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CUT LOCI OF POINTS AT INFINITY

GEORGE M. LEWIS

In a G-space R if B is a co-ray to A then the union of all co-rays to A that contain B is either a straight line or a co-ray to A maximal in that it is properly contained in no other co-ray to A. In the latter case, the initial point of the maximal co-ray is a copoint to A. The concept of co-point is an analogue to that of minimum point in a sense made precise. On certain non-compact G-surfaces of finite connectivity, including those with non-positive curvature, we characterize the locus of co-points to a given ray and obtain bounds for the number of components of this locus, the number of co-rays emanating from a co-point and the number of co-points that are origins of more than two co-rays.

1. Introduction. A G-space can be described as a metric space any two of whose points can be joined by a segment, and in which any segment may be prolonged uniquely to a geodesic. The theory of G-spaces is found in Busemann [1] hereafter quoted as GG.

In a G-space, as in a Riemannian space, a minimum point m of a point p may be defined as a point for which no segment $T(p, m)$ can be prolonged beyond m. We shall be concerned with the analog to m when p lies at infinity in the sense made precise below.

A co-ray B from a point p to a ray A is the limit of a converging sequence of segments $T(p_n, z_n)$ where $p_n \to p$ and z_n tends to infinity on A. Obviously B is also a co-ray to any ray contained in or containing A as a sub-ray. Furthermore, the limit of a converging sequence of co-rays to a ray A is likewise a co-ray to A. Less trivial is the fact (see GG, p. 136) that the co-ray to A from any point of B other than p is unique and a sub-ray of B.

Given a ray A in a straight space and a co-ray B to A, the union of all co-rays to A containing B is an asymptote to the oriented straight line containing A as a positive sub-ray. In an arbitrary non-compact G-space such a union is either an oriented straight line, any positive sub-ray of which is a co-ray to A, or it is a co-ray to A that is not a proper sub-ray of any co-ray to A. This leads to the following terminology in non-compact G-spaces.

DEFINITION 1. Given a ray A in a G-space

(a) An asymptote to A is an oriented straight line any positive sub-ray of which is a co-ray to A.

(b) A maximal co-ray to A is a co-ray to A that is not a proper sub-ray of any co-ray to A.

(c) A co-point to A is the origin of a maximal co-ray to A. We denote by $C(A)$ the set of co-points to A.

A point p_∞ at infinity in a G-space is a maximal set of rays such that a co-ray to one ray in p_∞ is a co-ray to each ray in p_∞. For $A, B \in p_\infty$, $C(A) = C(B)$; hence the locus of co-points depends only on p_∞. Nevertheless it is convenient to retain the notation $C(A)$. The concept of co-point is thus a natural analog to the concept of minimum point in the finite case.

The study of $C(A)$ was initiated by Nasu [4, 5, 6], who by "asymptote" means "maximal co-ray or asymptote" and uses "asymptotic conjugate point" instead of "co-point".

It is our purpose to extend and clarify much of Nasu's work. In §6 we characterize $C(A)$ on certain G-surfaces of finite connectivity (including those with nonpositive curvature) obtaining bounds for the number of components of $C(A)$, the number of co-rays emanating from a point and the number of points that are origins of more than two co-rays to A.

2. Preliminaries. This section consists of results used in later proofs. We begin with a proposition of Nasu [4].

PROPOSITION 2. *Given a ray A in a G-space and a point $p \in C(A)$, there exists for each $\delta > 0$ a positive $\varepsilon \leqq \delta$ such that each co-ray to A with origin exterior to $S(p, \delta)$ fails to intersect $S(p, \varepsilon)$. In particular, there is an $\varepsilon' > 0$ such that no asymptote to A intersects $S(p, \varepsilon')$.*

Proof. Otherwise there is a $\delta > 0$ and a sequence $p_n \to p$ such that each p_n lies on a co-ray B_n to A whose origin $q_n \notin S(p, \delta)$.

There is an index N such that $pp_n < \delta/2$ for $n \geqq N$. Thus for $n \geqq N$, $q_np_n \geqq q_np - pp_n > \delta/2$. Choose $q'_n \in B_n$ such that $(q_nq'_np_n)$ and $q'_np_n = \delta/4$. The co-ray B'_n from q'_n to A is unique and a sub-ray of B_n. A sub-sequence of B'_n converges to a co-ray B to A containing p in its interior—a contradiction.

What follows is a modification of a theorem of Busemann [2, p. 18].

PROPOSITION 3. *For each $x \in S(p, \rho_0)$ let a ray $A(x)$ with origin x be defined which depends continuously on x. If the spheres $K(p, \rho)$, $0 < \rho < \rho(p)$, are not contractible then, for some $x \in S(p, \rho_0)$, $x \neq p$ and $p \in A(x)$.*

Proof. Let $0 < \delta < \min(\rho_0, \rho(p))$. $A(p)$ intersects $K = K(p, \delta)$ in exactly one point w the antipode of which on K we denote by w'.

Let $u \in A(p)$ with $up = \delta/2$. The projection P of $S(u, \delta/4)$ on K

by segments from p is a proper sub-set of K and in particular does not include w'.

For $0 < \varepsilon < \delta$, let $K_\varepsilon = (p, \varepsilon)$. Let $V = \{A(v)\colon v \in K_\varepsilon\}$ and let $z(t, v)$ represent $A(v)$ with $z(0, v) = v$. Choose $\varepsilon > 0$ so small that for $v \in V$, $z(\delta/2, v) \in S(u, \delta/4)$ and $T(z(\delta/2, v), v)$ lies in $S(p, \delta)$.

If $z(t\delta/2, v) \neq p$ for $0 \leqq t \leqq 1$, define v_t by $(pz(t\delta/2, v)v_t)$ and $v_t p = \delta$.

For $t = 0$ we have (pvv_0) and if $p \in V$ then v_0 traverses K as v traverses K_ε. For $t = 1$ we have $(pz(\delta/2, v)v_1)$ and $z(\delta/2, v) \in S(u, \delta/4)$. Hence $v_1 \in P$.

The point $z(t\delta/2, v)$ depends continuously on both t and v. Thus if $p \notin V$ then v_t defines a deformation of K onto a proper sub-set of itself. This in turn can be deformed to a point thus contradicting the non-contractibility of K.

It follows that $p \in V$ yet $p \notin K_\varepsilon$ which proves the assertion.

Although it is not presently known whether the non-contractibility of small spheres holds in general, it is shown in Busemann [2] to hold in finite dimensional G-spaces.

The set $C(A)$ is not necessarily closed (see Nasu [4]). In the event that $C(A)$ is closed we have the following:

PROPOSITION 4. *Let A be a ray in a G-space R such that $C(A)$ is closed. Let $x_n \to x_0$ where x_n and x_0 lie on maximal co-rays to A. If x'_n and x'_0 are the co-points to A determined by x_n and x_0 respectively, then $x'_n \to x'_0$.*

Proof. We show first that the sequence x'_n is bounded. Otherwise there is a sub-sequence x'_m such that $x_m x'_m \to \infty$. Then there are, for sufficiently large m, co-rays B_m to A containing x_m whose initial points q_m satisfy $x_0x'_0 + 2 > x_m q_m > x_0 x'_0 + 1$.

