $\mathscr{A}$, of subsets of $\mathscr{F}$ into $\mathscr{F}$. $\mathscr{A}$ must satisfy some compatibility conditions. Many of the proofs in this development parallel those for analytic continuation and lead to the introduction of a manifold on which the generalized continuation is single valued. A generalized continuation of function elements (f_α, S_α) is achieved when all the f_α's are complex valued functions defined on S_α and some examples are given.

In §1 $\mathscr{A}$-continuation is developed for $\mathscr{F}$. A manifold $M(\mathscr{F}, \mathscr{A})$ is developed on which $\mathscr{A}$-continuation is single valued and the complete $\mathscr{A}$-function is introduced which is similar to the complete analytic function of Weierstrass. Theorem 11 states a necessary and sufficient local condition that $M(\mathscr{F}, \mathscr{A})$ and $M(\mathscr{H}, \mathscr{B})$ be holomorphic. In section 2 $\mathscr{A}$-continuation is specialized to sets, $\mathscr{F}$, where f_α is a function with S_α as its domain of definition. Then (f_α, S_α) is referred to as a function element. For function elements a compatible set of maps can be considered as a generalization of direct analytic continuation of power series. An indicator function is defined to help describe a complete $\mathscr{A}$-function. Direct analytic continuation and continuation of the coefficients of a linear Weierstrass polynomial are given as examples.

Given in §3 is the more intricate example of continuing the normalized B_3-associate of the Bergman-Whittaker Integral Operator. Using Theorem 11 this generalized continuation is shown to be equivalent to analytically continuating the harmonic function represented by the B_3-associate. This is the example which motivated the study of generalized continuation.

1. **Generalized continuation.** Let Φ be a set and with each f_α in Φ associate ball, S_α, in C^n and let $\mathscr{F} = \{(f_\alpha, S_\alpha) \mid f_\alpha \in \Phi\}$. Let x_α denote the center of S_α and consider a set of operators or maps $\mathscr{A} =$

$\{A_x \mid x \in C^n\}$ such that

$$A_x\colon \{f_\alpha \mid x \in S_\alpha\} \rightarrow \{f_\alpha \mid x_\alpha = x\} \text{ .}$$

In this paper the statement, "a property holds for an expression for every α and x" means that this property holds for all α and x for which the indicated expression is defined.

DEFINITION. A set of operators, $\mathscr{A}$, is called a compatible set of operators for $\mathscr{F}$ if $\mathscr{A}$ satisfies

(i) for every α, x and y, $f_\alpha A_x = f_\alpha A_y A_x$

(ii) if $f_\alpha A_x = f_\beta$ then $r_\beta \geqq r_\alpha - d(x_\alpha, x_\beta)$

(iii) for every α, $f_\alpha A_{x_\alpha} = f_\alpha$.

In the preceding definition $d(x, y)$ denotes the distance between the two points $x = (x^1, \cdots, x^n)$ and $y = (y^1, \cdots, y^n)$ in C^n given by

$$d(x, y) = \left(\sum_{j=1}^{n} |x^j - y^j|^2\right)^{1/2}$$

and r_α is the radius of S_α.

DEFINITION. If $\mathscr{A}$ is a compatible set of operators and $f_\alpha A_x = f_\beta$, then f_β is called a direct generalized $\mathscr{A}$-continuation of f_α or simply a direct $\mathscr{A}$-continuation of f_α.

As in the case of an analytic function of one complex variable an analytic manifold is introduced on which $\mathscr{A}$-continuation is single-valued. First, the following definitions are given.

1. A finite sequence of balls, $S_1, \cdots, S_n$ is called a chain if the center a_{i+1} of S_{i+1} lies in S_i.

2. If $f_i A_{a_{i+1}} = f_{i+1}$ for $i = 1, \cdots, n-1$, then f_1 is said to have been $\mathscr{A}$-continued along the chain of balls.

3. A curve or path C on C^n is a continuous mapping, μ, of the closed unit interval, I, into C^n and is denoted by $C = (\mu(t), I)$. The inverse curve C^{-1} of $C = (\mu(t), I)$ is the curve (δ, I) where $\delta(t) = \mu(1-t)$ for $t \in I$.

