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ON CLOSE-TO-CONVEX FUNCTIONS OF ORDER β

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For $\beta \geqq 0$, denote by $K(\beta)$ the class of normalized functions f, regular and locally schlicht in the unit disc, which satisfy the condition that for each $r < 1$, the tangent to the curve $C(r) = \{f(re^{i\theta}): 0 \leqq \theta < 2\pi\}$ never turns back on itself as much as $\beta\pi$ radians. $K(\beta)$ is called the class of close-to-convex functions of order β. The purpose of this paper is to investigate the asymptotic behavior of the integral means and Taylor coefficients of $f \in K(\beta)$. It is shown that the function F_β, given by $F_\beta(z) = (1/(2(\beta+1)))\{((1+z)/(1-z))^{\beta+1} - 1\}$, is in some sense extremal for each of these problems. In addition, the class $B(\alpha)$ of Bazilevic functions of type $\alpha > 0$ is related to the class $K(1/\alpha)$. This leads to a simple geometric interpretation of the class $B(\alpha)$ as well as a geometric proof that $B(\alpha)$ contains only schlicht functions.

Let f be regular in $U = \{z: |z| < 1\}$ and be given by

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots . \tag{1.1}$$

Following an argument due to Kaplan [9], we see that $f \in K(\beta)$ iff, for some normalized convex function φ and some constant c with $|c| = 1$, we have for all $z \in U$ that

$$\left| \arg \frac{cf'(z)}{\varphi'(z)} \right| \leqq \beta\pi/2 . \tag{1.2}$$

Equivalently,

$$cf'(z) = p(z)^\beta \varphi'(z) , \tag{1.3}$$

where $p(z) = \sum_{n=0}^{\infty} p_n z^n$, $|p_0| = 1$, has positive real part in U.

It is geometrically clear that for $0 \leqq \beta \leqq 1$, $K(\beta)$ contains only schlicht functions. However, for any $\beta > 1$, Goodman [3] has shown that $K(\beta)$ contains functions of arbitrarily high valence. $K(0)$ is the class of convex functions, and $K(1)$ is the class of close-to-convex functions introduced by Kaplan [9]. For $0 \leqq \alpha \leqq 1$, Pommerenke [13, 14] has studied m-fold symmetric functions of class $K(\alpha)$. The following theorem shows that the study of these functions is closely related to the study of $K(\beta)$ for arbitrary $\beta \geqq 0$.

THEOREM 1.1. *Let $\beta \geqq 0$ and m be a positive integer. Then $f \in K(\beta)$ iff there exists an m-fold symmetric function $g \in K(\beta/m)$ such that $f'(z^m) = g'(z)^m$.*

Proof. Suppose $f \in K(\beta)$, and define g by $g'(z) = f'(z^m)^{1/m}$. From (1.3) it follows that

$$g'(z) = c^{-1/m} p(z^m)^{\beta/m} \psi'(z)$$

where the convex function ψ is defined by $\psi'(z) = \varphi'(z^m)^{1/m}$. Hence $g \in K(\beta/m)$, and g is clearly m-fold symmetric. To prove the converse implication, we merely reverse the above procedure.

Finally, for $k \geqq 2$ denote by V_k the class of normalized functions with boundary rotation at most $k\pi$. From the proof of [2, Theorem 2.2], it follows that $V_k \subset K(k/2 - 1)$. However, $f \in V_k$ implies that f is at most $k/2$ valent [2], so $K(k/2 - 1)$ is in general a much larger class than V_k. The results in § 2 and 3 of this paper are extensions to $K(\beta)$ of results of the author [10] for the class V_k. These results also generalize and improve some of the results of Pommerenke [13] for $K(\alpha)$, $0 \leqq \alpha \leqq 1$.

2. Behavior of the coefficients.

We begin by studying $M(r, f') = \max_{|z|=r} |f'(z)|$.

THEOREM 2.1. *Let $f \in K(\beta)$. Then $((1 - r)/(1 + r))^{\beta+2} M(r, f')$ is a decreasing function of r, and hence $\omega = \lim_{r \to 1} (1 - r)^{\beta+2} M(r, f')$ exists and is finite. If $\omega > 0$ and f is given by (1.3), then there exists θ_0 such that $\varphi'(z) = (1 - ze^{-i\theta_0})^{-2}$ and $\omega = \lim_{r \to 1} (1 - r)^{\beta+2} |f'(re^{i\theta_0})|$.*

Proof. Since for each $\beta \geqq 0$, $K(\beta)$ is a linear-invariant family of order $\beta + 1$ in the sense of Pommerenke [12] (See [4, Theorem 3] for a proof.), the first two statements of the theorem follow. Also, if φ' is not of the stated form, then $\varphi'(z) = O(1)(1 - r)^{-\delta}$ for some $0 < \delta < 2$, and hence from (1.3) we see $\omega = 0$. Finally, if $\omega > 0$, then $\varphi'(z) = (1 - ze^{-i\theta_0})^{-2}$, and just as in the proof of [10, Theorem 3.1] we see that $\omega = \lim_{r \to 1} (1 - r)^{\beta+2} |f'(re^{i\theta_0})|$.

We now begin to study the coefficient behavior. Our method is the major-minor arc technique used by Hayman [5], and the proofs are similar to the proofs of the corresponding results for the class V_k [10]. Hence we omit details wherever possible. We first require two lemmas.

LEMMA 2.1 *Let $f \in K(\beta)$ and $\omega = \lim_{r \to 1} (1 - r)^{\beta+2} |f'(re^{i\theta_0})| > 0$. Then given $\delta > 0$, we may choose $C = C(\delta) > 0$ and $r_0 = r_0(\delta) < 1$ such that for $r_0 \leqq r < 1$ we have*

$$\int_E |f'(re^{i\theta})| \, d\theta < \frac{\delta}{(1 - r)^{\beta+1}}$$

where $E = \{\theta \colon C(\delta)(1 - r) \leqq |\theta - \theta_0| \leqq \pi\}$.

