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NORMPRESERVING EXTENSIONS IN SUBSPACES OF $C(X)$

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If B is a subspace of $C(X)$ and F is a closed subset of X, this note gives sufficient conditions in order that every function in the restriction subspace $B|_F$ has an extension in B with no increase in norm.

Introduction. Let X be a compact Hausdorff space, $C(X)$ the Banach algebra of all continuous complex-valued functions on X and let B be a closed linear subspace of $C(X)$ separating the points of X and containing the constants. A closed subset F of X is said to have the normpreserving extension property w.r.t. B if any function b_0 in the restriction subspace $B|_F$ has an extension $b \in B$ (i.e. $b|_F = b_0$) such that $||b|| = ||b_0||_F(||\cdot||$(resp. $||\cdot||_F$) denotes the supremum norm on X (resp. F)). The main result is the following:

Let F be a closed subset of X and suppose there is a map T (not necessarily linear) from $M(X)$ into $M(X)$ satisfying the following conditions

(i) $m - Tm \in B^\perp$ *for all* $m \in M(X)$

(ii) $T\lambda$ *is a probability measure when* λ *is*

(iii) *If* $s_i \in C$ *and* $m_i \in M(X)$ $i = 1, \cdots, n$ *and* $\sum_{i=1}^n s_i m_i \in k(F)^\perp$ *then* $\sum_{i=1}^n s_i(Tm_i)|_{X\setminus F} \in B^\perp$.

Then F has the normpreserving extension property.

$M(X)$ denotes the set of regular Borel measures on X, and if A is a subset of B then $A^\perp$ is the set of those measures in $M(X)$ which annihilate A. $k(F)$ consists of those functions in B which are identically 0 on F. Also if G is a Borel subset of X and $m \in M(X)$ then $m|_G$ is the measure $\chi_G m$ where χ_G is the characteristic function for G.

Two conditions, either of which is known to imply that a closed subset F of X has the normpreserving extension property are the following:

Condition 1. *For all* $\sigma \in B^\perp, \sigma|_F \in B^\perp$.

Condition 2. *F is a compact subset of the Choquet boundary Σ_B for B and for all* $\sigma \in M(\Sigma_B) \cap B^\perp, \sigma|_F \in B^\perp$.

($M(\Sigma_B)$ denotes the set of those $\sigma \in M(X)$ for which the total variation $|\sigma|$ is maximal in Choquet's ordering for positive measures (see [1]

Ch. I §3 and [6] p. 24)).

In Chapter 2 of this note we show that when either Condition 1 or Condition 2 is satisfied there exists a map T with the above properties.

Actually, when Condition 1 or Condition 2 is satisfied stronger extension properties than the normpreserving one hold. (In the case of Condition 1 see [4] Theorem 3 and [5] Theorem 4.8 in the case of Condition 2 see [2] Theorem 4.5 and [3] Theorem 2). But as we show in Chapter 2 these stronger extension properties are corollaries to theorems based on the existence of a map T described above. Thus we are able to deal simultaneously with Conditions 1 and 2.

1. A condition for the normpreserving extension property.

Throughout this chapter F is a fixed closed subset of X and T is a map from $M(X)$ into $M(X)$ satisfying

(i) $m - Tm \in B^{\perp}$ *for all* $m \in M(X)$

(ii) $T\lambda$ *is a probability measure when* λ *is.*

(iii) *If* $s_i \in \boldsymbol{C}$ *and* $m_i \in M(X)$ *and* $\sum_{i=1}^{n} s_i m_i \in k(F)^{\perp}$ *then*

$$\sum_{i=1}^{n} s_i (Tm_i)|_{X \setminus F} \in B^{\perp} .$$

REMARK 1.1. It follows from conditions (i) and (iii) that if $\Sigma s_i \sigma_i \in B^{\perp}$ then $\Sigma s_i (T\sigma_i)|_F \in B^{\perp}$. Also if λ is a probability measure and $\lambda = \lambda|_F$ then $T\lambda = (T\lambda)|_F$, because $\lambda \in k(F)^{\perp}$ hence by (iii) $(T\lambda)|_{X\setminus F} \in B^{\perp}$. Since B contains the constants and $T\lambda$ is a positive measure $(T\lambda)|_{X\setminus F} = 0$.

We let S_B denote the state space of B s.e. $S_B = \{p \in B^*\colon \|p\| = p(1) = 1\}$. S_B is a convex set which is compact in the w^*-topology and the natural map of X into S_B is a homeomorphism. We shall frequently think of X as embedded in S_B. A representing measure for $p \in S_B$ is a probability measure v_p on X such that $p(f) = \int f dv_p$ for all $f \in B$.