4. Let $C = (\mu, I)$ be a curve in C^n with an element (f_t, S_t) in $\mathscr{F}$ associated with each $t \in I$ such that the center of S_t is $\mu(t)$. If for every t_0 and t_1 such that $\mu(t)$ lies in S_{t_0} for all t in the interval $t_0 \leqq t \leqq t_1$ we have f_{t_1} is a direct $\mathscr{A}$-continuation of f_{t_0}, then f_1 is said to be the $\mathscr{A}$-continuation of f_0 along the curve C.

In order to construct the analytic manifold some properties of $\mathscr{A}$-continuation are needed. These results are contained in the following Theorems. Some of the proofs are similar to the proofs of the corresponding properties in one complex variable and these proofs are omitted and the reader is referred to [8, pages 63-69]. For the rest of this section it is assumed that $\mathscr{A}$ is a compatible set of operators.

LEMMA 1. *If* $x_1, \cdots, x_n \in S_\alpha$ *and* $f_\alpha A_{x_1} \cdots A_{x_1}$ *is defined then*

$$f_\alpha A_{x_1} \cdots A_{x_n} = f_\alpha A_{x_n} .$$

The proof of this Lemma is by induction and (i) of the definition of compatibility.

For a given f_0 and any $x_\alpha \in S_0$ define $r(x_\alpha)$ to be the radius of S_α, the ball associated with $f_0 A_{x_\alpha}$. Using (ii) of compatibility the following Theorem can be proven.

THEOREM 1. *$r(x_\alpha)$ is either identically infinite or is a continuous function of x_α.*

THEOREM 2. *Let $f_\beta = f_\alpha A_x$ and let $C = (\mu, I)$ be a curve such that $|C| \subset S_\alpha$, $\mu(0) = x_\alpha$, and $\mu(1) = x_\beta$. Then there exist $x_1, \cdots, x_n$ on $|C|$ such that*

$$f_\beta A_{x_1} \cdots A_{x_n} = f_\alpha .$$

Lemma 1 is used in the proof of Theorem 2. This Theorem says that if f_β is a direct $\mathscr{A}$-continuation of f_α and C is a path in S_α which joins x_β to x_α then there exists an $\mathscr{A}$-continuation of f_β along a chain $S_1, \cdots, S_n$ to obtain f_α where the centers of the S_j's, $j = 1, \cdots, n$ lie on $|C|$.

THEOREM 3. *If $f_\alpha A_x = f_\beta A_x = f_\gamma$ and if $x_\alpha = x_\beta$, then $f_\alpha = f_\beta$.*

Proof. By Theorem 2 there exists $z_1, \cdots, z_n = y = x_\alpha = x_\beta$ on the line segment between y and x such that

$$f_\gamma A_{z_1} \cdots A_{z_n} = f_\alpha .$$

Also substituting for f_γ and using (iii) of compatibility and Lemma 1

$$(f_\beta A_x) A_{z_1} \cdots A_{z_n} = f_\beta .$$

Hence, $f_\alpha = f_\beta$.

COROLLARY 1. *If $f_\alpha A_{x_1} \cdots A_{x_n} = f_\beta A_{x_1} \cdots A_{x_n}$ and if $x_\alpha = x_\beta$, then $f_\alpha = f_\beta$.*

THEOREM 4. *Let $\{f_t\}$ be the elements of an $\mathscr{A}$-continuation of f_0 along the path C to obtain f_1. Then $\{f_{1-t}\}$ are the elements of an*

$\mathscr{A}$-continuation of f_1 along the path C^{-1} and this continuation gives f_0.

THEOREM 5. *$\mathscr{A}$-continuation of a given element f_0 along a given curve C always leads to the same element f_1.*

THEOREM 6. *If an $\mathscr{A}$-continuation of f_0 along a path C is possible, it can always be accomplished by $\mathscr{A}$-continuation along a finite chain of balls.*

THEOREM 7. *Let $S_1, \cdots, S_n$ be a chain of balls with centers $x_1, \cdots, x_n$ and $C = (\mu, I)$ be a path from x_1 to x_n and passing through $x_2, \cdots, x_{n-1}$ such that $\mu(t) \in S_j$ for all t, $t_j \leqq t \leqq t_{j+1}$ where $\mu(t_j) = x_j$. Then if $f_1, \cdots, f_n$ is an $\mathscr{A}$-continuation along this chain, there exists an $\mathscr{A}$-continuation of f_0 along C which gives f_n at x_n.*

The desired $\mathscr{A}$-continuation along the curve C for Theorem 7 is given by: for each $t \in [0, 1]$ associate the element $f_t = f_{x_j} A_{\mu}(t)$ where $t_j \leqq t \leqq t_{j+1}$.