Proof. Without loss of generality we may assume $\theta_0 = 0$, so from Theorem 2.1 and (1.3) we find, with $z = re^{i\theta}$,

$$|f'(z)| = |p(z)|^\beta |1 - z|^{-2} .$$

Hence, with $C > 0$ and E as above, we find

$$\int_E |f'(z)|\, d\theta = \frac{O(1)}{(1-r)^\beta} \int_{C(1-r)}^{\pi} \theta^{-2} d\theta = O(1)\, \frac{1}{C}\, \frac{1}{(1-r)^{\beta+1}} ,$$

and the lemma now follows upon choosing C sufficiently large.

LEMMA 2.2. *Let* $f \in K(\beta)$, $\omega = \lim_{r\to 1} (1 - r)^{\beta+2} |f'(re^{i\theta_0})| > 0$, $r_n = 1 - 1/n$, $\omega_n = (1 - r_n)^{\beta+2} f'(r_n e^{i\theta_0})$, *and*

$$f_n'(z) = \frac{\omega_n}{(1 - ze^{-i\theta_0})^{\beta+2}} .$$

Let S be a fixed but arbitrary Stolz angle with vertex $e^{i\theta_0}$, and let $D_n = \{z \in S \colon |e^{i\theta_0} - z| < 2/n\}$. Then as $n \to \infty$, $f_n' \sim f'$ uniformly for $z \in D_n$.

Proof. Again assuming $\theta_0 = 0$, we have from (1.3) $cf'(z) = p(z)^\beta (1 - z)^{-2}$, and so

$$f_n'(z) = \frac{[(1 - r_n)p(r_n)]^\beta}{c(1 - z)^{\beta+2}} .$$

Thus, to prove the lemma it suffices to show that as $n \to \infty$,

$$\frac{(1 - r_n)p(r_n)}{(1 - z)p(z)} \longrightarrow 1 \tag{2.1}$$

uniformly for $z \in D_n$.

By a theorem of Hayman [6, Theorem 2], $\lim_{r\to 1} (1 - r)p(r) = L$ exists, and it is clear that $(1 - z)p(z)$ is bounded as $|z| \to 1$, providing $z \in S$. By a theorem of Lindelöf [8, p. 260], we have for $z \in S$ that $\lim_{z\to 1} (1 - z)p(z) = L$ where the limit is approached uniformly as $|z| \to 1$. But $0 < \omega = \lim_{r\to 1} (1 - r)^{\beta+2} |f'(r)| = \lim_{r\to 1} [(1 - r)|p(r)|]^\beta$, so $L \neq 0$. Combining these remarks with the inequality

$$\left|\frac{(1 - z)p(z)}{(1 - r^n)p(r_n)} - 1\right| \leqq \frac{1}{|(1 - r_n)p(r_n)|} \{|(1 - z)p(z) - L| + |L - (1 - r_n)p(r_n)|\} ,$$

we see that (2.1) holds, so the proof is complete.

We can now determine the asymptotic behavior of a_n as $n \to \infty$.

THEOREM 2.2. *Let $f \in K(\beta)$ be given by (1.1), and let $\omega = \lim_{r\to 1}(1-r)^{\beta+2}M(r, f')$. Let Γ denote the gamma function. Then*

$$\lim_{n\to\infty} \frac{|a_n|}{n^\beta} = \frac{\omega}{\Gamma(\beta+2)}\,.$$

Also, if $\omega = \lim_{r\to 1}(1-r)^{\beta+2}|f'(re^{i\theta_0})| > 0$, then as $n \to \infty$

$$a_n \sim \frac{f'(r_n e^{i\theta_0})e^{-i(n-1)\theta_0}}{n^2\Gamma(\beta+2)}$$

where $r_n = 1 - 1/n$.

Proof. Suppose first that $\omega > 0$, and define

$$f_n'(z) = \omega_n \sum_{m=0}^{\infty} d^m e^{-im\theta_0} z^m$$

as in Lemma 2.2. We note that

$$d_m = \frac{\Gamma(m+\beta+2)}{\Gamma(m+1)\Gamma(\beta+2)}\,, \tag{2.2}$$

so $d_m \sim m^{\beta+1}/\Gamma(\beta+2)$ as $m \to \infty$. Computation shows that

$$na_n - \omega_n d_{n-1}e^{-i(n-1)\theta_0} = \frac{1}{2\pi r^{n-1}}\int_{-\pi}^{\pi}\{f'(re^{i\theta}) - f_n'(re^{i\theta})\}e^{-i(n-1)\theta}d\theta\,. \tag{2.3}$$

Given $\delta > 0$, we choose $C = C(\delta)$ and E as in Lemma 2.1, and we let $r_n = 1 - 1/n$. With n sufficiently large, Lemma 2.1 gives

$$\int_E |f'(r_n e^{i\theta})|\,d\theta < \delta n^{\beta+1}\,,$$

and clearly this inequality is also true for f_n'. Hence, we see that

$$\left|\int_E \{f'(r_n e^{i\theta}) - f_n'(r_n e^{i\theta})\}e^{-i(n-1)\theta}d\theta\right| < 2\delta n^{\beta+1} \tag{2.4}$$

for n sufficiently large. We now choose a Stolz angle S, depending on δ, such that $\{r_n e^{i\theta}\colon \theta \in E'\} \subset S$ for large n, where $E' = [-\pi, \pi]\backslash E$. By Lemma 2.2, we have as $n \to \infty$ and with $\theta \in E'$,

$$\begin{aligned} f'(r_n e^{i\theta}) - f_n'(r_n e^{i\theta}) &= o(1)\{f_n'(r_n e^{i\theta})\} \\ &= o(1)n^{\beta+2}\,, \end{aligned}$$

where $o(1)$ is uniform for $\theta \in E'$, and hence as $n \to \infty$, we have

$$\left|\int_{E'}\{f'(r_ne^{i\theta}) - f'_n(r_ne^{i\theta})\}e^{-i(n-1)\theta}d\theta\right| \leqq o(1)2C(\delta)(1-r_n)n^{\beta+2} = o(1)n^{\beta+1} . \tag{2.5}$$

Note that although $o(1)$ depends on δ, $o(1)\to 0$ as $n\to\infty$ once δ has been fixed.