Since q_m is bounded, a sub-sequence B_i of B_m converges to a co-ray B_0 to A containing x_0 with initial point q_0 satisfying $x_0q_0 \geqq x_0x'_0 + 1$. This, however, is impossible since $x'_0 \in C(A)$. Therefore x'_n is bounded.

If x'_n does not converge to x'_0 then there is a sub-sequence x'_j of x'_n and a $\delta > 0$ such that each $x'_j \notin S(x'_0, \delta)$. Let H_j be a maximal co-ray to A containing x_j. Since x'_n is bounded, a sub-sequence H_k of H_j converges to a co-ray H to A containing x_0. Hence the corresponding sequence x'_k of co-points converges to the initial point of H which, since $C(A)$ is closed, must be x'_0—a contradiction.

We conclude this section with the following separation property.

PROPOSITION 5. *Let A be a ray. The complement of $C(A)$ has no bounded component, and no compact, sub-set of $C(A)$ separates the space.*

Proof. Let $p \notin C(A)$ and let B be the co-ray from p to A. Then $B \cap C(A) = \emptyset$ and the component determined by p contains B.

Suppose a compact sub-set K of $C(A)$ separates the space R. Then all points of $A - \{p\}$ lie in the same component of $R - K$. Let p lie in a different component. Consider a sequence x_n on A with $px_n \rightarrow \infty$ such that a sequence of segments $T(p, x_n)$ converges to a co-ray B from p to A. Each $T(p, x_n)$ intersects K in a point y_n, and, since K is compact, K contains an accumulation point y_0 of y_n. It follows that $y_0 \in K \cap B$ which is impossible.

3. The universal covering surface. While the preceding section concerned arbitrary G-spaces the remainder of this article is concerned with G-surfaces. In this section we generalize results of Nasu [5, 6] proved under the stronger hypothesis of nonpositive curvature.

A tube in a G-surface R is a closed domain bounded by a geodesic polygon P and homeomorphic to a disk punctured at one point. A ray A in R is said to ultimately lie in a tube T if A or some sub-ray of A lies in T.

THEOREM 6. *Let R be a G-space surface and A a ray in R. If the universal covering surface $\bar{R}$ is straight and if A ultimately lies in a tube T then the number of co-rays to A from any point p is finite.*

Proof. Assume without loss of generality that the initial point q of A is on P, the polygon bounding T, and is the only point in which A intersects P. Assume further that p is exterior to T. Let $\lambda =$ length of $P, \gamma = \max\{px\colon x \in P\}$ and $0 < \varepsilon < pP$. Consider the class of oriented geodesic polygons of the form $T(q, p) \cup T(p, x) \cup T(x, z) \cup T(z, q)$ where $T(q, p)$ is fixed, $z \in A - \{q\}$ and $x \in S(p, \varepsilon)$. We show that the class of such polygons determines only a finite number of homotopy classes in R.

Given such a polygon, there is a last point y in which $T(x, z)$ intersects P. Because T is homeomorphic to a punctured disk, there is a sub-arc $P'(q, y)$ of P from q to y such that $p'(q, y) \cup T(y, z) \cup T(z, q)$ is null homotopic. It follows that $T(q, p) \cup T(p, x) \cup T(x, z) \cup T(z, q)$ is homotopic to $T(q, p) \cup T(p, x) \cup T(x, y) \cup P'(y, q)$.

Fix $\bar{p} \in \bar{R}$ over p, and hence fix $T(\bar{q}, \bar{p})$ over $T(q, p)$. Let $T(\bar{p}, \bar{x})$ be the unique segment from $\bar{p}$ over $T(p, x)$, $T(\bar{x}, \bar{y})$ the unique segment from $\bar{x}$ over $T(x, y)$ and $\bar{P}'(\bar{y}, \bar{q}^*)$ the unique geodesic polygon from $\bar{y}$ over $\bar{P}'(y, q)$. The end-point $\bar{q}^*$ of $\bar{P}'(\bar{y}, \bar{q})$ then lies over q and $\overline{pq} \leqq \overline{px} + \overline{xy} +$ length $\bar{P}'(\bar{y}, \bar{q}) = px + xy +$ length $P'(y, q) \leqq px + px + py + \lambda < 2\varepsilon + \gamma + \lambda$.

The point $\bar{q}^*$ so construted are in one-to-one correspondence with the number of homotopy classes determined by the above class of geodesic polygons and are finite in number since they are all interior to $S(\bar{p}, 2\varepsilon + \gamma + \lambda)$.

Let $x_n \to p$ and let $z_n \in A - \{q\}$ be a sequence with $qz_n \to \infty$. Assume without loss of generality that $x_n p < \varepsilon$. Let $\Gamma_1, \cdots, \Gamma_k$ be the homotopy classes determined by the geodesic polygons $T(q, p) \cup T(p, x_n) \cup T(x_n, z_n) \cup T(z_n, q)$ where $T(q, p)$ is fixed, and let $\bar{q}_1, \cdots, \bar{q}_k \in \bar{R}$ over q be constructed as above. The end-points $\bar{z}_n$ of the unique geodesic polygons from $\bar{p}$ over $T(p, x_n) \cup T(x_n, z_n)$ then lie on one of the rays $\bar{A}_1, \cdots, \bar{A}_k$ over A originating from $\bar{q}_1, \cdots, \bar{q}_k$.

If $T(x_n, z_n)$ converges to a co-ray B to A then B is the image of a co-ray $\bar{B}$ from $\bar{p}$ to one of the rays $\bar{A}_1, \cdots, \bar{A}_k$. Since $\bar{R}$ is straight, the co-ray from $\bar{p}$ to any given ray is unique and the theorem follows.

We saw in the preceding proof that given $\bar{p} \in \bar{R}$ over p, the co-rays from p to A are the images of the co-rays from $\bar{p}$ to certain rays $\bar{A}_1, \cdots, \bar{A}_k$ over A. The following tells us that the choice of $\bar{A}_i$ is, to an extent, uniform.

THEOREM 7. (Nasu [5]). *Under the hypothesis of* (6), *if the asymptote relation in $\bar{R}$ is transitive and the co-rays from p to A are images of co-rays from $\bar{p} \in \bar{R}$ to rays $\bar{A}_1, \cdots, \bar{A}_m$ over A then there is a positive $\beta_p < \rho(p)/2$ such that each co-ray to A from $x \in S(p, \beta_p)$ is the image of a co-ray from $\bar{x} \in S(\bar{p}, \beta_p)$ over x to one of the rays $\bar{A}_1, \cdots, \bar{A}_m$.*

Proof. Assume otherwise. There is then a sequence $p_n \to p$ with $pp_n < \min(\rho(p)/2, pP/2)$ such that each p_n is the origin of a co-ray B_n to A which is not the image of a co-ray from $\bar{p}_n \in S(\bar{p}, \rho(p)/2)$ over p_n to any of the rays $\bar{A}_1, \cdots \bar{A}_m$.