DEFINITION 1.2. For each $b_0 \in B|_F$ we define a function $\bar{b}_0$ on S_B as follows. If $p \in S_B$ put

$$\bar{b}_0(p) = \int_F b_0 d Tv_p$$

where v_p is any representing measure for p on X.

REMARK 1.3. The above definition is meaningful because if v'_p is another representing measure for p on X then $v_p - v'_p \in B^{\perp}$; hence by Remark 1.1 $(Tv_p)|_F - (Tv'_p)|_F \in B^{\perp}$.

LEMMA 1.4. $\bar{b}_0$ *has the following properties*:

(1) $\bar{b}_0$ *is an affine function*

(2) $|\bar{b}_0(p)| \leqq \|b_0\|_F$ *for all* $p \in S_B$

(3) $\bar{b}_0(p) = b_0(p)$ *if* $p \in F$

(4) $\bar{b}_0$ *is a linear combination of upper semicontinuous affine functions.*

(5) $\int \bar{b}_0 d\sigma = 0$ *for all* $\sigma \in B^\perp$.

Proof 1. follows from the definition of $\bar{b}_0$ and remark 1.1. (2) is trivial: To prove (3) observe that if $x \in F$ then by remark 1.1 $T\delta_x = (T\delta_x)|_F$ (δ_x is point mass at x). But $T\delta_x$ is a representing measure for x. (4) Observe that if $b_0 \in B|_F$ and $f_0 = Reb_0$, we can define $\bar{f}_0$ in exactly the same way as we defined $\bar{b}_0$. Then $\bar{f}_0$ is affine on S_B and $\bar{f}_0 = Re\bar{b}_0$. First assume that $f_0 \geqq 0$. We want to show that $\bar{f}_0$ is upper semi-continuous. For each $t \geqq 0$ put $K_t = \{p \in S_B: \bar{f}_0(p) \geqq t\}$ we must show that K_t is closed. Let $\{p_\alpha\}$ be a net from K_t with limit point p_0, and v_α a representing measure for p_α on X for each α. Write $Tv_\alpha = u_\alpha + w_\alpha$ where $u_\alpha = (Tv_\alpha)|_F$. Let u_0 be a w^*-clusterpoint for $\{u_\alpha\}$ and let $\{u_\beta\}$ be a subnet from $\{u_\alpha\}$ converging to u_0. Also let w_0 be a clusterpoint for $\{w_\beta\}$. Then $v_0 = u_0 + w_0$ is a representing measure for p_0 and since

$$u_0 = u_0|_F,\ T\left(\frac{u_0}{\|u_0\|}\right) = T\left(\frac{u_0}{\|u_0\|}\right)\Big|_F .$$

(Remark 1.1). Using this and Remark 1.1 once more we get:

$$\bar{f}_0(p_0) = \int_F f_0 dTv_0 = \|u_0\| \int_F f_0 dT\left(\frac{u_0}{\|u_0\|}\right) + \|w_0\| \cdot \int_F f_0 dT\left(\frac{w_0}{\|w_0\|}\right)$$

$$\geqq \|u_0\| \int_F f_0 dT\left(\frac{u_0}{\|u_0\|}\right) = \int_F f_0 du_0 \geqq t. \quad \text{Hence } p_0 \in K_t .$$

In general take a positive number k such that $f_0 + k \geqq 0$. Then $\bar{f}_0 = \overline{f_0 + k} - \bar{k}$ is the difference of upper semi-continuous functions. Since this holds for any $f_0 \in ReB|_F$ (4) is proved.

Since $\bar{b}_0$ is a linear combination of real valued affine upper semi-continuous functions it satisfies the barycenter formula i.e. if $p \in S_B$ and v_p is a representing measure for p then

$$\int \bar{b}_0 dv_p = \bar{b}_0(p)$$

(See [1] Cor. I 1.4)

Now we consider a measure $\sigma \in B^\perp$ with a decomposition $\sigma = \sum_{i=1}^4 t_i \sigma_i$ into probability measures σ_i representing points $p_i \in S_B$ for

$i = 1, 2, 3, 4$. By axiom (i) the measure $T\sigma_i$ also represent p_i for $i = 1, 2, 3, 4$. Applying the above result together with the definition of $\bar{b}_0$ and axiom (iii), we obtain:

$$\begin{aligned}\int \bar{b}_0 d\sigma &= \sum_{i=1}^{4} t_i \int \bar{b}_0 d\sigma_i = \sum_{i=1}^{4} t_i \bar{b}_0(p_i) \\ &= \sum_{i=1}^{4} t_i \int_F b_0 d(T\sigma_i) = \int_F b_0 d\Big[\sum_{i=1}^{4} t_i(T\sigma_i)\Big] = 0 .\end{aligned}$$

This completes the proof of (5).

