PACIFIC JOURNAL OF MATHEMATICS
Vol. 44, No. 1, 1973

EXISTENCE OF SPECIAL K-SETS IN CERTAIN LOCALLY COMPACT ABELIAN GROUPS

FRANK B. MILES

In all that follows, G is an infinite, nondiscrete, locally compact T_0 abelian group with character group X and Δ is a nonempty subset of X. In a standard proof of the existence of infinite (in fact, perfect) Helson sets (see for example Hewitt and Ross) it is shown that each nonvoid open subset of an arbitrary G contains a K-set (terminology of Hewitt and Ross) homeomorphic to Cantor's ternary set (or, in the terminology of Rudin, a Kronecker set or a set of type K_a homeomorphic to the Cantor set). In this paper, it is shown that $K_{0,\Delta}$-sets or $K_{a,\Delta}$-sets homeomorphic to the Cantor set exist in profusion in a large class of infinite nondiscrete locally compact T_0 abelian groups G, provided that $\bar{\Delta}$ is not compact. (A nonvoid subset E of G is called a $K_{0,\Delta}$-set if for every continuous function from E to T, the circle group, and every $\varepsilon > 0$, there is a $\gamma \in \Delta$ such that $|\gamma(x) - f(x)| < \varepsilon$ for all $x \in E$. Let a be an integer greater than one. A nonvoid subset E of G is called a $K_{a,\Delta}$-set if it is totally disconnected and every continuous function on E with values in the set of a th roots of unity is the restriction to E of some $\gamma \in \Delta$.)

The following theorems will be proved.

THEOREM I. *Let G be compact. Let Δ be infinite. Suppose that, except for the character which is identically* 1, *$\Delta\Delta^{-1}$ consists solely of elements of infinite order. (This condition is satisfied automatically if G is connected, for then X is torsion-free.) Then every nonvoid open set in G contains a $K_{0,\Delta}$-set homeomorphic to the Cantor set.*

THEOREM II. *Let G be locally connected. Suppose that $\bar{\Delta}$ is not compact. Then every nonvoid open set in G contains a $K_{0,\Delta}$-set homeomorphic to the Cantor set.*

THEOREM III. *Let G be a compact torsion group. Let Δ be infinite. Then there is an integer $a \geqq 2$ such that every nonvoid open set in G contains a translate of a $K_{a,\Delta}$-set homeomorphic to the Cantor set.*

1. Preliminaries.

NOTATION 1.1. We denote Haar measure on G by m, with $m(G) = 1$ when G is compact. When H is a subgroup of G, we write

$\Delta|_H$ for $\{\gamma|_H : \gamma \in \Delta\}$. $M(P)$ denotes the set of all (finite) regular Borel measures on the compact subset P of G.

$C(A, B)$ denotes the set of all continuous functions from A to B, where A and B are topological spaces. If $B = \mathbf{C}$, the set of complex numbers, we write $C(A)$ instead of $C(A, \mathbf{C})$.

$\mathbf{Z}$ is the group of integers. $\mathbf{R}$ is the group of real numbers. $\mathbf{Q}$ is the (discrete) group of rational numbers. $\mathbf{N}$ is the set of positive integers. When a is an integer greater than one, $\mathbf{Z}_a$ is the additive group of integers modulo a and $\mathbf{T}_{(a)}$ is the multiplicative group of ath roots of unity.

1 is the identity element of X.

$\prod^*_{\iota \in I} G_\iota$ is the weak direct product of the groups G_ι.

REMARKS 1.2.

(a) In §5, we give examples which show some of the limitations of Theorems I, II, and III.

(b) The hypothesis on $\Delta\Delta^{-1}$ in Theorem I is related to connectedness, as will be shown in Theorem 2.1.

(c) When G is compact, a $K_{0,\Delta}$-set (or $K_{a,\Delta}$-set) E is a Δ-Helson set—i.e., a set with the property that every $f \in C(E)$ has the form $f = \check{g}|_E$ for some $g \in L_1(X)$ which vanishes off Δ. When G is not compact, a $K_{0,\Delta}$-set need not be a Δ-Helson set as the example $G = X = \mathbf{R}$ and $\Delta = \mathbf{Q}$ shows.