Assume without loss of generality that the co-rays B_n converge to a co-ray B from p to A. Let $\gamma_n = \max\{p_n x: x \in P\}$. Each B_n is the image of a co-ray from $\bar{p}_n$ to a ray $\bar{A}'_n$ over A with initial point $\bar{q}'_n$ satisfying $\bar{p}_n\bar{q}'_n \leqq \gamma_n + \lambda$ (since ε in the proof of (6) can be made arbitrarily small). Also $\gamma_n \leqq \gamma + pp_n$ hence $\bar{p}\bar{q}'_n \leqq \bar{p}\bar{p}_n + \bar{p}\bar{q}'_n \leqq \bar{p}\bar{p}_n + \gamma + pp_n + \lambda = \gamma + \lambda + 2pp_n$. It follows that there are only a finite number of distinct points $\bar{q}'_n$ We can therefore assume, by selecting an appropriate sub-sequence, that each B_n is the image of the co-ray from $\bar{p}_n$ to $\bar{A} \neq \bar{A}_1, \cdots, \bar{A}_m$ over A.

B is then the image of the co-ray $\bar{B}$ from $\bar{p}$ to $\bar{A}$. $\bar{B}$ is also a co-ray to one of the rays $\bar{A}_i$, say $\bar{A}_1$. It follows from the transitivity (and implied symmetry) of the asymptote relation that $\bar{A}$ and $\bar{A}_1$ are co-rays to each other. Then $\bar{B}_n$ is a co-ray to $\bar{A}_1$-a contradiction.

We note that an example due to Busemann (*GG*, pp. 265–66) shows the hypothesis that A ultimately lie in a tube to be essential.

COROLLARY 8. *Under the hypothesis of* (7), *if* $p \in C(A)$ *then* p *is the origin of at least two co-rays to* A. *Furthermore,* $C(A)$ *is closed.*

Proof. Assume that the co-ray B from $p \in C(A)$ *to* A *is unique.* It follows from (7) that the co-ray from each $x \in S(p, \beta_p)$ is unique. By (3) there is an $x \in S(p, \beta_p)$ such that $x \neq p$ and p lies on the co-ray from x to A—a contradiction.

On the other hand if $p \notin C(A)$ then the co-ray from p to A is unique and is thus unique for each $x \in S(p, \beta_p)$. Hence $S(p, \beta_p) \cap C(A) = \emptyset$ and the complement of $C(A)$ is open.

4. The local structure of $C(A)$. In this section we describe the local topological structure of $C(A)$. As in the previous section our results generalize results of Nasu [5, 6].

LEMMA 9. *Under the hypothesis of* (7), *if* $p \in C(A)$ *then there is a* $\gamma_p > 0$ *such that no point of* $\bar{S}(p, \gamma_p)$, *with the possible exception of* p, *is the origin of more than two co-rays to* A.

Proof. Choose $\gamma_p > 0$ such that $\gamma_p < \beta_p$, no asymptote to A intersects $\bar{S}(p, \gamma_p)$ and $\bar{S}(p, \gamma_p)$ is homeomorphic to the closed unit disk in E^2. Denote by B_i, $1 \leqq i \leqq m$, the maximal co-rays to A from p and by x_i the intersection of B_i with $K(p, \gamma_p) = \{x \mid px = \gamma_p\}$. Let the indexing be such that x_{i+1} follows x_i where $x_{m+1} = x_1$. The points x_i partition $K(p, \gamma_p)$ into sub-arcs K_i, $1 \leqq i \leqq m$, where K_i has end-points x_i and x_{i+1}. These arcs with the co-rays B_i partition $\bar{S}(p, \gamma_p)$ into closed simply connected regions $D_1, \cdots, D_m$ with non-empty mutually disjoint interiors such that each D_i is bounded by $B_i \cap \bar{S}(p, \gamma_p)$, K_i and $B_{i+1} \cap \bar{S}(p, \gamma_p)$.

Choose $\bar{p} \in \bar{R}$ over p. Since $\gamma_p < \beta_p \leqq \rho(p)/2$, the covering map sends $\bar{S}(\bar{p}, \gamma_p)$ isometrically onto $\bar{S}(p, \gamma_p)$. Let $\bar{B}_i$ with initial point $\bar{p}$ lie over B_i. $\bar{B}_i$ is then a co-ray to a ray $\bar{A}_i$ over A. $\bar{S}(\bar{p}, \gamma_p)$ is partitioned into closed simply connected regions $\bar{D}_i$ over D_i where $\bar{D}_i$ is bounded by $\bar{B}_i \cap \bar{S}(\bar{p}, \gamma_p)$, $\bar{K}_i$ over K_i and $\bar{B}_{i+1} \cap \bar{S}(\bar{p}, \gamma_p)$.

For each $x \in S(p, \gamma_p)$ a co-ray from x to A is the image of a co-ray from $\bar{x} \in \bar{S}(\bar{p}, \gamma_p)$ over x to one of the rays $\bar{A}_i$, $1 \leqq i \leqq m$. Since $\bar{R}$ has a transitive and hence symmetric asymptote relation, we can say that a co-ray from x to A is the image of the co-ray from $\bar{x}$ to one of the rays $\bar{B}_i$, $1 \leqq i \leqq m$.

Consider $x \in D_i$, $x \neq p$. We assert that if γ_p is sufficiently small then any co-ray from x to A is the image of the co-ray from $\bar{x} \in \bar{D}_i$

over x to one of the rays $\bar{B}_i$ or $\bar{B}_{i+1}$. Assume otherwise and fix $\gamma_p < \beta_p$. There is then a sequence of points $x_n \to p$ in the interior of D_i such that each x_n is the origin of a co-ray H_n to A where H_n is the image of $\bar{H}_n$, the co-ray from $\bar{x}_n \in D$ over x_n to some $\bar{B}_j$, $j \neq i, i+1$ (we can assume without loss of generality that each $\bar{H}_n$ is a co-ray to the same $\bar{B}_j$). A sub-sequence of the $\bar{H}_n$ then converges to $\bar{B}_j$ which is impossible since $\bar{B}_i \cup \bar{B}_{i+1}$ separates $\bar{B}_j$ from each $\bar{H}_n$. The assertion thus follows and hence the lemma.

Continuing in this manner, we prove the following result.

THEOREM 10. *Let R be a G-surface and A a ray in R. If $\bar{R}$ is straight and has a transitive asymptote relation, and if A ultimately lies in a tube, then for each $p \in C(A)$ there is a closed region V containing p in its interior that is homeomorphic to a closed disk D in such a way that p corresponds to the center of D and $C(A) \cap V$ to the union of a number of radii of D equal to the number of co-rays from p to A.*

Proof. We begin where the proof of (9) ends. Each $x \in K_i$ determines a unique co-point $\phi(x)$ to A. It follows from (4) that the map $\phi: K_i \to \phi(K_i)$ is continuous.