Combining (2.3), (2.4), and (2.5), we find

$$|na_n - \omega_n d_{n-1}e^{-i(n-1)\theta_0}| < \{2\delta + o(1)\}n^{\beta+1}$$

for sufficiently large n. Since $\delta > 0$ is arbitrary and since $o(1)\to 0$ once δ has been fixed, we have

$$a_n = \omega_n \frac{d_{n-1}}{n} e^{-i(n-1)\theta_0} + o(1)n^\beta .$$

From (2.2) and the definition of ω_n we see that as $n\to\infty$,

$$a_n \sim \omega_n e^{-i(n-1)\theta_0}n^\beta/\Gamma(\beta+2) \sim \frac{f'(r_ne^{i\theta_0})e^{-i(n-1)\theta_0}}{n^2\Gamma(\beta+2)} .$$

In particular,

$$\lim_{n\to\infty}\frac{|a_n|}{n^\beta} = \frac{\omega}{\Gamma(\beta+2)} .$$

We now suppose $\omega = 0$. We shall subsequently prove (Theorem 3.1 with $\lambda = 1$) that if $\omega = 0$, then

$$\lim_{r\to 1}(1-r)^{\beta+1}\int_0^{2\pi}|f'(re^{i\theta})|d\theta = 0 .$$

Using a standard inequality relating coefficients and integral means [7, p. 11] we have $\lim_{n\to\infty}|a_n|/n^\beta = 0$. This completes the proof of the theorem. Note that if $\omega > 0$, then it follows easily from the theorem that $\lim_{n\to\infty} a_{n+1}/a_n = e^{-i\theta_0}$, and so the radius of maximal growth can be determined from the coefficients.

We now consider the problem of determining

$$\max\{|a_n| : f\in K(\beta)\} .$$

It is natural to conjecture that for each $n \geqq 2$ this problem is solved by the function

$$F_\beta(z) = \frac{1}{2(\beta+1)}\left\{\left(\frac{1+z}{1-z}\right)^{\beta+1} - 1\right\} = z + \sum_{j=2}^{\infty} A_j(\beta)z^j .$$

Toward this end we have the following theorem.

THEOREM 2.3. *Let $f \in K(\beta)$ be given by* (1.1) *and let F_β be as above.*

(i) *There exists an integer n_0 depending on f such that $|a_n| \leqq A_n(\beta)$ for $n \geqq n_0$.*

(ii) *If $n \leqq \beta + 2$, then $|a_n| \leqq A_n(\beta)$.*

(iii) *If β is an integer, then $|a_n| \leqq A_n(\beta)$ for all n.*

Note that since $V_k \subset K(\beta)$ with $\beta = k/2 - 1$, we have from (ii) that $|a_n| \leqq A_n(\beta)$ for $n \leqq k/2 + 1$ and from (iii) that $|a_n| \leqq A_n(\beta)$ for all n whenever k is an even integer.

Proof. We have from (1.3), with $|c| = 1$,

$$cf'(z) = p(z)^\beta \varphi'(z) ,$$

where p has positive real part and φ is convex. Suppose that $p(z) = \sum_{n=0}^\infty p_n z^n$, $|p_0| = 1$, and $p(z)^\beta = \sum_{n=0}^\infty q_n z^n$. Then it is easily verified by induction that for $m \geqq 1$,

$$q_m = \frac{1}{m!} \sum_{j=1}^{m} \beta(\beta - 1) \cdots (\beta - (j-1)) p_0^{\beta - j} D_j(p)$$

where $D_j(p)$ is a polynomial, with nonnegative coefficients, in the variables $p_0, p_1, \cdots, p_m$.

Therefore, if β is an integer, $|q_m|$ is maximal for all $m \geqq 1$ when $p_0 = 1$ and $p_j = 2$ for $j \geqq 1$, which implies $p(z) = (1+z)/(1-z)$. Also, for any $\beta \geqq 0$, we see as above that if $n \leqq \beta + 2$, then $|q_m|$ is maximal for $1 \leqq m \leqq n - 1$ when $p(z) = (1+z)/(1-z)$. In addition, if $\varphi'(z) = 1 + \sum_{j=2}^\infty u_j z^{j-1}$, it is well-known that $|u_j| \leqq j$ for all j, with equality for $\varphi'(z) = (1-z)^{-2}$. But when $p(z) = (1+z)/(1-z)$ and $\varphi'(z) = (1-z)^{-2}$, we have $cf'(z) = F'(z)$. Hence, since

$$cna_n = \sum_{j=0}^{n-1} q_j u_{n-j}$$

where we define $u_1 = 1$, we see that (ii) and (iii) are proved.

We now prove (i). We first note that as $n \to \infty$,

$$A_n(\beta) \sim \frac{2^\beta n^\beta}{\Gamma(\beta + 2)} . \tag{2.6}$$

Let $\omega = \lim_{r \to 1} (1-r)^{\beta+2} M(r, f')$. If $\omega = 0$, then Theorem 2.2 shows $a_n = o(1) n^\beta$, and so it is clear from (2.6) that (i) holds. We now suppose $\omega = \lim_{r\to 1} (1-r)^{\beta+2} |f'(re^{i\theta_0})| > 0$, and we recall that in this case $\omega = \lim_{r \to 1} [(1-r)|p(re^{i\theta_0})|]^\beta$. Hence, from [6, Theorem 2], it follows easily that $\omega \leqq 2^\beta$ with equality only if

$$p(z) = \frac{1 + ze^{-i\theta_0}}{1 - ze^{-i\theta_0}} .$$

But $\omega > 0$ implies also that $\varphi'(z) = (1 - ze^{-i\theta_0})^{-2}$, and thus we have $\omega \leqq 2^\beta$ with equality only if $cf'(z) = F'_\beta(e^{-i\theta_0}z)$, in which case $|a_n| = A_n(\beta)$ for all n, since $|c| = 1$. Thus we may suppose $\omega < 2^\beta$, and using Theorem 2.2 and (2.6) we see that (i) holds. This completes the proof of Theorem 2.3.

To conclude this section we examine the asymptotic behavior of the quantity $||a_{n+1}| - |a_n||$ for $f \in K(\beta)$.