(d) Our proof of Theorem II for the case where G is metrizable uses a technique due to Kaufman, [6, p. 184–185 and 7]. The general case follows from the case where G is metrizable and from Theorem I. Our proofs of Theorems I and III depend on the notion of an equidistributed sequence in a compact group. This notion for the case $G = \mathbf{T}$ is due to Weyl [9]. The notion has been generalized by Eckmann [2] and Hlawka [5]. Eckmann's work offers more than enough generality for our purposes; relevant parts are given below in 1.3 and 1.4.

DEFINITION 1.3. Let H be a compact abelian group with Haar measure μ and $\mu(H) = 1$. Let $\{\alpha_j\}_{j=1}^\infty$ be a sequence in H. For $F \subset H$, let $n(F)$ be the number of α_j with index $j \leqq n$ which are in F. The sequence $\{\alpha_j\}_{j=1}^\infty$ is said to be equidistributed in H if $\lim_{n\to\infty} n(F)/n = \mu(F)$ for all closed F with the property that $\mu(\text{boundary } F) = 0$.

THEOREM 1.4. *Let H be a compact abelian group with Haar measure μ and $\mu(H) = 1$. Let $\{\alpha_j\}_{j=1}^\infty$ be a sequence in H. The following are equivalent:*

(i) *$\{\alpha_j\}_{j=1}^\infty$ is equidistributed in H;*

(ii) *for every continuous character* γ *of* H *such that* $\gamma \not\equiv 1$, *we have* $\lim_{n\to\infty} n^{-1} \Sigma_{j=1}^{n} \gamma(\alpha_j) = 0$.

REMARKS 1.5.

(a) In the proofs of Theorems I and III we will use the equivalence of (i) and (ii) in Theorem 1.4 for the cases $H = \mathbf{T}$ and $H = \mathbf{T}_{(a)}$ respectively. If $H = \mathbf{T}$ we have Weyl's original result: The sequence $\{\alpha_j\}_{j=1}^{\infty} \subset \mathbf{T}$ is equidistributed in $\mathbf{T}$ if and only if $\lim_{n\to\infty} n^{-1} \Sigma_{j=1}^{n} \alpha_j^r = 0$ for all nonzero integers r (or, equivalently, for all $r \in \mathbf{N}$) [9]. If $H = \mathbf{T}_{(a)}$, we have: The sequence $\{\alpha_j\}_{j=1}^{\infty} \subset \mathbf{T}_{(a)}$ is equidistributed in $\mathbf{T}_{(a)}$ if and only if for every integer $r \in \{1, 2, \cdots, a-1\}$ we have $\lim_{n\to\infty} n^{-1} \Sigma_{j=1}^{n} \alpha_j^r = 0$.

(b) Eckmann's definition differs from Definition 1.3 in that he omits the restriction μ(boundary F) $= 0$. This restriction is necessary, as has been pointed out [3].

2. Proof of Theorem I.

2.1. We first investigate the hypothesis on $\Delta\Delta^{-1}$ in the statement of Theorem I and find that it is related to connectedness.

THEOREM. *Let* G *be compact. Let* Δ *be a countably infinite subset of* X. *The following are equivalent*:

(i) $\Delta\Delta^{-1}\backslash\{1\}$ *consists solely of elements of infinite order*;

(ii) G *contains a compact connected metrizable subgroup* H *with the property that* $\delta \to \delta|_H$ *is a one-to-one map from* Δ *to the character group of* H.

Proof. (ii) implies (i): Let δ_1 and δ_2 be distinct elements of Δ. Then $\delta_1|_H \neq \delta_2|_H$, so $\delta_1\delta_2^{-1}|_H \not\equiv 1$. Since H is connected, its character group is torsion-free. Hence, $\delta_1\delta_2^{-1}|_H$ has infinite order and therefore so does $\delta_1\delta_2^{-1}$.

(i) implies (ii): Let Γ be a maximal torsion-free independent subset of Δ. (Clearly, Δ contains at most one element of finite order, so Γ is nonvoid.) We have $\Gamma = \{\gamma_1, \cdots, \gamma_p\}$ for some positive integer p or $\Gamma = \{\gamma_1, \gamma_2, \cdots\}$. If Γ is finite, let $P = \mathbf{Q}^p$. If not, let P be the weak direct product of countably many copies of $\mathbf{Q}$. (In either case, P is countable.) For $n \in \mathbf{N}$ (and $n \leqq p$ if Γ is finite) let e_n be that element of P with nth coordinate equal to 1 and all other coordinates equal to zero. Let Y be the subgroup of X generated by Γ. Since Γ is independent, the map $\gamma_n \to e_n$ extends to a (one-to-one) homomorphism from Y to P. Since P is divisible, this homomorphism extends to a homomorphism $\phi: X \to P$. Hence $W = X/\ker\phi$ is isomorphic to a subgroup of P. Let H be the annihilator of $\ker\phi$ in G. Then H

