DISTRIBUTING TENSOR PRODUCT OVER DIRECT PRODUCT

K. R. GOODEARL

This paper is an investigation of conditions on a module A under which the natural map

$$A \otimes (\Pi C_{\alpha}) \longrightarrow \Pi(A \otimes C_{\alpha})$$

is an injection. The investigation leads to a theorem that a commutative von Neumann regular ring is self-injective if and only if the natural map

$$(\Pi F_{\alpha}) \otimes (\Pi G_{\beta}) \longrightarrow \Pi (F_{\alpha} \otimes G_{\beta})$$

is an injection for all collections $\{F_{\alpha}\}$ and $\{G_{\beta}\}$ of free modules. An example is constructed of a commutative ring R for which the natural map

$$R[[s]] \otimes R[[t]] \longrightarrow R[[s, t]]$$

is not an injection.

R denotes a ring with unit, and all R-modules are unital. All tensor products are taken over R.

We state for reference the following theorem of H. Lenzing [2, Satz 1 and Satz 2]:

THEOREM L. (a) A right R-module A is finitely generated if and only if for any collection $\{C_{\alpha}\}$ of left R-modules, the natural map $A \otimes \Pi C_{\alpha} \rightarrow \Pi (A \otimes C_{\alpha})$ is surjective.

(b) A right R-module A is finitely presented if and only if for any collection $\{C_{\alpha}\}$ of left R-modules, the natural map $A \otimes \Pi C_{\alpha} \rightarrow \Pi(A \otimes C_{\alpha})$ is an isomorphism.

THEOREM 1. For any right R-module A, the following conditions are equivalent:

(a) If $\{C_{\alpha}\}$ is any collection of flat left R-modules, then the natural map $A \otimes \Pi C_{\alpha} \to \Pi (A \otimes C_{\alpha})$ is an injection.

(b) There is a set X of cardinality at least card (R) such that the natural map $A \otimes R^x \to A^x$ is an injection.

(c) If B is any finitely generated submodule of A, then the inclusion $B \rightarrow A$ factors through a finitely presented module.

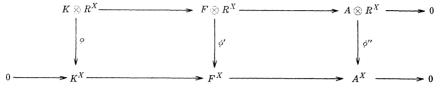
Note that condition (c) always holds when R is right noetherian, for then all finitely generated submodules of A are finitely presented.

Proof. (b) \Rightarrow (c): If R is finite, then it is right noetherian and

(c) holds. Thus we may assume that R is infinite.

Let $f: F \to A$ be an epimorphism with F_R free, and set $K = \ker f$. There is a finitely generated submodule G of F such that fG = B.

We have a commutative diagram with exact rows as follows (Diagram I):





Since G is finitely generated, $G^x \leq \phi'(F \otimes R^x)$. A short diagram chase (using the injectivity of ϕ'') shows that $(G \cap K)^x \leq \phi(K \otimes R^x)$.

card $(G) \leq \text{card } (R)$ because R is infinite, hence $\text{card } (G \cap K) \leq \text{card } (X)$. Thus there is a surjection $\alpha \mapsto g_{\alpha}$ of X onto $G \cap K$. The element $g = \{g_{\alpha}\}$ in $(G \cap K)^{x}$ must be the image under ϕ of some element $h_{1} \otimes r_{1} + \cdots + h_{n} \otimes r_{n}$ in $K \otimes R^{x}$. It follows easily that $G \cap K$ is contained in the submodule H of K generated by h_{1}, \cdots, h_{n} . Note that $G \cap H = G \cap K$.

G + H is contained in some finitely generated free submodule F_0 of F. The map f induces a monomorphism of $G/(G \cap H)$ into A, and this monomorphism factors through the finitely presented module F_0/H . Since fG = B, the inclusion $B \rightarrow A$ also factors through F_0/H .

(c) \Rightarrow (a): Consider any x belonging to the kernel of the natural map $\phi: A \otimes \Pi C_{\alpha} \rightarrow \Pi(A \otimes C_{\alpha})$. There is a finitely generated submodule

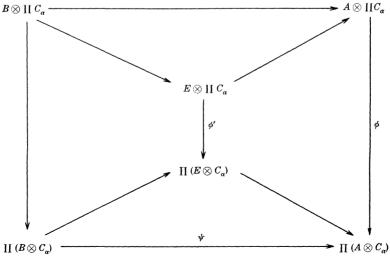


DIAGRAM II

108

B of A such that x is in the image of the map $B \otimes \Pi C_{\alpha} \to A \otimes \Pi C_{\alpha}$. By (c), the inclusion $B \to A$ factors through some finitely presented module E.

We have a commutative diagram as follows (Diagram II):

 ϕ' is an isomorphism by Theorem L, and ψ is a monomorphism because all the C_{α} 's are flat. Another diagram chase now shows that x = 0.