THEOREM 2.4. *Let $f \in K(\beta)$ be given by* (1.1). *If $\omega > 0$, then*

$$\lim_{n\to\infty} \frac{||a_{n+1}| - |a_n||}{n^{\beta-1}} = \frac{\beta\omega}{\Gamma(\beta + 2)} .$$

The theorem is in general false when $\omega = 0$.

Proof. If $\beta = 0$ and $\omega > 0$, then from (1.3) it follows that $cf'(z) = (1 - ze^{-i\theta_0})^{-2}$, so $|a_n| = 1$ for all n, and the theorem is trivially true. Thus, we may assume without loss of generality that $\beta > 0$. The proof will be divided into three parts.

We first claim that given $\delta > 0$, there exists $C(\delta) > 0$ such that

$$\left|\frac{1}{2\pi}\int_E (1 - re^{i(\theta-\theta_0)})f'(re^{i\theta})d\theta\right| < \frac{\delta}{(1 - r)^\beta} \tag{2.7}$$

where θ_0 is as in Theorem 2.1 and $E = \{\theta: C(\delta)(1 - r) \leqq |\theta - \theta_0| \leqq \pi\}$. To prove (2.7), we note that $\omega > 0$ implies that

$$cf'(z) = p(z)^\beta(1 - z)^{-2} ,$$

where we have assumed without loss of generality that $\theta_0 = 0$. Also, for notational ease, we assume $c = 1$ and $p(0) = 1$, so

$$(1 - z)f'(z) = p(z)^\beta/(1 - z) .$$

Choose $\lambda > 1$ such that $\lambda\beta > 1$, and let $1/\lambda + 1/\lambda' = 1$. If C is an arbitrary positive constant, we have from Hölder's inequality that

$$\int_E |(1 - z)f'(z)|\, d\theta \leqq \left\{\int_E |p(z)|^{\lambda\beta} d\theta\right\}^{1/\lambda}\left\{\int_E |1 - z|^{-\lambda'} d\theta\right\}^{1/\lambda'} . \tag{2.8}$$

Since p is subordinate to $(1 + z)/(1 - z)$, and since $\lambda\beta > 1$,

$$\left\{\int_0^{2\pi} |p(z)|^{\lambda\beta} d\theta\right\}^{1/\lambda} = O(1)\frac{1}{(1 - r)^{\beta-1/\lambda}} . \tag{2.9}$$

Also, as in the proof of Lemma 2.1, we have (since $\lambda' > 1$)

$$\int_E |1 - z|^{-\lambda'} d\theta = O(1) \frac{1}{C^{\lambda'-1}} \frac{1}{(1-r)^{\lambda'-1}} . \tag{2.10}$$

Hence, combining (2.8), (2.9), and (2.10), we find

$$\left| \int_E (1 - z) f'(z) d\theta \right| = O(1) \frac{1}{C^{1/\lambda}} \frac{1}{(1-r)^{\beta}} ,$$

which gives (2.7) if we choose C sufficiently large.

From this point on we proceed essentially as in the proof of [11, Theorem 2], and thus we merely sketch the proof. We define ω_n as in Lemma 2.2, $\lambda_n = \arg \omega_n$, and

$$f_n'(z) = \frac{\omega e^{i\lambda_n}}{(1 - ze^{-i\theta_0})^{\beta+2}} = \omega e^{i\lambda_n} \sum_{m=0}^{\infty} d_m e^{-im\theta_0} z^m .$$

Since $\omega_n = [(1 - r_n) p(r_n e^{i\theta_0})]^{\beta}$, $\lim_{n\to\infty} \lambda_n$ exists by [6, Theorem 2]. As in [11, Lemma 3] we find that as $n \to \infty$,

$$a_n - e^{-i\theta_0} a_{n-1} = -\frac{e^{-i\theta_0} a_{n-1}}{n} + \frac{\omega e^{i(\lambda_n - (n-1)\theta_0)}}{\Gamma(\beta + 1)} n^{\beta-1} + o(1) n^{\beta-1} , \tag{2.11}$$

and hence as $n \to \infty$,

$$\frac{a_n - e^{-i\theta_0} a_{n-1}}{n^{\beta-1}} = \frac{\omega e^{i(\lambda_n - (n-1)\theta_0)}}{\Gamma(\beta + 1)} \left[1 - \frac{1}{\beta + 1} (1 + o(1) \right] + o(1) , \tag{2.12}$$

where we have used (2.11) and Theorem 2.2. Theorem 2.2 also implies that as $n \to \infty$,

$$\arg e^{-i\theta_0} a_n = \arg \omega e^{i(\lambda_n - n\theta_0)} + o(1) ,$$

and since $\lim_{n\to\infty} \lambda_n$ exists we have as $n \to \infty$ that

$$\arg e^{-i\theta_0} a_{n-1} = \arg w e^{i(\lambda_n - (n-1)\theta_0)} + o(1) . \tag{2.13}$$

Combining (2.12) with (2.13), we find

$$\frac{||a_n| - |a_{n-1}||}{n^{\beta-1}} = \frac{\beta\omega}{\Gamma(\beta + 2)} + o(1)$$

as $n \to \infty$, which proves the theorem.

We now show that the theorem is false when $\omega = 0$. Let $\beta \geqq 0$ be given, and define $f \in K(\beta)$ by

$$f'(z) = \frac{1}{(1 - z^2)^{\beta+1}} .$$

Clearly f is an odd function, and it is easily verified that $a_{2n+1} \sim n^{\beta-1}/2\Gamma(\beta + 1)$ as $n \to \infty$, so

$$\lim_{n\to\infty}\frac{||a_{2n+1}|-|a_{2n}||}{n^{\beta-1}}=\lim_{n\to\infty}\frac{|a_{2n+1}|}{n^{\beta-1}}=\frac{1}{2\Gamma(\beta+1)}\,.$$

However, $\omega=\lim_{r\to1}(1-r)^{\beta+2}M(r,f')=\lim_{r\to1}(1-r)/(1+r)^{\beta+1}=0$, so the theorem is false when $\omega=0$. This is in sharp contrast to the corresponding result [11] for V_k, where the result is true for all $k>2$ even if $\omega=0$.