DEFINITION. For every α and β define

$$\mathscr{R}_{\alpha}^{\beta} = \{x \,|\, f_{\alpha} A_x = f_{\beta} A_x\} .$$

THEOREM 8. *$\mathscr{R}_{\alpha}^{\beta} = \emptyset$ or $\mathscr{R}_{\alpha}^{\beta} = S_{\alpha} \cap S_{\beta}$.*

Proof. Both $\mathscr{R}_{\alpha}^{\beta} \subset S_{\alpha} \cap S_{\beta}$ and $(S_{\alpha} \cap S_{\beta}) \backslash \mathscr{R}_{\alpha}^{\beta}$ are open sets. The theorem follows since $S_{\alpha} \cap S_{\beta}$ is connected.

$\mathscr{A}$-continuation need not be possible along a given curve $C = (\mu, I)$. The point $\mu(t_0)$ is a singular point or an $\mathscr{A}$-singular point relative to C and f_0 if the element f_0 can be continued along the segment 0 to t for all $t < t_0$ but not along the segment if $t > t_0$.

DEFINITION. The (complete) $\mathscr{A}$-function is the set F of all elements obtainable from a given element by $\mathscr{A}$-continuation.

From this definition and Theorem 4, it is clear that each element of F can be obtained from any other element of F by $\mathscr{A}$-continuation. Furthermore, two $\mathscr{A}$-functions F_1 and F_2 which have a single element in common are identical. Let

$$M_F = \{(x_{\alpha}, f_{\alpha}) \,|\, f_{\alpha} \in F \text{ and } x_{\alpha} \text{ is the center of } f_{\alpha}\}$$

and for $\rho < r(x_{\alpha})$ let

$$K_{\rho}(x_{\alpha}, f_{\alpha}) = \{(y, g) \,|\, g = f_{\alpha} A_y \quad \text{and} \quad d(y, x_{\alpha}) < \rho\} .$$

Let $\{K_\rho(x_\alpha, f_\alpha)\}$ be the base for a topology on M_F and the projection map of $K_\rho(x_\alpha, f_\alpha)$, $(y, g) \to y$ be the coordinate map for M_F.

THEOREM 9. *$\Gamma = (\mu(t), I)$ where $\mu(t) = (x_t, f_t)$, is a path on M_F if and only if f_t is an $\mathscr{A}$-continuation along the path $C = (x_t, I)$.*

Proof. Clearly (x_t, I) is a path. For any t_0, let t_1 be such that $x_t \in S_{t_0}$ for all t, $t_0 \leqq t \leqq t_1$. In particular $x_{t_1} \in S_{t_0}$ and $(x_{t_1}, f_{t_1}) \in K_\rho(x_{t_0}, f_{t_0})$ of some $\rho > r_{t_0}$. Hence, $f_{t_1} = f_{t_0} A_{x_{y_1}}$ and we have an $\mathscr{A}$-continuation.

DEFINITION. The union of all M_F is called the manifold of $\mathscr{F}$ with respect to $\mathscr{A}$-continuation and is denoted by $M(\mathscr{F}, \mathscr{A})$.

THEOREM 10. *M_F is a connected analytic manifold.*

DEFINITION. Given $\mathscr{A}$-continuation for $\mathscr{F}$ and $\mathscr{B}$-continuations for $\mathscr{G}$ a mapping ψ from $M(\mathscr{F}, \mathscr{A})$ to $M(\mathscr{G}, \mathscr{B})$ is called an $\mathscr{A}\mathscr{B}$-morphism if

(i) $\psi(x_\alpha, f_\alpha) = (y_\alpha, g_\alpha)$ implies $x_\alpha = y_\alpha$

(ii) $\psi(x_\alpha, f_\alpha) = (x_\alpha, g_\alpha)$ implies $\psi(x_\alpha, f_\alpha A_x) = (x_\alpha, g_\alpha B_x)$ if both $f_\alpha A_x$ and $g_\alpha B_x$ are defined.