is a closed subgroup of G and has character group W, which is torsion-free and countable. Hence, H is connected and metrizable. Now $\delta_1|_H = \delta_2|_H$ if and only if $\delta_1\delta_2^{-1} \in \ker\phi$. Let δ_1 and δ_2 be distinct elements of Δ. It is sufficient to show that $\delta_1\delta_2^{-1} \notin \ker\phi$. Since Γ is a maximal torsion-free independent subset of Δ, there exist nonzero integers r_1 and r_2 such that $\delta_1^{r_1}$ and $\delta_2^{r_2}$ are in Y. Therefore there is a nonzero integer r such that $(\delta_1\delta_2^{-1})^r \in Y$. By the hypothesis on $\Delta\Delta^{-1}$, we have $(\delta_1\delta_2^{-1})^r \neq 1$. Since ϕ is one-to-one on Y, $r\phi(\delta_1\delta_2^{-1}) = \phi((\delta_1\delta_2^{-1})^r)$ is not the identity of P. Hence $\delta_1\delta_2^{-1} \notin \ker\phi$ and the proof is complete.

LEMMA 2.2. *Let G be compact. Let $\Delta = \{\gamma_1, \gamma_2, \cdots\}$ be a countably infinite set of distinct elements of X arranged in any fixed order. Suppose that $\Delta\Delta^{-1}\backslash\{1\}$ consists solely of elements of infinite order. Then for m-almost all $x \in G$, the sequence $\{\gamma_j(x)\}_{j=1}^{\infty}$ is equidistributed in* $\mathbf{T}$.

Proof. Our proof follows Weyl [9]. For $x \in G$, $n \in \mathbf{N}$, and $r \in \mathbf{N}$, define $f_{nr}(x) = n^{-1}\Sigma_{j=1}^{n}\gamma_j^r(x)$. From our hypothesis on $\Delta\Delta^{-1}$ we find that $\gamma_j^r\overline{\gamma_k^r} = 1$ implies that $\gamma_j = \gamma_k$. Since G is compact, $\int_G \gamma(x)dm(x) = 0$ when $\gamma \neq 1$. Thus, we have

$$\int_G |f_{nr}|^2 dm = n^{-2}\int_G \Sigma_{j,k=1}^{n}\gamma_j^r(x)\overline{\gamma_k^r(x)}dm(x) = n^{-1} .$$

Therefore we have $\Sigma_{n=1}^{\infty}\|f_{n^2,r}\|_2^2 < \infty$ and hence $f_{n^2,r}(x) \to 0$ as $n \to \infty$ for m-almost all $x \in G$. Suppose that $f_{n^2,r}(x) \to 0$ as $n \to \infty$ for all $x \notin A_r$ where $m(A_r) = 0$.

For $n \in \mathbf{N}$, let $\lambda(n)$ be the positive integer such that $\lambda^2 \leqq n < (\lambda + 1)^2$. Then we have $|nf_{nr}(x) - \lambda^2 f_{\lambda^2,r}(x)| \leqq 2\lambda$ and hence

$$\left|f_{nr}(x) - \frac{\lambda^2}{n}f_{\lambda^2,r}(x)\right| \leqq 2/\sqrt{n} .$$

Let $\varepsilon > 0$. Fix $x \notin A_r$. Then there is a positive integer M such that $|f_{\lambda^2,r}(x)| < \varepsilon/2$ whenever $\lambda \geqq M$. Let $n \geqq M^2$ and $n > 16/\varepsilon^2$. Let λ be such that $\lambda^2 \leqq n < (\lambda + 1)^2$. Then $\lambda^2/n \leqq 1$, $2/\sqrt{n} < \varepsilon/2$, and $\lambda^2 \geqq M^2$, so we have

$$|f_{nr}(x)| \leqq \left|f_{nr}(x) - \frac{\lambda^2}{n}f_{\lambda^2,r}(x)\right| + \frac{\lambda^2}{n}|f_{\lambda^2,r}(x)| < 2/\sqrt{n} + \varepsilon/2 < \varepsilon .$$

Hence, $f_{nr}(x) \to 0$ as $n \to \infty$ for all $x \notin A_r$.