3. Behavior of the integral means. In this section we shall investigate the behavior of $I_\lambda(r,f')$ and $I_\lambda(r,f)$, where for $\lambda>0$ we define

$$I_\lambda(r,g)=\frac{1}{2\pi}\int_0^{2\pi}|g(re^{i\theta})|^\lambda d\theta\,.$$

Our results again include as special cases previous results of the author [10] for the class V_k as well as generalizing results of Pommerenke [13] for the classes $K(\alpha)$, $0\leqq\alpha\leqq1$. Although the details of the proofs given here are slightly more involved than those for V_k, we refer to [10] whenever possible. We first need two lemmas, the first of which is proved in exactly the same way as [10, Lemma 4.1].

LEMMA 3.1. *Let $f\in K(\beta)$, $\omega=\lim_{r\to1}(1-r)^{\beta+2}|f'(re^{i\theta_0})|>0$. Let $C>0$ and $\lambda>0$ be fixed, and for $0<R<1$ define $E=\{\theta: C(1-R)\leqq|\theta-\theta_0|\leqq\pi\}$, $E'=[-\pi,\pi]\backslash E$. Define $\omega(R)=(1-R)^{\beta+2}|f'(Re^{i\theta_0})|$ and*

$$f_R'(z)=\frac{\omega(R)}{(1-ze^{-i\theta_0})^{\beta+2}}\,.$$

Then as $R\to1$,

$$\int_{E'}|f_R'(Re^{i\theta})|^\lambda d\theta\sim\int_{E'}|f'(Re^{i\theta})|^\lambda d\theta\,.$$

LEMMA 3.2. *Let $f\in K(\beta)$, $\omega>0$, and f_R' be as above. If $\lambda(\beta+2)>1$, then as $r\to1$,*

$$I_\lambda(r,f')=I_\lambda(r,f_r')+o(1)(1-r)^{1-\lambda(\beta+2)}\,.$$

Proof. By definition, with $z=re^{i\theta}$, we have

$$2\pi|I_\lambda(r,f')-I_\lambda(r,f_r')|\leqq\int_E|f'(z)|^\lambda d\theta+\int_E|f_r'(z)|^\lambda d\theta$$
$$+\int_{E'}\left\{|f'(z)|^\lambda-|f_r'(r)|^\lambda\right\}d\theta\,,$$

where E and E' are as in Lemma 3.1. If $\beta=0$, then $\omega>0$ implies

$f'(z) = (1-z)^{-2}$, and so the lemma is trivial. With $\beta > 0$, let $\gamma = 1 + 2/\beta$ and $\gamma' = 1 + \beta/2$, so $1/\gamma + 1/\gamma' = 1$. Recalling that in (1.3) we have $\varphi'(z) = (1-z)^{-2}$ since $\omega > 0$, we have from Hölder's inequality that

$$\int_E |f'(z)|^\lambda d\theta \leqq \left\{\int_E |p(z)|^{\lambda(\beta+2)} d\theta\right\}^{\beta/(\beta+2)} \left\{\int_E |1-z|^{-\lambda(\beta+2)} d\theta\right\}^{2/(\beta+2)} .$$

As in the proof of (2.9) and (2.10) it follows that

$$\int_E |p(z)|^{\lambda(\beta+2)} d\theta = O(1)(1-r)^{1-\lambda(\beta+2)} .$$

Also, with $\delta > 0$, it follows that

$$\int_E |1-z|^{-\lambda(\beta+2)} d\theta < \frac{\delta}{(1-r)^{\lambda(\beta+2)-1}}$$

for $C(\delta)$ depending on δ and for $\lambda(\beta+2) > 1$, and therefore

$$\int_E |f'(z)|^\lambda d\theta < \frac{\delta}{(1-r)^{\lambda(\beta+2)-1}}$$

for r sufficiently close to 1. Clearly this inequality also holds for f'_r, and so using Lemma 3.1 we have for r sufficiently close to 1 that

$$\begin{aligned} 2\pi\,|I_\lambda(r, f') - I_\lambda(r, f'_r)| &< \frac{2\delta}{(1-r)^{\lambda(\beta+2)-1}} + o(1)\int_{E'} |f'_r(z)|^\lambda d\theta \\ &< \frac{2\delta}{(1-r)^{\lambda(\beta+2)-1}} + \frac{o(1)\omega(r)^\lambda}{(1-r)^{\lambda(\beta+2)}}\int_0^{(1-r)C(\delta)} d\theta \\ &< \frac{2\delta}{(1-r)^{\lambda(\beta+2)-1}} + \frac{o(1)\omega(r)^\lambda C(\delta)}{(1-r)^{\lambda(\beta+2)-1}} . \end{aligned}$$

Since $\delta > 0$ was arbitrary and since $o(1)$ approaches zero once δ has been fixed, the lemma follows.

We can now determine the asymptotic behavior of $I_\lambda(r, f')$ when $\lambda(\beta+2) > 1$. For notational convenience, define

$$G(\lambda, \beta) = \frac{\Gamma(\lambda(\beta+2)-1)}{2^{\lambda(\beta+2)-1}\Gamma^2\{(\lambda(\beta+2))/2\}} .$$

THEOREM 3.1. *Let $f \in K(\beta)$ and $\lambda(\beta+2) > 1$. Then*

$$\lim_{r\to 1}(1-r)^{\lambda(\beta+2)-1} I_\lambda(r, f') = \omega^\lambda G(\lambda, \beta) .$$