On K_i choose y_i so close to x_i that no point of the sub-arc $K(x_i, y_i)$ of K_i joining x_i and y_i is a co-point to A and so that $L_i = \phi[K(x_i, y_i)]$ is, with the exception of $p = \phi(x_i)$, interior to D_i. This is possible since ϕ is continuous and $C(A)$ is closed.

Let $\bar{y}_i \in D_i$ lie over y_i and $\bar{K}(\bar{x}_i, \bar{y}_i)$ over $K(x_i, y_i)$ be the subarc of $\bar{K}_i$ joining $\bar{x}_i$ and $\bar{y}_i$. By (7) if y_i if chosen sufficiently close to x_i then the co-ray from each $\bar{x} \in \bar{K}(\bar{x}_i, \bar{y}_i)$ to $\bar{B}_i$ lies over a co-ray to A. Let $\bar{H}_i$ be the co-ray from $\bar{\phi}(\bar{y}_i)$ over $\phi(y_i)$ to $\bar{B}_i$. Then $\bar{H}_i$ lies over a co-ray to A from $\phi(y_i)$.

$\phi(y_i)$ is the origin of exactly two maximal co-rays to A. Let U_i denote the remaining maximal co-ray to A. Since $\bar{H}_i$ is the co-ray from $\bar{\phi}(y_i)$ to $\bar{B}_i$, the ray $\bar{U}_i$ over U_i from $\bar{\phi}(\bar{y}_i)$ is a co-ray to $\bar{B}_{i+1}$.

Denote by z_i the intersection of U_i with K_i. The choice of y_i guarantees that $z_i \notin K(x_i, y_i)$. Let $K(z_i, x_{i+1})$ be the sub-arc of K_i joining z_i and x_{i+1}. It follows that $K(x_i, y_i)$ and $K(z_i, x_{i+1})$ have no points in common. Let x be an interior point of $K(x_i, y_i)$. $\phi(x)$ is the origin of exactly two maximal co-rays to A. If $\bar{\phi}(x) \in \bar{D}_i$ lies over $\phi(x)$ then the co-ray $\bar{H}_x$ from $\bar{\phi}(x)$ to $\bar{B}_i$ and the co-ray $\bar{U}_x$ from $\bar{\phi}(x)$ to $\bar{B}_{i+1}$ lie over the maximal co-rays to A from $\phi(x)$. y_i was chosen so that $\bar{U}_x$ cannot intersect $\bar{K}(\bar{x}_i, \bar{y}_i)$. Neither can $\bar{U}_x$ intersect $\bar{B}_i, \bar{B}_{i+1}, \bar{H}_i$ or $\bar{U}_i$. $\bar{U}_x$ must then intersect $\bar{K}(\bar{z}, \bar{x}_{i+1})$ over $K(z_i, x_{i+1})$ and U_x intersects $K(z_i, x_{i+1})$. It follows that ϕ restricted to $K(x_i, y_i)$ is one-to-one and $L_i = \phi[K(x_i, y_i)]$ is an arc joining p and $\phi(y_i)$.

We know that each $x \in L_i$ is the origin of exactly two maximal co-rays to A. One of these, H_x, intersects $K(x_i, y_i)$ and the other, U_x, intersects $K(z_i, x_{i+1})$. With $x \in L_i$ associate $\sigma(x) = U_x \cap K(z_i, x_{i+1})$. The continuity of the map $\sigma: L_i \to K(z_i, x_{i+1})$ can be shown by a standard argument. $\sigma(L_i)$ is then a connected sub-set of $K(z_i, x_{i+1})$ that contains both z_i and x_{i+1}. Thus $\sigma(L_i) = K(z_i, x_{i+1})$ and $L_i = \phi[K(z_i, x_{i+1})]$.

Consider the closed region V_i bounded by $B_i \cap \bar{S}(p, \gamma_p)$, $K(x_i, y_i)$, $H_i \cap \bar{S}(p, \gamma_p)$, $U_i \cap \bar{S}(p, \gamma_p)$, $K(z_i, x_{i+1})$ and $B_{i+1} \cap \bar{S}(p, \gamma_p)$. $V_i \cap C(A) = L_i$ and $V = V_1 \cup \cdots \cup V_m$ is then the desired closed region.

We note that since $\gamma_p > 0$ can be arbitrarily small we can find such a V contained in any neighborhood of p. This implies that $C(A)$ is locally arc-wise connected and that the arc-wise connected components of $C(A)$ are closed in $C(A)$ and hence are closed in R.

We conclude this section with some remarks on the applicability of the preceding results.

A G-surface R of finite connectivity can be regarded as a subspace of a compact manifold M of finite genus γ. As such, it is obtained from M by excluding a finite number of points a_i. There are simple closed pairwise disjoint geodesic polygons P_i in R each of which bounds a closed region M_i in M homeomorphic to a disk and containing a_i, but no other a_j, in its interior. The set $T_i = M_i - \{a_i\}$ is then a tube in R.

Each ray in R must ultimately lie in one of the tubes T_i and the preceding results apply to the extent that the universal covering surface has the appropriate properties. For example, if R has convex capsules then the universal covering surface $\bar{R}$ is straight. If in addition $\bar{R}$ has the divergence property then the asymptote relation is transitive. In particular this is the case when R has nonpositive curvature (see *GG*, pp. 249–50).

5. The covering map and the co-ray relation. In view of (7) it is natural to ask when a co-ray to $\bar{A}$ in $\bar{R}$ over A lies over a co-ray to A in R. In this section we present some partial answers to this question primarily for use in establishing our principal results in the following section.

Let R be a G-surface whose universal covering surface $\bar{R}$ is straight and let A with origin q be a ray in R which ultimately lies in a tube T. Fix p in R and $\bar{A}$ in $\bar{R}$ over A with origin $\bar{q}$. It can be seen from the proof of (6), by applying covering motions of $\bar{R}$ to the rays $\bar{A}_i$ if necessary, that if B is a co-ray from p to A then there is a sequence $x_n \to p$, a sequence z_n on A with $qz_n \to \infty$ and a sequence of segments $T(x_n, z_n) \to B$ such that if $\bar{z}_n$ on $\bar{A}$ lies over z_n then the segments $T(\bar{x}_n, \bar{z}_n)$ over $T(x_n, z_n)$ converge to a co-ray $\bar{B}$ to $\bar{A}$ over B.

If $\bar{p}$ is the origin of $\bar{B}$ then $\bar{p}$ lies over p and since $\bar{x}_n\bar{z}_n = x_n z_n$ we have $\alpha(\bar{A}, \bar{p}) = \alpha(A, p)$ (see *GG*, p. 31).