Since an $\mathscr{A}\mathscr{B}$-morphism leaves the first entry in (x_α, f_α) fixed it is convenient to write ψf_α in place of $\psi(x_\alpha, f_\alpha)$. Using this convention (ii) can be stated as:

(ii)′ $\psi(f_\alpha A_x) = (\psi f_\alpha) B_x$.

LEMMA 2. *ψ a bijective $\mathscr{A}\mathscr{B}$-morphism implies ψ^{-1} is a $\mathscr{B}\mathscr{A}$-morphism.*

Proof. Let $\psi f_\alpha = g_\alpha$ have their center at x and assume $f_\alpha A_y = f_\beta$ and $g_\alpha B_y = g_\beta$ both exist.

$$\psi f_\beta = \psi(f_\alpha A_y) = (\psi f_\alpha) B_y = g_\alpha B_y = g_\beta .$$

Hence,

$$\psi^{-1}(g_\alpha B_y) = \psi^{-1} g_\beta = f_\beta = f_\alpha A_y = (\psi^{-1} g_\alpha) A_y .$$

THEOREM 11. *Let ψ be a bijective mapping from $M(\mathscr{F}, \mathscr{A})$ to $M(\mathscr{H}, \mathscr{B})$ such that $\psi(x, f) = (x, h)$. ψ is a homeomorphism if and only if ψ is an $\mathscr{A}\mathscr{B}$-morphism.*

Proof. Assume ψ is an $\mathscr{A}\mathscr{B}$-morphism, $\psi f_\alpha = h_\alpha$, $(f_\alpha, S_\alpha) \in \mathscr{F}$, $(h_\alpha, T_\alpha) \in \mathscr{F}$, and $U = S_\alpha \cap T_\alpha$. Then

$$\psi\{(x, f_\alpha A_x) \mid x \in U\} = \{(x, h_\alpha B_x) \mid x \in U\}$$

implies that ψ and ψ^{-1} are continuous.

Assume ψ is a homeomorphism and using the same notation

$$E \equiv \{(y, h_\alpha B_y) \mid y \in U\}$$

is a basic open set in $M(\mathscr{H}, \mathscr{B})$ and ψ a homeomorphism implies $\psi^{-1}(E)$ contains a basic open set of the form

$$\{(y, f_\alpha A_y) \mid y \in N_\alpha\}$$

where $N_\alpha \subset U$ is a ball. ψ preserves first coordinate and is injective implies

$$(1) \qquad \psi(y, f_\alpha A_y) = (y, h_\alpha B_y)$$

for all y in N_α. Hence, $\psi f_\alpha = h_\alpha$ implies there exists a ball N_α such that (1) holds for y in N_α.

Let z be in S and L denote the line segment from x_α to z. For each x in L let $f_x = f_\alpha A_x$ and N_x be the ball where (1) holds for f_x. Let M_x be the ball concentric with N_x and having a radius which is one fourth the radius of N_x. L compact implies there exist $\{M_{x_j} \mid j = 1, \cdots, n\}$ which covers L. Then assuming $x = x_1, x_2, \cdots, x_n = z$ are ordered along L then x_j is in $N_{x_{j-1}}$. Hence,

$$f_\alpha A_z = f_\alpha A_{x_2} A_{x_3} \cdots A_{x_n}$$

and since (1) holds for f_{x_j} in N_{x_j}

$$\begin{aligned} \psi(f_\alpha A_z) &= \psi[(f_\alpha A_{x_2} \cdots A_{x_{n-1}}) A_{x_n}] = [\psi(f_\alpha A_{x_2} \cdots A_{x_{n-1}})] B_{x_n} \\ &= (\psi f_\alpha) B_{x_2} \cdots B_{x_n} = (\psi f_\alpha) B_z \, . \end{aligned}$$

Therefore, (1) holds in S which is the ball in which both $f_\alpha A_x$ and $h_\alpha B_x$ are defined.