PROPOSITION 1.5. *$B|_F$ is closed in $C(F)$*

Proof. Let $\sigma \in B^\perp$, and consider a $b_0 \in B|_F$ such that $||b_0||_F \leqq 1$. By statement (5) of Lemma 1.4:

$$0 = \int \bar{b}_0 d\sigma = \int_F b_0 d\sigma + \int_{X\setminus F} \bar{b}_0 d\sigma .$$

Hence

$$\left|\int_F b_0 d\sigma\right| = \left|\int_{X\setminus F} b_0 d\sigma\right| \leqq ||\sigma|_{X\setminus F}|| ,$$

and so $|\sigma|_F \leqq ||\sigma|_{X\setminus F}||$.

By a result of Gamelin [4] and Glicksberg [5] (see also [3, Prop. 1]) this implies that $B|_F$ is almost normpreserving, or what is equivalent, that $B/_{k(F)}$ is isometric to $B|_F$. Hence $B|_F$ is complete in uniform norm, and we are done.

PROPOSITION 1.6. *Let $b_0 \in B|_F$ and let ψ be a strictly positive lower semi-continuous function on X such that $\psi(x) > |\bar{b}_0(x)|$ for all $x \in X$. Then there is a function $b \in B$ such that $b|_F = b_0$ and $|b(x)| < \psi(x)$ for all $x \in X$.*

Proof. Apply Theorem 2.2 of [2].

THEOREM 1.7. *Let F and T be as in the beginning of this chapter and let $b_0 \in B|_F$ with $||b_0||_F \leqq 1$ and let ψ be a strictly positive lower semi-continuous function such that $\psi(x) > |\bar{b}_0(x)|$ for all $x \in X$. Then there is a function $b \in B$ such that*

$$b|_F = b_0,\ ||b|| = ||b_0||_F \textit{ and } |b(x)| < \psi(x) \textit{ for all } x \in X .$$

Proof. The proof is exactly the same as proof of [3] Theorem 2 after replacing the function A from [3] by $\bar{b}_0$ and Lemma 1 of [3] by Proposition 1.6 of this note.

COROLLARY 1.8. *F and T as before. Then F has the normpreserving extension property w.r.t. B.*

THEOREM 1.9. *Let F and T be as before let $b_0 \in B|_F$ and let ψ be a strictly positive lower semi-continuous function such that $\psi(x) \geqq |\bar{b}_0(x)|$ for all $x \in X$. Suppose furthermore that $\psi(x) \geqq \int \psi dT\lambda_x$ for all $x \in X \backslash F$ for which $\bar{b}_0(x) \neq 0$ (λ_x is a representing measure for x). Then there is a function $b \in B$ such that*

$$b|_F = b_0 \text{ and } |b(x)| \leqq \psi(x) \text{ for all } x \in X .$$

Proof. The proof is the same as the proof of [2] Theorem 4.5 replacing in the proof of Theorem 2.1 of [2] by Proposition 1.6 of this note.

2. Relations to conditions 1 and 2. We start by showing the equivalence of condition 1 to a condition involving $k(F)^\perp$

PROPOSITION 2.1. *Let F be a closed subset of X. Then the following conditions are equivalent:*

1. *For all $\sigma \in B^\perp$, $\sigma|_F \in B^\perp$*

1′. *For all $\sigma \in k(F)^\perp$, $\sigma|_{X \backslash F} \in B^\perp$.*

Proof. Condition 1′ trivially implies 1. Suppose Condition 1 is satisfied and let $\sigma \in k(F)^\perp$. Let $b_0 \in B|_F$ and let $b \in B$ be any extension of b_0. Since $\sigma \in k(F)^\perp$ the quantity $\int b d\sigma$ is independent of the choice of the extension b. Thus $b_0 \to \int b d\sigma$ is a well defined linear functional on $B|_F$. By [4] Theorem 1, $B|_F$ is closed in $C(F)$. It then follows from the open mapping theorem that $b_0 \to bd\sigma$ is a continuous linear functional. Thus we can find a measure $\sigma_1 = \sigma_1|_F$ such that $\sigma_1 - \sigma \in B^\perp$. But then $\sigma|_{X \backslash F} = (\sigma_1 - \sigma)|_{X \backslash F} \in B^\perp$.

Let again F be a closed subset of X and suppose that Condition 1 is satisfied. Let T be the identity map from $M(X)$ to $M(X)$. By the above proposition T satisfies requirements (i) (ii) and (iii) from the beginning of Chapter 1. In this case if $b_0 \in B|_F$, $\bar{b}_0(x) = 0$ for all $x \in X \backslash F$. From Theorem 1.9 we can then deduce the following well known theorem.