Let $A = \cup A_r$. Then $m(A) = 0$ and for $x \notin A$ we have for all $r \in \mathbf{N}$ that $f_{nr}(x) \to 0$ as $n \to \infty$. Therefore, by 1.5(a), $\{\gamma_j(x)\}_{j=1}^{\infty}$ is equidistributed in $\mathbf{T}$ for all $x \notin A$.

LEMMA 2.3. *Let G and Δ be as in Theorem 1. Let $V_1, \cdots, V_k$ be nonvoid open subsets of G. Then there exist $x_j \in V_j (1 \leqq j \leqq k)$ with the property that for every $\varepsilon > 0$ and for all $z_1, \cdots, z_k \in \mathbf{T}$ there is a $\gamma \in \Delta$ such that $|\gamma(x_j) - z_j| < \varepsilon (1 \leqq j \leqq k)$, i.e., there exist $x_j \in V_j$ $(1 \leqq j \leqq k)$ such that $\{x_1, \cdots, x_k\}$ is a $K_{0,\Delta}$-set.*

Proof. We may suppose that Δ is countable. Let $q \in \{1, 2, \cdots, k\}$. Let "$P(q)$ holds for $x_1, \cdots, x_q$" mean "$x_j \in V_j (1 \leqq j \leqq q)$ and $\{x_1, \cdots, x_q\}$ is a $K_{0,\Delta}$ set." By Lemma 2.2, there is an $x_1 \in V_1$ such that $P(1)$ holds for x_1. Suppose that $1 \leqq r \leqq k - 1$ and that $P(r)$ holds for $x_1, \cdots, x_r$. It is sufficient to show there is an $x_{r+1} \in V_{r+1}$ such that $P(r + 1)$ holds for $x_1, \cdots, x_{r+1}$. Let $A = \{w \in V_{r+1} | P(r + 1)$ does not hold for $x_1, \cdots, x_r, w\}$. It is sufficient to show that $m(A) = 0$. Let S be a countable dense subset of $\mathbf{T}$. Then $w \in A$ if and only if $w \in V_{r+1}$ and there exist $p \in \mathbf{N}$ and $s_1, \cdots, s_{r+1} \in S$ such that for all $\gamma \in \Delta$ either $|\gamma(x_j) - s_j| \geqq p^{-1}$ for some $j (1 \leqq j \leqq r)$ or $|\gamma(w) - s_{r+1}| \geqq p^{-1}$, i.e., we have

$$A = \bigcup_{p \in \mathbf{N}} \bigcup_{s_1 \in S} \cdots \bigcup_{s_{r+1} \in S} A(p, s_1, \cdots, s_{r+1})$$

where $A(p, s_1, \cdots, s_{r+1}) = \bigcap_{\gamma \in \Delta} \{y \in V_{r+1}: |\gamma(y) - s_{r+1}| \geqq p^{-1}$ or at least one $|\gamma(x_j) - s_j| \geqq p^{-1}\}$.

Let

$$\tilde{\Delta}(p, s_1, \cdots, s_r) = \{\gamma \in \Delta: |\gamma(x_j) - s_j| < p^{-1}, 1 \leqq j \leqq r\} .$$

Then we have

$$\begin{aligned} &A(p, s_1, \cdots, s_{r+1}) \\ &= \{y \in V_{r+1}: |\gamma(y) - s_{r+1}| \geqq p^{-1} \text{ for all } \gamma \in \tilde{\Delta}(p, s_1, \cdots, s_r)\} . \end{aligned}$$

Hence, it is sufficient to show that each $\tilde{\Delta}(p, s_1, \cdots, s_r)$ is infinite (for then, by Lemma 2.2, each $A(p, s_1, \cdots, s_{r+1})$ is m-null and therefore so is A).