COROLLARY. *If ψ is a bijective $\mathscr{A}\mathscr{B}$-morphism and $\psi(f_0, x_0) = (g_0, x_0)$ then M_F is homeomorphic to M_G where F and G are the $\mathscr{A}$-function and $\mathscr{B}$-function of f_0 and g_0, respectively.*

THEOREM 12. *Let ψ be a bijective $\mathscr{A}\mathscr{B}$-morphism and $C = (\alpha(t), I)$ be a path in C^n. $\{f_t \mid t \in I\}$ is an $\mathscr{A}$-continuation along C if and only if $\{g_t \mid \psi(x(t), f_t) = (x(t), g_t)$ and $t \in I\}$ is a $\mathscr{B}$-continuation along C.*

Proof. From Theorem 9 $\{f_t \mid t \in I\}$ an $\mathscr{A}$-continuation along C is equivalent to $\{(x(t), f_t) \mid t \in I\}$ being a path on $M(\mathscr{F}, \mathscr{A})$. Since ψ is homeomorphism $\{\psi(x(t)), f_t) \mid t \in I\}$ is a path on $M(\mathscr{G}, \mathscr{B})$ and this is equivalent to $\{g_t \mid \psi(x(t), f_t) = (x(t), g_t), t \in I\}$ being a $\mathscr{B}$- continuation along C.

2. Examples of generalized continuation of function elements. The elements of Φ are called function elements if (f_α, S_α) in $\mathscr{F}$ implies f_α is a complex valued function whose domain of definition is S_α or $S_\alpha \times T$ where T is fixed (see §3). In general, for y in $S_\alpha \cap S_\beta$, where (f_α, S_α) and $(f_\alpha A_x, S_\beta)$ are in $\mathscr{F}$

$$(f_\alpha A_x)(y) \neq f_y(y)$$

as can be seen in the examples. The Complete Weierstrass Analytic is quite similar to the complete $\mathscr{A}$-continuation of function elements except the values of a function element do not have to agree with its direct $\mathscr{A}$-continuation.

DEFINITION. Let F be a complete $\mathscr{A}$-function generated by a function element then the single-valued function, f, defined on M_F by

$$f[(x_\alpha, f_\alpha)] = f_\alpha(x_\alpha)$$

is called the indicator function of F.

In the case of $\mathscr{A}$-continuation of function elements the Law of Permance of Functional Equations can be applied, however, the functional equations to which it applies depends on the particular $\mathscr{A}$-continuation. Two examples of generalized continuation of function elements are given.

1. *Analytic Continuation*: Let Φ denote the set of absolutely convergent power sesies of one complex variable with positive radius of convergence,

$$\Phi = \left\{P_\alpha(z) = \sum_{n=0}^{\infty} a_n^{(\alpha)}(z - z_\alpha)^n\right\},$$

and for P_α in Φ let S_α be its disc of convergence so that

$$\mathscr{F} = \{(P_\alpha, S_\alpha)\}.$$

Analytic continuation can be represented by

$$\mathscr{A} = \{A_z\}_{z \in C}$$

where A_z is the operator which expresses a function element defined in a neighborhood of z as a power series about the point z. In this case it is known that $\mathscr{F}$ and $\mathscr{A}$ satisfy the conditions for being a compatible set of operators. Indeed $\mathscr{A}$ is referred to as a direct analytic continuation. The indicator function in this example is the multivalued analytic function which is generated by the power series.

3. Continuing the coefficients of linear Weierstrass Polynomials. Let Φ be the functions defined by a power series with positive radius of convergence and which have the value zero at the center of their disc of convergence,

$$\Phi = \left\{f_y(z) = \sum_{n=1}^{\infty} a_n^{(\alpha)}(z - z_0)^n\right\},$$

and for f_α in Φ let S_α be its disc of convergence. Now a set of operators, $\mathscr{W}$, can be defined on Φ by fW_{z_1} is defined by the power series of $f(z) - f(z_1)$ with center z_1 whenever z_1 is in the disc of convergence of f. This set of operators is compatible and hence gives a generalized continuation, $\mathscr{W}$-continuation, on $\mathscr{F}$. Note, that indicator function of any complete $\mathscr{W}$-function is

$$(fW_z)(z) \equiv 0 .$$

For $\mathscr{W}$-continuation the Law of Permance of Functional Equations is quite similar to that of analytic continuation. For instance the $\mathscr{W}$-continuation of an algebraic function element is again an algebraic function element.