Proof. If $\omega > 0$, then the theorem is an immediate consequence of Lemma 3.2 and Pommerenke's result [13] that as $r \to 1$,

$$\frac{1}{2\pi}\int_0^{2\pi} |1 + re^{i\theta}|^{-m}\, d\theta \sim \frac{\Gamma(m-1)}{2^{m-1}\Gamma^2(m/2)}(1-r)^{1-m} \tag{3.1}$$

whenever $m > 1$. Hence, we now assume $\omega = 0$, and we divide the proof into two cases. We first assume that in (1.3) φ' is not of the form $(1 - ze^{-i\theta})^{-2}$. Then, as is well known, $M(r, \varphi') = O(1)(1-r)^{-\gamma}$ for some $0 < \gamma < 2$. Without loss of generality we assume $\gamma\lambda(\beta+2)/2 > 1$. As in the proof of Lemma 3.2, we find

$$\int_0^{2\pi} |f'(z)|^{\lambda}\, d\theta \leqq \left\{\int_0^{2\pi} |p(z)|^{\lambda(\beta+2)}\, d\theta\right\}^{\beta/(\beta+2)} \left\{\int_0^{2\pi} |\varphi'(z)|^{(\lambda(\beta+2))/2}\, d\theta\right\}^{2/(\beta+2)}$$

and

$$\left\{\int_0^{2\pi} |p(z)|^{\lambda(\beta+2)}\, d\theta\right\}^{\beta/(\beta+2)} = O(1)(1-r)^{\beta/(\beta+2)-\lambda\beta}\,.$$

Also, since φ is convex, $z\varphi'$ is starlike and schlicht, so from [7, Theorem 3.2] we have

$$\left\{\int_0^{2\pi} |\varphi'(z)|^{(\lambda(\beta+2))/2}\, d\theta\right\}^{2/(\beta+2)} = O(1)(1-r)^{2/(\beta+2)-\gamma\lambda}\,.$$

Hence

$$\int_0^{2\pi} |f'(z)|^{\lambda}\, d\theta = O(1)(1-r)^{1-\lambda(\beta+\gamma)}\,,$$

and since $\gamma < 2$ we have as $r \to 1$

$$(1-r)^{\lambda(\beta+2)-1} I_\lambda(r, f') \longrightarrow 0\,.$$

It remains only to consider the case $\omega = 0$ and $\varphi'(z) = (1 - ze^{-i\theta_0})^{-2}$ for some θ_0. Assuming without loss of generality that $\theta_0 = 0$, we find from (1.3) and our hypothesis $\omega = 0$ that

$$0 = \lim_{r\to 1}(1-r)p(r)\,.$$

As in Lemma 2.2, it now follows that for z in a Stolz angle with vertex at 1, we have $\lim_{|z|\to 1}(1-z)p(z) = 0$ where the limit is approached uniformly as $|z| \to 1$. Hence, since $(1-r)|p(z)| \leqq |1-z||p(z)|$,

$$|p(z)| \leqq \frac{h(r)}{1-r}$$

for z in the Stolz angle, where $h(r) \to 0$ as $r \to 1$. Thus, given $C > 0$,

$$\int_0^{C(1-r)} |f'(z)|^{\lambda}\, d\theta = \int_0^{C(1-r)} |p(z)|^{\lambda\beta} |1-z|^{-2\lambda}\, d\theta$$

$$\leqq \left\{\int_0^{C(1-r)} |p(z)|^{\lambda(\beta+2)}\, d\theta\right\}^{\beta/(\beta+2)} \left\{\int_0^{C(1-r)} |1-z|^{-\lambda(\beta+2)}\, d\theta\right\}^{2/(\beta+2)} \tag{3.2}$$

$$\leqq \frac{(Ch(r))^{\beta\lambda}}{(1-r)^{\beta\lambda-\beta/(\beta+2)}} \cdot \frac{O(1)}{(1-r)^{2\lambda-2/(\beta+2)}}$$
$$= \frac{o(1)}{(1-r)^{\lambda(\beta+2)-1}}$$

where we have used (3.1). Exactly as in the proof of Lemma 3.2 we also have, given $\delta > 0$,

$$\int_{C(1-r)}^{\pi} |f'(z)|^{\lambda} d\theta < \frac{\delta}{(1-r)^{\lambda(\beta+2)-1}} \tag{3.3}$$

for an appropriate choice of $C = C(\delta)$, and hence from (3.2) and (3.3)

$$\lim_{r\to 1} (1-r)^{\lambda(\beta+2)-1} I_{\lambda}(r, f') = 0 ,$$

which completes the proof of Theorem 3.1.

To complete this section, we examine $I_{\lambda}(r, f)$.

THEOREM 3.2. *Let $f \in K(\beta)$ and let $G(\lambda, \beta)$ be as in Theorem* 3.1.

(i) *If $\lambda \geqq 1$, then*

$$\liminf_{r\to 1} (1-r)^{\lambda(\beta+1)-1} I_{\lambda}(r, f) \geqq \frac{\omega^{\lambda} G(\lambda, \beta)}{2^{\lambda(\beta+2)-1}} .$$

(ii) *If $\lambda \geqq 1$ and $\lambda(\beta + 1) > 1$, then*

$$\limsup_{r\to 1} (1-r)^{\lambda(\beta+1)-1} I_{\lambda}(r, f) \leqq \frac{\omega^{\lambda} G(\lambda, \beta)}{(\beta+1-(1/\lambda))^{\lambda}} .$$

Note that when $\omega = 0$, $\lim_{r\to 1} (1-r)^{\lambda(\beta+1)-1} I_{\lambda}(r, f) = 0$, and when $\omega > 0$ the growth of $I_{\lambda}(r, f)$ is regular in the sense that $\limsup_{r\to 1}$ *and* $\liminf_{r\to 1}$ *are either both positive or both zero.*

Proof. The proof of (i) is very similar to that of [10, Theorem 4.4], and so we omit the details. To prove (ii), we first note that

$$f(re^{i\theta}) = \int_0^r f'(te^{i\theta}) dt .$$

Since $\lambda \geqq 1$, a generalization of Minkowski's inequality [15, p. 260] gives

$$I_{\lambda}(r, f)^{1/\lambda} \leqq \int_0^r I_{\lambda}(t, f')^{1/\lambda} dt .$$

Since Theorem 3.1 gives us the asymptotic behavior of $I_{\lambda}(t, f')$ as $t \to 1$, a straightforward argument shows that whenever $\lambda(\beta + 1) > 1$,

$$\limsup_{r\to 1}\,(1-r)^{\lambda(\beta+1)-1}I_\lambda(r,f) \leqq \frac{\omega^\lambda G(\lambda,\beta)}{(\beta+1-1/\lambda)^\lambda}\,.$$

In conclusion, it should be noted that the basic result underlying the theorems of §§2 and 3 is the existence of $\omega = \lim_{r\to 1}(1-r)^{\alpha+1}M(r, f')$, where $\alpha = \beta + 1$. Since this limit exists whenever f belongs to a linear-invariant family of order α, it is interesting to speculate as to whether the results of the previous sections remain true if we assume only that f belong to such a linear-invariant family. Nothing seems to be known concerning this question. The similarity between the results of the previous sections and results of Hayman [5] on mean p-valent functions should also be noted. In this direction, W. E. Kirwan has recently shown (unpublished) that given $f \in V_k$ with $2 \leqq k \leqq 4$, there exists a constant $d(f)$ such that $f - d(f)$ is circumferentially mean-$k/4$ valent.