LEMMA 11. *Let A be a ray in a G-surface R. If the universal covering surface $\bar{R}$ is straight and if A ultimately lies in a tube then for any $p \in R$ and any ray $\bar{A}$ in $\bar{R}$ over A there is no point $\bar{p}_1 \in \bar{R}$ over p with $\alpha(\bar{A}, \bar{p}_1) < \alpha(A, p)$.*

Proof. Assume otherwise. Let $\bar{p}, x_n, \bar{x}_n$ and $\bar{z}_n$ be as above and let $\bar{t}_n \to \bar{p}_1$ where $\bar{t}_n$ lies over x_n. For sufficiently large n, $\bar{t}_n\bar{z}_n < \bar{x}_n\bar{z}_n$ which contradicts the assumption that $T(\bar{x}_n, \bar{z}_n)$ lies over a segment.

We present now a sufficient condition for a co-ray to $\bar{A}$ in $\bar{R}$ to lie over a co-ray to A in R.

THEOREM 12. *Let R be a G-surface whose universal covering surface is straight and let the ray A ultimately lie in a tube. For any $p \in R$ and any ray $\bar{A}$ in $\bar{R}$ over A,* if *$\bar{p} \in \bar{R}$ lies over p and $\alpha(\bar{A}, \bar{p}) = \alpha(A, p)$ then the co-ray $\bar{B}$ from $\bar{p}$ to $\bar{A}$ lies over a co-ray B to A.*

Proof. We show first that $\bar{B}$ lies over a ray B. If $\bar{x} \in \bar{B}$ then $\bar{p}$ is a foot of $\bar{x}$ on the limit sphere $K_\infty(\bar{A}, \bar{p})$ (see *GG*, p. 135). Since no points over p are interior to $K_\infty(\bar{A}, \bar{p})$, $T(\bar{p}, \bar{x})$ lies over a segment. Since $\bar{x} \in \bar{B}$ is arbitary, $\bar{B}$ lies over a ray.

Let $x \neq p$ be any point of B and let $\bar{x}$ on $\bar{B}$ lie over x. Choose z_n on A with $qz_n \to \infty$ and let $\bar{z}_n$ on $\bar{A}$ lie over z_n. Since $\bar{R}$ is straight, $T(\bar{p}, \bar{z}_n)$ converges to $\bar{B}$. For sufficiently large n we can choose $\bar{x}_n$ in $T(\bar{p}, \bar{z}_n)$ such that $\bar{p}\bar{x}_n = \bar{p}\bar{x} = px$. Then $\bar{x}_n \to \bar{x}$ and, letting $\bar{x}_n$ lie over x_n, $x_n \to x$.

We then have the following:

(a) Limit $(\bar{p}\bar{z}_n - pz_n) = 0$ since $\bar{z}_n\bar{q} = z_n q$ and limit $(\bar{p}\bar{z}_n - \bar{z}_n\bar{q}) = \alpha(\bar{A}, \bar{p}) = \alpha(A, p) = \text{limit}\,(pz_n - z_n q)$.

(b) Limit $(\bar{p}\bar{x}_n - px_n) = 0$ since $\bar{p}\bar{x}_n \to \bar{p}\bar{x} = px$ and $px_n \to px$.

(c) $\bar{x}_n\bar{z}_n = \bar{p}\bar{z}_n - \bar{p}\bar{x}_n$ and $x_n z_n \geqq pz_n - px_n$.

From (c) we have $0 \leqq \bar{x}_n\bar{z}_n - x_n z_n = (\bar{p}\bar{z}_n - \bar{p}\bar{x}_n - x_n z_n) \leqq \bar{p}\bar{z}_n - \bar{p}\bar{x}_n - pz_n + px_n = (\bar{p}\bar{z}_n - pz_n) - (\bar{p}\bar{x}_n - px_n)$.

This inequality in conjunction with (a) and (b) yields limit $(\bar{x}_n\bar{z}_n - x_n z_n) = 0$. We then have $\alpha(A, x) = \text{limit}\,(x_n z_n - z_n q) = \text{limit}\,[(\bar{x}_n\bar{z}_n - z_n q) + (x_n z_n - \bar{x}_n\bar{z}_n)] = \text{limit}\,(\bar{x}_n\bar{z}_n - z_n q) = \alpha(\bar{A}, \bar{x}) = \alpha(\bar{A}, \bar{p}) - \bar{p}\bar{x} = \alpha(A, p) - px$. The assertion then follows from a result of Busemann (see *GG*, p. 136).

In his thesis (University of Southern California, 1970) the author believed he had carried the above line of reasoning further and obtained, under the hypothesis of (12), a negative answer to a still unsolved problem of Busemann: can a maximal co-ray be a proper sub-ray of another ray? Unfortunately this assertion with its implication of transitive co-rays in a certain class of G-surfaces was reported in Busemann [2, p. 89 (13) and p. 90 (15)] before an error in the proof was discovered by the author.

Let P be a geodesic polygon that bounds T the tube containing A. We may assume without loss of generality that P contain q but no other points of A. Given $\bar{q}$ in $\bar{R}$ over q, there is exactly one ray $\bar{A}$ over A with origin $\bar{q}$ and exactly one geodesic polygon $\bar{P}$ over P with initial point $\bar{q}$. The end-point $\bar{q}'$ of $\bar{P}$ also lies over q and is the origin of exactly one ray $\bar{A}'$ over A. $\bar{A}$, $\bar{P}$ and $\bar{A}'$ bound a simply connected region $\bar{T}$ over T on the interior of which the covering map is one-to-one.

PROPOSITION 13. *Let λ denote the length of P and hence of $\bar{P}$. If $\bar{p}$ lies in the interior of $\bar{T}$ with $\bar{p}\bar{A} < \bar{p}\bar{P} - \lambda$ then the co-ray $\bar{B}$ from $\bar{p}$ to $\bar{A}$, the co-ray $\bar{B}'$ from $\bar{p}$ to $\bar{A}'$ or both lie over a co-ray to A.*

Proof. Let z_n be a sequence on A with $qz_n \to \infty$ and let $\bar{z}_n$ on $\bar{A}$ lie over z_n. Assume that $\bar{z}''_n$ exterior to $\bar{T}$ also lies over z_n and that $T(\bar{p}, \bar{z}''_n)$ lies over a segment. $T(\bar{p}, \bar{z}''_n)$ can intersect neither $\bar{A}$ nor $\bar{A}'$ and so must intersect $\bar{P}$. Since $T(\bar{z}_n, \bar{q})$ lies over the unique segment $T(z_n, q)$, $\bar{p}\bar{z}''_n \geqq \bar{z}''_n\bar{P} + p\bar{P} \geqq \bar{z}''_n\bar{q} - \lambda + \bar{p}\bar{P} > \bar{z}_n\bar{q} - \lambda + \bar{p}\bar{P}$ for all n. On the other hand for sufficiently large n, $\bar{p}\bar{z}_n \leqq \bar{p}\bar{A} + \bar{z}_n\bar{q} < \bar{p}\bar{P} - \lambda + \bar{z}_n\bar{q}$. Thus for sufficiently large n, $T(\bar{p}, \bar{z}''_n)$ does not lie over a segment.