This example can be generalized to C^n by letting S_α be the largest ball in which the power series converges absolutely. Then f_0 in Φ can be considered as the coefficient of a linear Weierstrass Polynomial which is regular in W,

$$P(w, z) = (w - w_0) + f_0(z)$$

which has center (w_0, z_0). [6, page 68]. If (w_1, z_1) is a zero of P and z_1 is S_0 then representing the zero set of P in a neighborhood of (w_1, z_1) is the Weierstrass Polynomial with center (w_1, z_1), namely,

$$(w - w_1) + (fW_{z_1})(z) .$$

Hence, $\mathscr{W}$-continuation continues the coefficient of a linear Weierstrass Polynomial.

4. Continuing the normalized B_3-associate of the Bergman Integral Operator. Let $\mathscr{L} = \{\zeta \mid |\zeta| = 1\}$ and set $X = (x, y, z)$ in E^3

$$u = u(X, \zeta) = x + Z\zeta + Z^*\zeta^{-1}$$
$$Z = \frac{1}{2}(iy + z) , \quad \text{and} \quad Z^* = \frac{1}{2}(iy - z) . \tag{1}$$

Bergman introduced the integral operator

$$H(X) = \frac{1}{2\pi}\int_{\mathscr{L}} f(u, \zeta)\frac{d\zeta}{\zeta} \tag{2}$$

where f is an analytic function of the complex variables u and ζ

having at most a finite number of isolated singularities [1]. The integral operator defined in (2) is called the Bergman-Whittaker Integral Operator. Bergman has shown that in a neighborhood of X_0 (2) represents a harmonic function [1, 2]. The function f in (2) is called the B_3-associate of the harmonic function which it defines. B_3-associates of the form

$$f(u, \zeta) = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} a_{n,k} u^n \zeta^k \tag{3}$$

where (3) converges absolutely for u in a neighborhood, N, of zero and uniformly for u in a compact subset of N and ζ on $\mathscr{L}$ are called normalized B_3-associates. Then (2) gives a one to one correspondence between normalized B_3-associates and harmonic functions which are regular in a neighborhood of the origin [3, 4]. A translation of the origin in (1) gives

$$u(X - X_0, \zeta) = (x - x_0) + (Z - Z_0)\zeta + (Z^* - Z_0^*)\zeta^{-1} \tag{4}$$

$$f_\alpha(X, \zeta) \equiv f_\alpha(u(X - X_0, \zeta), \zeta) = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} a_{n,k}^{(\alpha)} [u(X - X_0, \zeta)]^n \zeta^k . \tag{5}$$

Then (2) gives a one-to-one correspondence between normalized B_3-associates centered at $X_0, f_\alpha(u(X - X_0, \zeta), \zeta)$, and harmonic functions regular in a neighborhood of X_0.

The B_3-associate may be defined for all u and ζ but the Bergman integral operator only represents the harmonic function in a domain, called the domain of association, which is usually not all of E^3 [4]. Rational B^3-associates generate harmonic functions which are not in general regular throughout E^3. The space is divided by surfaces of separation into a finite number of regions. As X moves from one domain of association to another, a new harmonic function is defined. If X changes from one demain of association to another the singular points of $f(u, \zeta)$ may enter or leave the interior of the curve of integration. In this section the generalized continuation developed for normalized B_3-associates overcomes this difficulty. That is, generalized continuations of a normalized B_3-associate generate the same harmonic function.

Let $\Phi = \{f_\alpha\}$ be the set of all normalized B_3-associates with centers X_α in C^3. That is, in (1) continue x, y, and z to complex numbers $x = x_1 + ix_2$, $y = y_1 + iy_2$, and $z = z_1 + iz_2$. Set $\mathscr{N} = \{(f_\alpha, S_\alpha)\}$ where f_α is in Φ and S_α is the largest ball with center X_α such that for any compact subset M of S_α, (5) converges absolutely and uniformly on $M \times \mathscr{L}$. Hence, f_α is defined on $S_\alpha \times \mathscr{L}$. The compatible set of operators $\mathscr{B}$ defined on $\mathscr{N}$ is a generalization of analytic continuation such as one finds in Hille [7: page 128]. Assume

$$(6)\qquad f_\alpha(X,\zeta)=f_\alpha(u(X,\zeta),\zeta)=\sum_{n=0}^{\infty}\sum_{k=-n}^{n}a_{n,k}[u(X,\zeta)]^n\zeta^k$$

is normalized B_3-associate which is centered at the origin. To apply B_{X_0} to obtain a normalized B_3-associate which is centered at X_0 the steps are:

(A) In (6) express $u(X,\zeta)$ as

$$u(X-X_0,\zeta)+u(X_0,\zeta)=u(X-X_0,\zeta)+x_0+Z_0\zeta+Z^*\zeta^{-1}$$

and then expand this last four termed expression in a multinomial expansion to obtain

$$(7)\qquad \sum_{n=0}^{\infty}\sum_{k=-1}^{n}a_{n,k}\Big(\sum_{r=0}^{n}\sum_{s=0}^{n-r}\sum_{t=0}^{n-r-s}b_{r,s,t}x_0^qZ_0^sZ_0^{*t}[u(X-X_0,\zeta)]^r\zeta^{s-t}\Big)\zeta^k$$

where $b_{r,s,t}$ is the multinomial coefficient and $q=n-r-s-t$.

(B) If (7) converges absolutely as a multiple series we can add the series in any admissible manner [8; page 114]. In particular (7) can be expressed as

$$(8)\qquad f_\alpha^*(X,\zeta)=\sum_{r=0}^{\infty}\sum_{v=-\infty}^{\infty}c_{r,v}[u(X-X_0,\zeta)]^r\zeta^v$$

where $c_{r,v}$ is obtained by adding all the coefficients for a fixed r and v.

(C) Normalize f^*, that is, remove all the terms from (8) for which $|v|>r$. This gives the direct $\mathscr{B}$-continuation

$$(9)\qquad f_\alpha B_{X_0}=\sum_{r=0}^{\infty}\sum_{v=-r}^{r}c_{r,v}[u(X-X_0,\zeta)]^r\zeta^v\ .$$

Note that f_α^* is an analytic continuation of f_α, hence, the integrals of f_α^* and f_α defined in (2) will be equal for X in the intersection of the domains of definition of f_α^* and f_α. Moreover, normalizing f_α^* does not change the value of the integral (2) as can be seen by applying the Residue Theorem to a term by term integration of the series. This implies that Bergman's Integral operator carries direct-$\mathscr{B}$-continuation of normalized B_3-associates over into analytic continuation of their respective harmonic functions.

To show that $\mathscr{B}$ is a compatible set of operators it is necessary to show that

$$(10)\qquad r_0\geqq r_\alpha-d(X_0,0)$$

where r_0 is the radius of the ball of definition of $f_0=f_\alpha B_{X_0}$. First, note that

$$(11)\qquad |u(X-X_\beta,\zeta)|\leqq\sqrt{2}\,d(X,X_\beta)$$

and that for every R there exists a $\hat{X}$ and $\hat{\zeta}$ such that

$$|u(X-\hat{X},\hat{\zeta})| = \sqrt{2}\,d(X,\hat{X}) = \sqrt{2R}\ .$$

Hence, if r_α is the radius of S_α then $\sqrt{2}\,r_\alpha$ is the radius of convergence of

$$\sum_{n=0}^{\infty}\Big(\sum_{k=-n}^{n}|a_{n,k}|\Big)u^n\ . \tag{12}$$

Second, if $X=(x,y,z)$ and is represented by (x,Z,Z^*) then $\tilde{X}=(|x|,|Z|,|Z^*|)$ has the property that $d(X,0)=d(\tilde{X},0)$.

In examining the absolute convergence of (7)

$$|b_{r,s,t}x_0^q Z_0^s Z_0^{*t}[u(X-X_0,\zeta)]^r\zeta^{s-t}| \equiv C_{r,s,t}$$

are the terms in the expansion of

$$[u(X-X_0,\zeta)+u(\tilde{X}_0,1)]^n\ .$$

Hence, (7) converges absolutely for $d(X,X_0)<r_\alpha-d(X_0,0)$ since (11) and (12) imply that

$$\begin{aligned}(14)\qquad &\sum_{n=0}^{\infty}\sum_{k=-n}^{n}\sum_{r=0}^{n}\sum_{s=0}^{n-r}\sum_{t=0}^{n-r-s}|a_{n,k}C_{r,s,t}|\\ &\leqq \sum_{n=0}^{\infty}\Big(\sum_{k=-n}^{n}|a_{n,k}|\Big)[|u(X-X_0,\zeta)|+u(\tilde{X}_0,1)]^n\\ &\leqq \sum_{n=0}^{\infty}\Big(\sum_{k=-n}^{n}|a_{n,k}|\Big)[\sqrt{2}\,d(X,X_0)+\sqrt{2}\,d(\tilde{X}_0,0)]^n\\ &\leqq \sum_{n=0}^{\infty}\Big(\sum_{k=-n}^{n}|a_{n,k}|\Big)[\sqrt{2}\,\rho]^n\end{aligned}$$

where $\rho<r_\alpha$. This convergence is uniform on compact sets of $S_\alpha\cap S_0$.