COROLLARY. Suppose that R is (von Neumann) regular. For any right R-module A, the following conditions are equivalent:

(a) If $\{C_{\alpha}\}$ is any collection of left R-modules, then the natural map $A \otimes \Pi C_{\alpha} \to \Pi (A \otimes C_{\alpha})$ is an injection.

(b) There is a set X of cardinality at least card (R) such that the natural map $A \otimes R^x \to A^x$ is injective.

(c) All finitely generated submodules of A are projective.

Proof. (b) \Rightarrow (c): If B is a finitely generated submodule of A, then Theorem 1 says that the inclusion $B \rightarrow A$ factors through a finitely presented module E. E is flat (because R is regular) and hence is projective. Thus B can be embedded in a projective module. Since R is semihereditary, B must be projective.

(c) \Rightarrow (a): All the C_{α} 's are flat (since R is regular), and all finitely generated submodules of A are finitely presented, so this follows directly from Theorem 1.

THEOREM 2. Assume that R is a commutative regular ring. Then the following conditions are equivalent:

(a) If $\{F_{\alpha}\}$ and $\{G_{\beta}\}$ are any collections of free R-modules, then the natural map $(\Pi F_{\alpha}) \otimes (\Pi G_{\beta}) \to \Pi(F_{\alpha} \otimes G_{\beta})$ is an injection.

(b) There is a set X of cardinality at least card (R) such that the natural map $R^x \otimes R^x \to R^{x \times x}$ is an injection.

(c) R is injective as a module over itself.

Proof. (b) \Rightarrow (c): By [1, Theorem 2.1], it suffices to show that any finitely generated nonsingular *R*-module *B* is projective.

[1, Lemma 2.2] says that we can embed B in a finite direct sum $Q_1 \bigoplus \cdots \bigoplus Q_n$, where each Q_i is a copy of the maximal quotient ring Q of R. Then B can be embedded in a direct sum $B_1 \bigoplus \cdots \bigoplus B_n$, where B_i is a finitely generated R-submodule of Q_i . Since R is semihereditary, B will be projective provided each B_i is projective. Thus without loss of generality we may assume that B is an R-submodule of Q.

Let b_1, \dots, b_n generate *B*. Since *R* is an essential submodule of *Q*, there is an essential ideal *I* of *R* such that $b_i I \leq R$ for all *i*.

Since R is commutative, the multiplications by the elements of I induce homomorphisms of B into R. Together, these homomorphisms induce a homomorphism $f: B \to R^{I}$. Q is a nonsingular R-module because it has the nonsingular R-module R as an essential submodule. Thus no nonzero element of B is annihilated by I; i.e., $f: B \to R^{I}$ is an injection. Since card $(I) \leq \text{card}(R) \leq \text{card}(X)$, there must also be an embedding of B into R^{X} .

Since the natural map $R^x \otimes R^x \to (R^x)^x$ is injective by (b), the corollary to Theorem 1 says that all finitely generated submodules of R^x are projective. Thus B must be projective.

(c) \Rightarrow (a): By [1, Theorem 2.1], all finitely generated nonsingular R-modules are projective. Since $R_{\mathbb{R}}$ is nonsingular, ΠF_{α} is nonsingular; thus all finitely generated submodules of ΠF_{α} are projective. By the corollary to Theorem 1, the natural map $(\Pi F_{\alpha}) \otimes (\Pi G_{\beta}) \rightarrow \Pi_{\beta}[(\Pi F_{\alpha}) \otimes G_{\beta}]$ is an injection. Likewise, each of the maps $(\Pi F_{\alpha}) \otimes G_{\beta} \rightarrow \Pi_{\alpha}(F_{\alpha} \otimes G_{\beta})$ is injective. Thus the map $(\Pi F_{\alpha}) \otimes (\Pi G_{\beta}) \rightarrow \Pi(F_{\alpha} \otimes G_{\beta})$ must be injective.

In particular, Theorem 2 asserts that if R is a countable commutative regular ring which is not self-injective, then the natural map $R^{X} \otimes R^{X} \rightarrow R^{X \times X}$ is not an injection for any infinite set X. For example, let F_1, F_2, \cdots be a countable sequence of copies of some countable field F; let R be the subalgebra of ΠF_n generated by 1 and $\bigoplus F_n$. R is obviously a countable commutative regular ring. Since ΠF_n is a proper essential extension of R_R, R_R is not injective.

If N is the set of natural numbers, then the natural map $R^{N} \otimes R^{N} \to R^{N \times N}$ is not an injection. Thus the tensor product of two one-variable power series rings, $R[[s]] \otimes R[[t]]$, is not embedded in R[[s, t]] by the natural map.

References

V. C. Cateforis, On regular self-injective rings, Pacific J. Math., 30 (1969), 39-45.
H. Lenzing, Endlich präsentierbare Moduln, Arch. der Math., 20 (1969), 262-266.

Received July 28, 1971.

UNIVERSITY OF WASHINGTON

Author's current address: UNIVERSITY OF UTAH