We assume that for some $p \in \mathbf{N}$ and $s_1, \cdots, s_r \in S$ the set $\tilde{\Delta} = \tilde{\Delta}(p, s_1, \cdots, s_r)$ is finite and use this to obtain a contradiction. A basic neighborhood of the point $\mathbf{z} = (z_1, \cdots, z_r) \in \mathbf{T}^r$ has the form $B(\mathbf{z}, \varepsilon) = \{\mathbf{w} = (w_1, \cdots, w_r): |z_j - w_j| < \varepsilon, 1 \leqq j \leqq r\}$ for some $\varepsilon > 0$. Let $\mathbf{s} = (s_1, \cdots, s_r)$ and $\mathbf{x} = (x_1, \cdots, x_r)$. For $\gamma \in \Delta$, let $\gamma(\mathbf{x}) = (\gamma(x_1), \cdots, \gamma(x_r))$. If $\tilde{\Delta}$ is finite, then $\{\gamma \in \Delta | \gamma(\mathbf{x}) \in B(\mathbf{s}, p^{-1})\}$ is finite. Then there exist $\mathbf{z} \in B(\mathbf{s}, p^{-1})$ and $\varepsilon > 0$ be such that $B(\mathbf{z}, \varepsilon) \subset B(\mathbf{s}, p^{-1})$ and $B(\mathbf{z}, \varepsilon)$ is disjoint from $\{\gamma(\mathbf{x}) | \gamma \in \Delta\}$. This contradicts the induction hypothesis that $P(r)$ holds for $x_1, \cdots, x_r$.

THEOREM 2.4. *Theorem I holds when G is metrizable.*

Proof. Repeat the proof of [4, (41.5), part I] choosing all characters in Δ and using Lemma 2.3 whenever [4] uses [4, (41.3)].

THEOREM 2.5. *Let G and Δ be as in Theorem I. Let U be a neighborhood of the identity in G. Then U contains a $K_{0,\Delta}$-set homeomorphic to the Cantor set.*

Proof. By Theorem 2.1, G contains a compact connected metrizable subgroup H with the property that $\Gamma = \Delta|_H$ is infinite. Let $V = U \cap H$. Since H is connected, its character group is torsion-free. Hence, by Theorem 2.4, V contains a $K_{0,\Gamma}$-set P homeomorphic to the Cantor set. Clearly, P is a $K_{0,\Delta}$-set contained in U.

THEOREM[1] 2.6. *Let P be a compact metrizable $K_{0,\Delta}$-set in G, where G is compact and $\Delta\Delta^{-1}\backslash\{1\}$ consists solely of elements of infinite order. Then for almost all $x \in G$, xP is a $K_{0,\Delta}$-set.*

Proof. Let $\{f_1, f_2, \cdots\}$ be a (uniformly) dense subset of $C(P, \mathbf{T})$. For each j, there is a sequence $\{\gamma_{ij}\}_{i=1}^{\infty}$ of elements of Δ such that $\gamma_{ij} \to f_j$ uniformly on P. By Lemma 2.2, there is an m-null set A_j such that $\{\gamma_{ij}(x)\}_{i=1}^{\infty}$ is equidistributed in $\mathbf{T}$ whenever $x \in G\backslash A_j$. Let $A = \cup A_j$. Then A is m-null. Let $x \in G\backslash A$. For each j, let $g_j(xy) = f_j(y)$. To show that xP is a $K_{0,\Delta}$-set, it is sufficient to show that each g_j is uniformly approximable by $\{\gamma_{ij}: i, j = 1, 2, \cdots\}$. Let $\varepsilon > 0$. Fix j. Then for some i_0, we have $|\gamma_{ij}(y) - f_j(y)| < \varepsilon/2$ for all $y \in P$ whenever $i > i_0$ and, since $\{\gamma_{ij}(x)\}_{i=1}^{\infty}$ is equidistributed in $\mathbf{T}$, there is an $i > i_0$ such that $|\gamma_{ij}(x) - 1| < \varepsilon/2$. For this i we have $|\gamma_{ij}(xy) - g_j(xy)| < \varepsilon$ for all $y \in P$.

Proof of Theorem I. 2.7. Immediate from Theorems 2.5 and Theorem 2.6.

3. Proof of Theorem II.

THEOREM 3.1. *Let G be locally connected. Let Δ be such that $\bar{\Delta}$ is not compact. Let U be a neighborhood of the identity in G. Then there is a γ in Δ such that $\gamma(U) = \mathbf{T}$.*

Proof. The topology on X is the restriction of the compact-open topology on $C(G)$ to the (closed) subspace X of $C(G)$. Hence, $\bar{\Delta}$ is

[1] In the original version of this paper, the conclusion of Theorem I was as follows: Every open set in G containing an element of finite order contains a $K_{0,\Delta}$-set homeomorphic to the Cantor set and, if G is metrizable, every nonvoid open set in G contains a $K_{0,\Delta}$-set homeomorphic to the Cantor set. Theorem 2.6 and the stronger version of Theorem I which it yields are due to Robert Kaufman [private communication, December, 1971].