4. Bazilevic functions and $K(\beta)$. For any $\alpha > 0$, define $B(\alpha)$ to be the class of functions g which are regular in U and which are given by

$$g(z) = \left\{\alpha\int_0^z \xi^{\alpha-1}p(\xi)\left(\frac{h(\xi)}{\xi}\right)^\alpha d\xi\right\}^{1/\alpha}, \tag{4.1}$$

where $p \in \mathscr{P}$, the class of functions P regular in U satisfying $\operatorname{Re} P(z) > 0$ and $P(0) = 1$, and where $h \in \mathscr{S}^*$, the class of normalized starlike functions. The powers appearing in (4.1) are meant as principal values. It is known [1] that $B(\alpha)$ contains only schlicht functions, and it is easy to verify that for various special choices of α, p, and h, the class $B(\alpha)$ reduces to the classes of convex, starlike, and close-to-convex functions. However, in general very little seems to be known about the geometry of $B(\alpha)$. In this section we shall relate $B(\alpha)$ to $K(1/\alpha)$. This relationship will allow us to give a simple geometric interpretation of $B(\alpha)$ as well as a simple geometric proof that $B(\alpha)$ contains only schlicht functions.

We first need a technical lemma.

LEMMA 4.1. *Let g be given by* (4.1). *Then g is locally schlicht and vanishes only at the origin.*

Proof. If $\alpha = 1$, then it is easily seen that g is close-to-convex, and hence the lemma is trivial. Thus we assume $\alpha \neq 1$. Let $z_0 \neq 0$ be given. We claim that $g(z_0) = 0$ iff $g'(z_0) = 0$. If $g(z_0) \neq 0$, then $(g(z)/z)^\alpha$ is regular in a neighborhood of z_0, and from (4.1)

$$(g(z)/z)^{\alpha-1}g'(z) = p(z)(h(z)/z)^\alpha\,. \tag{4.2}$$

Since neither p nor h vanish at z_0, it then follows that $g'(z_0) \neq 0$.

Suppose now that $g'(z_0) \neq 0$. We must show $g(z) \neq 0$. Since the zeros of g and g' are isolated, it is clear that we may choose (even if $g(z_0) = 0$) an arc γ ending at z_0 such that (4.2) holds for $z \in \gamma$, $z \neq z_0$, and such that $g'(z) \neq 0$ for $z \in \gamma$. Therefore, for $z \in \gamma$,

$$\lim_{z \to z_0} | g(z)/z |^{\alpha-1} = \left| \frac{p(z_0)}{g'(z_0)} \left(\frac{h(z_0)}{z_0} \right)^{\alpha} \right| ,$$

and hence (since $\alpha \neq 1$) $g(z_0) \neq 0$, which establishes our claim.

To prove the lemma, it is now sufficient to show that g vanishes only at the origin. Suppose not; that is, suppose $g(z) = (z - z_0)^m q(z)$ where $m \geqq 1$, $q(z_0) \neq 0$ and $z_0 \neq 0$. We choose an arc γ ending at z_0 such that for $z \in \gamma$ $(z \neq z_0)$ we have $g(z) \neq 0, g'(z) \neq 0$, and such that (4.2) holds. Then with $z \in \gamma$,

$$(z - z_0)^{m\alpha-1}\left(\frac{q(z)}{z}\right)^{\alpha-1}[(z - z_0)q'(z) + mq(z)] = p(z)\left(\frac{h(z)}{z}\right)^{\alpha} .$$

We now allow $z \to z_0$, and we find that $m\alpha = 1$. We now define G for $z \in U$ by $G(z)^m = g(z^m)$. From (4.1) it follows that G is close-to-convex with respect to H, given by $H(z)^m = h(z^m)$ where h is as in (4.1). But $G(z_0^{1/m})^m = g(z_0) = 0$ and $z_0^{1/m} \neq 0$, which contradicts the fact that G is schlicht. This proves the lemma.

We now define $K_0(\beta)$ to be that subclass of $K(\beta)$ such that in (1.3) we have $c = 1$ and $p(0) = 1$. Therefore, $f \in K_0(\beta)$ iff

$$f'(z) = p(z)^{\beta} \frac{h(z)}{z} \tag{4.3}$$

where $p \in \mathscr{P}$ and $h \in \mathscr{S}^*$. We also assume $\beta > 0$.

THEOREM 4.1. *If $f \in K_0(\beta)$, then $g \in B(1/\beta)$ where*

$$g(z) = \left\{ \frac{1}{\beta} \int_0^z (\xi f'(\xi))^{1/\beta} \xi^{-1} d\xi \right\}^{\beta} .$$

Conversely, if $g \in B(\alpha)$, then $f \in K_0(1/\alpha)$ where

$$f(z) = \int_0^z \left(\frac{g(\xi)}{\xi} \right)^{1-1/\alpha} (g'(\xi))^{1/\alpha} d\xi .$$

Proof. Suppose first that $f \in K_0(\beta)$ and is given by (4.3). Then

$$f'(z)^{1/\beta} = p(z)\left(\frac{h(z)}{z}\right)^{1/\beta} ,$$

and from the definition of $B(1/\beta)$ it follows that g defined as in the theorem belongs to $B(1/\beta)$.

Now we suppose $g \in B(\alpha)$, and we define f as in theorem. By Lemma 4.1 f is regular in U, and since $g \in B(\alpha)$ we have from the definition of f that

$$f'(z)^{\alpha} = p(z)\left(\frac{h(z)}{z}\right)^{\alpha}$$

where $p \in \mathscr{P}$ and $h \in \mathscr{S}^*$. Hence $f \in K_0(1/\alpha)$.