Let $\mathscr{H}=\{H_\alpha,S_\alpha)\}$ where H_α is a regular harmonic function represented by a power series whose largest ball of absolute convergence is S_α. The Bergman Integral Operator defines a map $\psi\colon M(\mathscr{N},\mathscr{B})\to M(\mathscr{H},\mathscr{A})$ where $\mathscr{A}$ is analytic continuation and ψf_α is given by (2). From previous statements it is noted that ψ is injective and as noted in (c) ψ is $\mathscr{B}\mathscr{A}$-morphism. Theorem 11 implies that $M(\mathscr{N},\mathscr{B})$ is homeomorphic to $M(\mathscr{H},\mathscr{A})$ and the Corollary implies that the manifold obtained by normalized continuation of f_α is the same as the manifold obtained by analytically continuing the harmonic function $H_\alpha=\psi f_\alpha$.

In particular when

$$f_\alpha(X,\zeta)=\sum_{n=0}^{\infty}\sum_{k=-n}^{n}a_{n,k}[u(X-X_\alpha,\zeta)]^n\zeta^k$$

with center X_α is $\mathscr{B}$-continued to the function

$$f_\alpha B_{X_0}(X, \zeta) = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} b_{n,k}[u(X - X_0, \zeta)]^n \zeta^k$$

with center X_0 the $b_{n,k}$'s can be calculated in the following cases.
(i) $b_{n,k} = \sum_{j=n}^{\infty} a_{n,k}(j!/n!(j-n)!)d^{j-n}$ when $X_0 - X_\alpha = (d, 0, 0)$
(ii) $b_{n,k} = \sum_{j=n}^{\infty} \sum_{h=b}^{j} a_{j,n+2h-j-k}(j!/(h-n)!n!(j-h)!)(id/2)^{j-n}$ when $X_0 - X_\alpha = (0, d, 0)$, and
(iii) $b_{n,k} = \sum_{j=n}^{\infty} \sum_{h=n}^{j} a_{j,n+2h-i-k}(-1)^{h-n}(j!/(h-n)!n!(j-h)!)(d/2)^{j-n}$ when $X_0 - X_\alpha = (0, 0, d)$.

For example if

$$f_0(X, \zeta) = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} [u(X, \zeta)]^n \zeta^k$$

which has center (0, 0, 0) is $\mathscr{B}$-continued using above expressions it is found that the $\mathscr{B}$-function determined by f_0 is

$$F = \{(X_\alpha, f_\alpha) \mid X_\alpha = (a, b, c)\ ,\quad a \neq 1 \quad \text{and} \quad b \neq 0\}\ ,$$

where

$$f_\alpha(X, \zeta) = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} \left(\frac{1}{1 - (a + bi)}\right)^{n+1} (u(X - X_\alpha, \zeta))^n \zeta^k\ .$$

Hence, f_0 is the B_3-associate of a harmonic function h_0 whose analytic extensions are single-valued since F is single-valued. Also the analytic continuation of h_0 is regular everywhere except for $\{(x, y, z) \mid x + iy = 1\}$.

Indeed it can be shown by using (2) that in a neighborhood of (0, 0, 0)

$$h_0(x, y, z) = \frac{1}{1 - (x + iy)}\ .$$

In a less tedious manner one can observe that

$$f_0(X, \zeta) = \frac{1}{1 - \zeta}\left\{\frac{1}{1 - u\zeta^{-1}} - \frac{\zeta}{1 - u\zeta}\right\}$$

and hence is the normalized B_3-associate of the same h_0 [5, Theorem 2.1].

For $\mathscr{B}$-continuation the indicator function of a complete $\mathscr{B}$-function generated by (f_α, S_α) is the complete $\mathscr{A}$-function generated by $(\psi f_\alpha, S_\alpha)$ as can be seen from (2). Hence, the indicator function for $\mathscr{B}$-continuation is the harmonic function obtained by the integral operators.

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