THEOREM 2.2. *Let F be a closed subset of X and suppose that $\mu|_F \in B^\perp$ for all $\mu \in B^\perp$. If $b_0 \in B|_F$ and ψ is a strictly positive lower semi-continuous function with $\psi(x) \geqq |b_0(x)|$ for all $x \in F$ then there is function $b \in B$ such that*

$$b|_F = b_0 \text{ and } |b(x)| \leqq \psi(x) \text{ for all } x \in X.$$

We now look at Condition 2. Let F be a compact subset of the Choquet boundary Σ_B and suppose Condition 2 is satisfied i.e. for all $\sigma \in B^\perp \cap M(\Sigma_B)$, $\sigma|_F \in B^\perp$. We need the following lemma

LEMMA 2.3. *Under the above hypotheses $B|_F$ is closed in $C(F)$.*

Proof. By [5] Theorem 3.1 we must show the existence of a constant $c \geqq 1$ such that $\|\mu - (B|_F)^\perp\| \leqq c\|\mu - B^\perp\|$ for all $\mu \in M(F)$. Let $\mu \in M(F)$ and $\sigma \in B^\perp$. We write $\sigma = \sigma|_F + \sigma|_{X\setminus F}$ and further write $\sigma|_{X\setminus F} = t_1\lambda_1 - t_2\lambda_2 + i(t_3\lambda_3 - t_3\lambda_4)$ where the t_i's are positive numbers and the λ's are probability measures such that λ_1 and λ_2 (resp. λ_3 and λ_4) live on disjoint subsets of X. For $i = 1, \cdots, 4$ let v_i be a maximal measure such that $\lambda_i - v_i \in B^\perp$. Put $w = t_1v_1 - t_2v_2 + i(t_3v_3 - t_4v_4)$. Then $\sigma_{X\setminus F} - w \in B^\perp$ and $\|w\| \leqq \sum_{i=1}^4 t_i\|v_i\| = \sum_{i=1}^4 t_i\|\lambda_i\| \leqq 2\|\sigma|_{X\setminus F}\|$. Now $\sigma|_F + w \in B^\perp \cap M(\Sigma_B)$ so that $\sigma|_F + w|_F \in B^\perp$. Hence $\|\mu - (A|_F)^\perp\| \leqq \|\mu - (\sigma|_F + w|_F)\| \leqq \|\mu - \sigma|_F)\| + 2\|\sigma\|_{X\setminus F}\| \leqq 2\|\mu - \sigma\|$. Thus we can take $c = 2$ and the lemma is proved.

As above let F be a compact subset of Σ_B and suppose that for all $\sigma \in M(\Sigma_B) \cap B^\perp$, $\sigma|_F \in B^\perp$. We define a map T from $M(X)$ to $M(X)$ as follows. If λ is a probability measure on X pick a maximal measure v with $\lambda - v \in B^\perp$ and put $T\lambda = v$. If λ is already maximal put $T\lambda = \lambda$. If $\sigma \in M(X)$ write $\sigma = t_1\lambda_1 - t_2\lambda_2 + i(t_3\lambda_3 - t_4\lambda_4)$ where the t_i's are positive numbers and where λ_1 and λ_2 (resp. λ_3 and λ_4) are probability measures living on disjoint subsets of X. Then put $T\sigma = t_1T\lambda_1 - t_2T\lambda_2 + i(t_3T\lambda_3 - t_4T\lambda_4)$. The map T from $M(X)$ to $M(X)$ we get in this way obviously has properties (i) and (ii) from the beginning of Chapter 1. Observe that $T\sigma = \sigma$ if $\sigma = \sigma|_F$ since $F \subset \Sigma_B$. To see that T also has property (iii) let $\Sigma s_i\sigma_i \in k(F)^\perp$. By Lemma 2.3 $B|_F$ is closed in $C(F)$. Just as in the proof of Proposition 2.1 we can find a measure $\mu = \mu|_F$ such that $\mu - \Sigma s_i\sigma_i \in B^\perp$. Then $\mu - \Sigma s_iT\sigma_i \in B^\perp \cap M(\Sigma_B)$ so that $\mu - \Sigma s_i(T\sigma_i)|_F \in B^\perp$, but then $\Sigma s_i(T\sigma_i)|_{X\setminus F} \in B^\perp$. We can then using Theorems 1.7 and 1.9 deduce the same interpolation theorems as in [2] and [3]. In particular we get from Theorem 1.7:

THEOREM 2.4. *Let F be a compact subset of the Choquet boundary Σ_B and suppose that for all $\sigma \in B^\perp \cap M(\Sigma_B)$, $\sigma|_F \in B^\perp$. Then F has the normpreserving extension property* w.r.t. B.

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