Let $\bar{z}'_n$ on $\bar{A}'$ lie over z_n. For sufficiently large n either $T(\bar{p}, \bar{z}_n)$, $T(\bar{p}, \bar{z}'_n)$ or both lie over a segment. It follows that $\alpha(A, p) = \alpha(\bar{A}, \bar{p})$, $\alpha(A, p) = \alpha(\bar{A}', \bar{p})$ or both. The proposition then follows from (12).

Observe that the conflicting inequalities arise because $T(\bar{p}, \bar{z}''_n)$ intersects $\bar{P}$. This means that if $T(\bar{p}, \bar{z}'_n)$ lies over a segment then, for sufficiently large n, it does not intersect $\bar{P}$. Thus if $\bar{B}'$ lies over a co-ray to A, then $\bar{B}'$ does not intersect $\bar{P}$. Likewise if $\bar{B}$ lies over a co-ray to A then $\bar{B}$ lies in $\bar{T}$.

DEFINITION 14. The distance from co-ray to ray is weakly bounded if for a co-ray B from p to A there is a sequence x_n on B with $x_n p \to \infty$ such that $x_n A$ is bounded.

In particular the distance from co-ray to ray is weakly bounded in a straight space with convex capsules where, in fact, both xA and yB are bounded for $x \in B$ and $y \in A$. An example of Busemann (*GG*, p. 137) shows that the latter do not necessarily follow from the distance from co-ray to ray being weakly bounded.

PROPOSITION 15. *Let R be a G-surface whose universal covering surface $\bar{R}$ is straight and has the distance from co-ray to ray weakly bounded. Let A, $\bar{A}$, q, $\bar{q}$, T and $\bar{T}$ be as in* (13). *Let $\bar{B}$ be a co-ray to $\bar{A}$ in $\bar{R}$ such that a sub-ray of $\bar{B}$ lies in $\bar{T}$. If $\bar{B}$ lies over a co-ray* *to A then there is a point $\bar{x}_0$ on $\bar{B}$ and a point $\bar{z}_0$ on $\bar{A}$ such that the sub-ray of $\bar{B}$ from $\bar{x}_0$, the sub-ray of $\bar{A}$ from $\bar{z}_0$ and $T(\bar{x}_0, \bar{z}_0)$ bound a sub-region of $\bar{T}$ the co-ray from each point of which to $\bar{A}$ lies over a co-ray to A.*

Proof. We may assume without loss of generality that the origin $\bar{p}$ of $\bar{B}$ is exterior to $\bar{T}$. There is a sequence $\bar{x}_n$ in $\bar{B}$ with $\bar{x}_n\bar{p} \to \infty$ and a constant $M > 0$ such that $\bar{x}_n\bar{A} < M$ for all n. Let $\bar{z}_n$ be a foot of $\bar{x}_n$ on $\bar{A}$. Then $\bar{z}_n\bar{q} \to \infty$ and $\bar{q}\bar{T}(\bar{x}_n, \bar{z}_n) \to \infty$.

Choose N such that for $n \geqq N$ if $\bar{y} \in T(\bar{x}_n, \bar{z}_n)$ then $M < \bar{y}\bar{p} - \lambda$. For each $n \geqq N$, if $\bar{y} \in T(\bar{x}_n, \bar{z}_n)$ then the co-ray $\bar{H}$ from $\bar{y}$ to $\bar{A}$ lies over a co-ray to A. Otherwise the co-ray $\bar{H}'$ from $\bar{y}$ to $\bar{A}'$ lies over a co-ray to A in which case $\bar{H}'$ would either co-incide with $\bar{H}$ or intersect $\bar{B}$, which is impossible.

Consider the sub-region of $\bar{T}$ bounded by $T(\bar{x}_N, \bar{z}_N)$, the sub-ray of $\bar{B}$ from $\bar{x}_N$ and the sub-ray of $\bar{A}$ from $\bar{z}_N$. Let $\bar{y}$ be any point in the interior of this sub-region. For sufficiently large $n > N$, $\bar{y}$ is in the interior of the region bounded by $T(\bar{x}_N, \bar{z}_N)$, $T(\bar{x}_N, \bar{x}_n)$, $T(\bar{z}_N, \bar{z}_n)$ and $T(\bar{x}_n, \bar{z}_n)$. Any ray $\bar{H}$ from $\bar{y}$ that lies over a co-ray A must intersect one of these segments and so must be a co-ray to $\bar{A}$.

Slight modification of the preceding proof yields the following:

PROPOSITION 16. *Under the assumptions of* (15), *if $\bar{R}$ has a transitive asymptote relation and $\bar{A}$ and $\bar{A}'$ are co-rays to each other then there are points $\bar{z}$ and $\bar{z}'$ on $\bar{A}$ and $\bar{A}'$ respectively such that the sub-ray of $\bar{A}$ from $\bar{z}$, the sub-ray of $\bar{A}'$ from $\bar{z}'$ and $T(\bar{z}, \bar{z}')$ bound a sub-region of $\bar{T}$ the co-ray from any point of which to $\bar{A}$ lies over a co-ray to A.*

We also obtain a result of Nasu [5].

COROLLARY 17. *Under the assumptions of* (16), *if $\bar{A}$ and $\bar{A}'$ are co-rays to each other then there is a sub-tube of T disjoint from $C(A)$. If $\bar{A}$ and $\bar{A}'$ are not co-rays to each other then no sub-tube of T is disjoint from $C(A)$.*

Proof. The first assertion is a direct consequence of (8) and (16). To prove the second assertion let $\bar{x}(t)$, $0 \leqq t \leqq 1$, be any curve in $\bar{T}$ with $\bar{x}(0)$ in $\bar{A}$ and $\bar{x}(1)$ in $\bar{A}'$. There is then a largest value of t, t_0, such that for $0 \leqq t \leqq t_0$ the co-ray from $\bar{x}(t)$ to $\bar{A}$ lies over a co-ray to A. Since $\bar{A}$ and $\bar{A}''$ are not co-rays to each other $0 < t_0 < 1$. Then $\bar{x}(t_0)$ is the origin of two rays lying over co-rays to A.

6. The structure of $C(A)$ in a class G-surfaces. In this section we analyze $C(A)$ in case R is a G-surface of finite connectivity with a straight universal covering surface with a transitive asymptote relation and the distance from co-ray to ray weakly bounded. We have mentioned previously that this includes all G-surfaces of finite connectivity whose universal covering surface has transitive asymptotes, and hence includes all G-surfaces of finite connectivity with non-positive curvature.

The following consequence of (16) is basic to our analysis.

PROPOSITION 18. *Let R be a G-surface of the above type. If A is a ray in R then $C(A)$ does not separate R.*

Proof. Since R has finite connectivity, A lies ultimately in a tube T. Assume that the proposition is false. $C(A)$ then separates R into at least two components. We consider two cases.

(i) The tube T or a sub-tube is contained in one of the components. Consider a point x in a different component. A co-ray B from x to A has a sub-ray contained in T. Thus B intersects $C(A)$ which is impossible.