Note that although for $\beta > 1$ f may be of arbitrarily high valence, it is always true that the corresponding g is schlicht. Also note that since $V_k \subset K(k/2 - 1)$, we have a relation between V_k and $B(2/(k-2))$.

We now investigate the geometry of $B(\alpha)$. We shall assume that g is regular and locally schlicht in U, is normalized as in (1.1), and vanishes only at the origin. Also, for $0 < r < 1$, we define the curve $C(r) = \{g(re^{i\theta})^{\alpha}: 0 \leqq \theta < 2\pi\}$.

THEOREM 4.2. *With the above notation and hypothesis on g, we have that $g \in B(\alpha)$ iff for all $0 < r < 1$ the tangent to $C(r)$ never turns back on itself as much as π radians.*

Proof. If $g \in B(\alpha)$, then we see from Theorem 4.1 that $f \in K_0(1/\alpha)$ where

$$f'(z) = \left(\frac{g(z)}{z}\right)^{1-1/\alpha}(g'(z))^{1/\alpha} .$$

Denote by $T(f, re^{i\theta})$ the tangent to the curve $f(|z| = r)$ at $f(re^{i\theta})$. Then with $z = re^{i\theta}$,

$$\arg T(f, re^{i\theta}) = (1 - 1/\alpha)\arg g(z) + (1/\alpha)\arg zg'(z) + \pi/2 ,$$

from which it follows by a standard argument that

$$\frac{\partial}{\partial\theta}\arg T(f, re^{i\theta}) = (1 - 1/\alpha)\operatorname{Re}\frac{zg'(z)}{g(z)} + \frac{1}{\alpha}\operatorname{Re}\left\{1 + \frac{zg''(z)}{g'(z)}\right\} .$$

Since $f \in K_0(1/\alpha)$,

$$\int_{\theta_1}^{\theta_2}\frac{\partial}{\partial\theta}\arg T(f, re^{i\theta})d\theta > -\pi/\alpha$$

for any $\theta_1 < \theta_2 < \theta_1 + 2\pi$, and so

$$(\alpha - 1)\int_{\theta_1}^{\theta_2}\operatorname{Re}\frac{zg'(z)}{g(z)}\,d\theta + \int_{\theta_1}^{\theta_2}\operatorname{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right)d\theta > -\pi . \tag{4.4}$$

Noting that locally we have $(g^{\alpha}(z))' = \alpha g(z)^{\alpha-1}g'(z)$, we see by a standard argument that (4.4) is equivalent to the fact that the tangent to $C(r)$ never turns back on itself by as much as π radians.

To prove the converse, we have from Lemma 4.1 that for $z \neq 0$, $(g(z))^\alpha$ is locally regular, so we may assume that (4.4) holds. If f is defined by

$$f(z) = \int_0^z \left(\frac{g(\xi)}{\xi}\right)^{1-1/\alpha} (g'(\xi))^{1/\alpha} d\xi ,$$

then f is regular in U and from (4.4) we have

$$\int_{\theta_1}^{\theta_2} \frac{\partial}{\partial \theta} \arg T(f, re^{i\theta}) d\theta > -\pi/\alpha \tag{4.5}$$

for any $\theta_1 < \theta_2 < \theta_1 + 2\pi$. Since f' never vanishes, an argument due to Kaplan [9] shows that (4.5) implies $f \in K_0(1/\alpha)$, and thus

$$f'(z) = p(z)^{1/\alpha} \frac{h(z)}{z}$$

where $p \in \mathscr{P}$ and $h \in \mathscr{S}^*$. We now see from the definition of f that

$$g(z) = \left\{\alpha \int_0^z \xi^{\alpha-1} p(\xi) \left(\frac{h(\xi)}{\xi}\right)^\alpha d\xi \right\}^{1/\alpha} ,$$

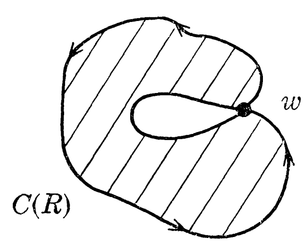
and so $g \in B(\alpha)$. This proves Theorem 4.2.

In conclusion, we prove geometrically that $B(\alpha)$ contains only schlicht functions.

COROLLARY 4.3. *$B(\alpha)$ contains only schlicht functions.*

Proof. Suppose $g \in B(\alpha)$ and g is not schlicht. For each $0 < r < 1$, let $C(r) = \{g(re^{i\theta}) : 0 \leqq \theta \leqq 2\pi\}$, and let $R = \inf\{r : C(r)$ is not a simple curve$\}$. Since $g'(0) = 1$, it is clear that $R > 0$. Also, $R < 1$, since it follows from the argument principle that there exists $r < 1$ such that g is not schlicht on $|z| = r$.

Consider now the curve $C(R)$. Clearly $C(R)$ is nonsimple, and g is schlicht in $\{z : |z| < R\}$. Hence we may choose w, $z_1 = Re^{i\theta_1}$, and $z_2 = Re^{i\theta_2}$ (with $\theta_1 < \theta_2$) such that $g(z_1) = g(z_2) = w$, and such that the curve $C(R)$ is simple for $\theta \in (\theta_1, \theta_2)$.



By Lemma 4.1 g is locally schlicht and vanishes only at the origin, so from Theorem 4.2, with $z = Re^{i\theta}$,

$$(\alpha - 1)\int_{\theta_1}^{\theta_2} d\arg g + \int_{\theta_1}^{\theta_2} d\arg zg'(z) > -\pi .$$

However, by the choice of θ_1 and θ_2 we have $\int_{\theta_1}^{\theta_2} d\arg g = 0$, and so

$$\int_{\theta_1}^{\theta_2} d\arg zg' > -\pi . \tag{4.6}$$

But it is clear geometrically that between θ_1 and θ_2 the argument of the tangent vector to $C(R)$ turns back on itself by π radians, which contradicts (4.6). Therefore g must be schlicht.

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