(ii) None of the components of $R - C(A)$ contains a sub-tube of T. Assume without loss of generality that the initial point q of A is not in $C(A)$. Then A lies in one of the components. Choose $x \notin T$ in a component that does not contain A. Then the co-ray B from x to A has a sub-ray contained in T.

Let P be the simple closed geodesic polygon bounding T. We may assume that q is the initial point of P. Choose $\bar{q} \in \bar{R}$ over q and let $\bar{A}$ and $\bar{P}$, each with initial point $\bar{q}$, lie over A and P. The end-point $\bar{q}'$ of $\bar{P}$ lies over q. Let $\bar{A}'$ with initial point $\bar{q}'$ lie over A. Then $\bar{A}$, $\bar{P}$ and $\bar{A}'$ bound a simply connected region $\bar{T}$ over T.

Let $\bar{B}$ with a sub-ray in $\bar{T}$ lie over B. $\bar{B}$ is a co-ray to $\bar{A}'$ or $\bar{A}$, say $\bar{A}$ to be definite. It follows from (16) that there is a segment $T(\bar{x}_0, \bar{y}_0)$ in $\bar{T}$ joining $\bar{B}$ to $\bar{A}$ no point of which lies over a co-ray to A. Thus $C(A)$ does not separate B and A which is a contradiction.

It was mentioned at the end of §4 that a G-surface of finite connectivity can be regarded as a sub-space of a compact manifold M of finite genus γ. As such it is obtained from M by the removal of a finite number of points a_i, $1 \leqq i \leqq N$, each of which corresponds to a tube T_i bounded by P_i, a simple geodesic polygon in R.

DEFINITION 19. Given a ray A in R, a G-surface of finite connectivity, denote by $C^*(A)$ the closure of $C(A)$ relative to M. $C(A)$ is said to occupy a tube T_j if the point a_j in M that determines T_j is in $C^*(A)$. Similarly a component $C_i^*(A)$ of $C^*(A)$ occupies T_j if a_j is in $C_i^*(A)$.

We note that $C^*(A)$ is obtained from $C(A)$ by adjoining those a_j that corresponds to tubes occupied by $C(A)$.

A sufficiently small deleted neighborhood of a_j in $C^*(A)$ can be thought of as a sub-tube of T_j. The next theorem extends (10) to include those a_j in $C^*(A)$. Ultimately this allows us to assert that $C^*(A)$ is triangulable as a one dimensional simplicial complex, a fact from which we derive the principal results of this section.

THEOREM 20. *Let R be a G-surface of finite connectivity with a straight universal covering surface $\bar{R}$ having a transitive asymptote relation and the distance from co-ray to ray weakly bounded. Let A be a ray in R and let T be a tube occupied by $C(A)$. The geodesic polygon P bounding T can be chosen so that $C(A) \cap T$ consists of a finite number of disjoint unbounded arcs emanating from P.*

Proof. We consider the case that T is not the tube that contains A or a sub-ray thereof. Let y_n be an unbounded sequence in T. Let B_n be a co-ray from y_n to A. Each B_n intersects P in a first point x_n. Since P is compact, the sequence x_n is bounded and a sub-sequence of B_n converges to an asymptote L^+ to A. Let q be the first point in which L^+ intersects P and let H be the negative sub-ray of L^+ with origin q. We note that H is contained in T and has only q in common with P.

In $\bar{R}$, the universe covering surface, choose a simply connected region $\bar{T}$ over T, bounded by $\bar{H}$ and $\bar{H}'$ over H with initial points $\bar{q}$ and $\bar{q}'$ respectively, and by $\bar{P}$ over P with $\bar{q}$ and $\bar{q}'$ as initial and final points. We note that the covering map is one-to-one on the interior of $\bar{T}$.

Assume without loss of generality that $\bar{P}$ is a segment. Then any ray that lies over a co-ray to A can intersect $\bar{P}$ at most once. Furthermore any ray lying over a co-ray to A that intersects $\bar{P}$ must originate from $\bar{T}$. This, with (10), implies that $\bar{P}$ contains at most a finite number of points that lie over co-points to A.

We show that $\bar{P}$ can be replaced by a geodesic polygon $\bar{P}'$ bounding a sub-region $\bar{T}'$ of $\bar{T}$ that lies over a sub-tube T' of T and is such that those points of $\bar{T}'$ that lie over $C(A) \cap T'$ form a finite number of unbounded disjoint arcs emanating from $\bar{P}'$.

Since $\bar{P}$ is compact it can be covered by a finite number of neighborhoods of the type in (10). It follows that there are a finite number of rays $\bar{A}_i$, $1 \leqq i \leqq m$, over A, none a co-ray to any other, to one of which any ray from a point of $\bar{P}$ that lies over a co-ray to A must be a co-ray. Furthermore, if $\bar{x} \in \bar{P}$ does not lie over a co-point to A, and if the co-ray from $\bar{x}$ to $\bar{A}_j$ lies over a co-ray to A then by

(10) there is a sub-segment of $\bar{P}$ the co-ray from any point of which to $\bar{A}_j$ lies over a co-ray to A.

Hence $\bar{P}$ can be partitioned into a finite number of non-overlapping segments $I_1, \cdots, I_k$ whose end-points are either end-points of $\bar{P}$ or lie over co-points to A. For each I_i there is an $\bar{A}_{j(i)}$ the co-ray to which from any point of I_i lies over a co-ray to A.

In I_i suppose two points $\bar{x}$ and $\bar{y}$ the asymptotes through which to $\bar{A}_{j(i)}$ lie over asymptotes to A and are denoted by $\bar{B}(\bar{x})$ and $\bar{B}(\bar{y})$ respectively. A ray lying over a co-ray to A in the strip bounded by $\bar{B}(\bar{x})$ and $\bar{B}(\bar{y})$ cannot intersect either $\bar{B}(\bar{x})$ or $\bar{B}(\bar{y})$. Such a ray must be a co-ray to $\bar{A}_{j(i)}$. Thus the asymptotes to $\bar{A}_{j(i)}$ through points of I_i between $\bar{x}$ and $\bar{y}$ lie over asymptotes to A. This implies that those points of I_i the asymptotes through which to $A_{j(i)}$ lie over asymptotes to A form a sub-segment of I_i. We note that this sub-segment might consist of a single point or be empty.

It follows that $\bar{P}$ contains a finite number of disjoint segments $K_0, \cdots, K_r$ whose points are the points of $\bar{P}$ that lie on straight lines that lie over asymptotes to A. Let $\bar{x}_i$ and $\bar{y}_{i+1}$, $0 \leqq i \leqq r$, denote the end-points of K_i indexed so that $\bar{q} = \bar{x}_0$, $\bar{q}' = \bar{y}_{r+1}$ and K_{i+1} follows K_i on $\bar{P}$. Let J_i, $1 \leqq i \leqq r$, denote the sub-segment of $\bar{P}$ joining $\bar{y}_i$ and $\bar{x}_i$. Then $\bar{P} = K_0 \cup J_1 \cup K_1 \cup \cdots \cup J_r \cup K_r$.

We will alter $\bar{P}$ by altering the segments J_i. Each point $\bar{z}$ in J_i determines a unique point $\phi(\bar{z})$ in $\bar{T}$ that lies over a co-point to A. By (4) ϕ is continuous. Consider in J_i a sequence $\bar{z}_n \rightarrow \bar{x}_i$. For sufficiently large n the co-ray $\bar{B}_n$ from $\phi(\bar{z}_n)$ to $\bar{A}_{j(i)}$ lies over a co-ray to A, and the sequence $\bar{B}_n$ converges to the asymptote to $\bar{A}_{j(i)}$ through $\bar{x}_i$. Furthermore, since $C(A)$ is closed in R, $\phi(\bar{z}_n)\bar{x}_i \rightarrow \infty$.

$\phi(\bar{z}_n)$ is the origin of at least one other ray $\bar{B}'_n$ that lies over a co-ray to A. Since $\phi(\bar{z}_n)\bar{x}_i \rightarrow \infty$, a sub-sequence of $\bar{B}'_n$ converges to an oriented straight line lying over an asymptote to A. The only possibility for the latter is the asymptote to $\bar{A}_{j(i-1)}$ through $\bar{y}_i$.

Thus if $\bar{z}_i$ in J_i is chosen sufficiently close to x_i then $\phi(\bar{z}_i)$ is the origin of exactly two rays that lie over co-rays to A: the co-ray $\bar{B}_i$ to $\bar{A}_{j(i)}$ and the co-ray $\bar{B}'_j$ to $\bar{A}_{j(i-1)}$. Let $\bar{z}'_i$ denote the intersection of $\bar{B}'_i$ with J_i. If $\bar{z}_i$ is sufficiently close to $\bar{x}_i$ then the image of $T(\bar{z}_i, \bar{x}_i) - \{\bar{x}_i\}$ under ϕ coincides with that of $T(\bar{y}_i, \bar{z}'_i) - \{\bar{y}_i\}$ under ϕ, and their common image is an unbounded arc emanating from $\phi(\bar{z}_i)$.

We replace J_i with $J'_i = T(\bar{y}_i, \bar{z}'_i) \cup T(\bar{z}'_i, \phi(\bar{z}_i)) \cup T(\phi(\bar{z}_i), \bar{z}_i) \cup T(\bar{z}_i, \bar{x}_i)$. When this is done for each i, $1 \leqq i \leqq r$, we have the desired geodesic polygon $\bar{P}'$.

The case in which T contains A or a sub-ray thereof is treated in a similar manner (although it involves a few more details).

Let $T_i = M_i - \{a_i\}$ where M_i is homeomorphic to a closed disk and contains a_i in its interior. If T_i is occupied by $C(A)$ it is clear from (20) that P_i may be chosen so that M_i is homeomorphic to a closed disk in such a way that a_i corresponds to the center and $C^*(A) \cap M$ to a finite number of radii. On the other hand, if T_i is not occupied by $C(A)$ then, by (17), P_i may be chosen so that T_i is disjoint from $C(A)$. If Int T_i denotes the interior of T_i then $R - \cup \operatorname{Int} T_i$ is compact and $C(A) \cap (R - \cup \operatorname{Int} T_i)$ can be covered by a finite number of neighborhoods of the type in (10). The following then holds.

COROLLARY 21. *Let A be a ray in R. $C^*(A)$ is triangulable as a one dimensional simplicial complex.*

DEFINITION 22. Given a ray A in R and p a co-point to A denote by $m(p)$ the number of co-rays from p to A minus two. If $m(p) > 0$ then p is called a multiple co-point to A.

It is clear that in any triangulation of $C^*(A)$ the multiple co-points will be vertices. In particular (20) implies that $C(A)$ contains only a finite number of such points. We now state the principal result of this section.

THEOREM 23. *Let R be an orientable (non-orientable) G-surface of finite connectivity with a straight universal covering surface $\bar{R}$ having a transitive asymptote relation and the distance from co-ray to ray weakly bounded. If A is a ray in R, let $\pi(A)$ denote the number of components of $C(A)$, $\mu(A)$ the number of multiple co-points to A, N the number of tubes in R and γ the genus of M, the compact surface of which R is a subspace. It then follows that $\mu(A) \leqq N - 2 + 2\gamma$ ($\mu(A) \leqq N - 2 + \gamma$), $\pi(A) \leqq N - 1 + 2\gamma$ ($\pi(A) \leqq N - 1 + \gamma$) and no co-point to A is the origin of more than $N + 2\gamma$ ($N + \gamma$) co-rays to A.*

Proof. We assume that $C(A) \neq \varnothing$. Let β_0 and β_1 be the first two Betti numbers of $C^*(A)$. The Euler-Poincare characteristic of $C^*(A)$ is $\chi(C^*(A)) = \beta_0 - \beta_1$.

$\beta_0 = \sigma(A)$, the number of components of $C^*(A)$. $C^*(A)$ can be regarded as a subcomplex of M which is likewise triangulable. Since $C(A)$ does not separate R, $C^*(A)$ does not separate M and no one cycle in $C^*(A)$ bounds in M. Thus $\beta_1 \leqq 2\gamma$. In the non-orientable case a bounding one cycle in $C^*(A)$ would correspond to a torsion element in $H_1(M)$ and the inequality is $\beta_1 \leqq \gamma$.

Consider a component $C_i^*(A)$ of $C^*(A)$. Let π_i be the number of components of $C(A)$ included in $C_i^*(A)$, let Δ_i be the number of tubes occupied by $C_i^*(A)$ and let $p(i, j)$, $1 \leqq j \leqq \delta_i$, be the multiple co-points in $C_i^*(A)$. The Euler-Poincaré characteristic of $C^*(A)$ is $\chi(C_i^*(A)) = (\Delta_i + \delta_i) - (\delta_i + \pi_i + \sum_j m(p(i, j)))$, $1 \leqq j \leqq \delta_i$.

Summing over $i = 1, \cdots, \sigma(A)$ we obtain $\chi(C^*(A)) = \sum_i \Delta_i - (\pi(A) + \sum_i \sum_j m(p(i, j))) = \beta_0 - \beta_1 \geqq \sigma(A) - 2\gamma$.

Then $2\gamma + N \geqq 2\gamma + \sum \Delta_i \geqq \sigma(A) + \pi(A) + \sum_i \sum_j m(p(i, j))$. This yields two inequalities: $2\gamma + N \geqq \sigma(A) + \pi(A) + \mu(A)$ and $2\gamma + N \geqq \sigma(A) + \pi(A) + \mu(A) - 1 + \max m(p(i, j))$.

Finally we obtain $2\gamma + N \geqq 1 + \pi(A)$, $2\gamma + N \geqq 2 + \mu(A)$ and $2\gamma + N \geqq 2 + \max m(p(i, j))$ which yield the desired results. In the non-orientable case 2γ is replaced by γ in the preceding inequalities.

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CALIFORNIA STATE POLYTECHNIC COLLEGE, SAN LUIS OBISPO