compact as a subspace of X if and only if it is compact as a subspace of $C(G)$ with the compact-open topology. Since by hypothesis $\bar{\Delta}$ is not compact, it follows from Ascoli's Theorem that Δ is not equicontinuous [1, p. 267] and, hence, that Δ is not equicontinuous at the identity of G. Therefore, there exists $\varepsilon > 0$ such that for every neighborhood W of the identity in G, there is an $x \in W$ and a $\gamma \in \Delta$ such that $|\gamma(x) - 1| \geqq \varepsilon$. Let $S = \{e^{it} | 0 \leqq t \leqq \varepsilon/2\}$. Let M be a positive integer with the property that $S^M = \mathbf{T}$. Let V be a connected neighborhood of the identity in G such that $V^M \subset U$. Then there exist $x \in V$ and $\gamma \in \Delta$ such that $|\gamma(x) - 1| \geqq \varepsilon$. Hence, $\gamma(V)$ contains an arc of length at least ε. Therefore we have $\mathbf{T} = \gamma(V)^M \subset \gamma(U) \subset \mathbf{T}$.

THEOREM 3.2. *Let G be locally connected and metrizable. Let Δ be such that $\bar{\Delta}$ is not compact. Let E be a compact totally disconnected subset of $\mathbf{R}$ or $\mathbf{T}$. Then there is a first category set $H \subset C(E, G)$ such that each $f \in C(E, G) \backslash H$ maps E homeomorphically onto a $K_{0,\Delta}$-set in G.*

Proof. Our proof follows the ideas of Kaufman [7] as given by Katznelson [6, p. 184–185].

For $h \in C(E, \mathbf{T}), f \in C(E, G)$, and $\varepsilon > 0$, let "(*) holds for h, f and ε" mean "there is a $\gamma \in \Delta$ such that $|\gamma(f(y)) - h(y)| < \varepsilon$ for all $y \in E$." Let $f \in C(E, G)$. Clearly, f is a homeomorphism of E onto $f(E)$ if and only if f is one-to-one. Also, if f is not one-to-one, it is clear that there exist $h \in C(E, \mathbf{T})$ and $\varepsilon > 0$ such that (*) fails for h, f, and ε. Hence, f is a homeomorphism of E onto $f(E)$ and $f(E)$ is a $K_{0,\Delta}$-set if and only if for every $h \in C(E, \mathbf{T})$ and every $\varepsilon > 0$, (*) holds for h, f, and ε.

Let d be an invariant metric on G compatible with the topology of G. For f and g in $C(E, G)$, let $D(f, g) = \sup\{d(f(y), g(y)) | y \in E\}$. Observe that $D(f, g) < \infty$ since E is compact.

Let $h \in C(E, \mathbf{T})$, $g \in C(E, G)$, $\varepsilon > 0$, and $\eta > 0$. We now show that there exist an $f \in C(E, G)$ such that (*) holds for h, f, and ε and $D(f, g) < \eta$. Let U be the open η-ball about the identity e of G. By Theorem 3.1, there is a $\gamma \in \Delta$ such that $\gamma(U) = \mathbf{T}$. Write $E = \bigcup_{j=1}^{n} E_j$, where the E_j are disjoint nonvoid open-closed subsets of E and $\gamma \circ g$ and h both vary by less than $\varepsilon/3$ on each E_j. (The E_j exist since E is totally disconnected.) Let $y_j \in E_j$ and suppose that $\gamma(g(y_j)) = \alpha_j$ and $h(y_j) = \beta_j$, $1 \leqq j \leqq n$. Let $x_j \in U$ be such that $\gamma(x_j) = \overline{\alpha_j}\beta_j$. Define $f \in C(E, G)$ by $f(y) = x_j g(y)$ when $y \in E_j$. We see that $D(f, g) = \max\{d(x_j, e)\} < \eta$ and for $y \in E_j$ we have

$$\begin{aligned} |\gamma(f(y)) - h(y)| &\leqq |\gamma(g(y))\gamma(x_j) - \gamma(g(y_j))\gamma(x_j)| \\ &+ |\gamma(g(y_j))\gamma(x_j) - h(y_j)| + |h(y_j) - h(y)| < \frac{\varepsilon}{3} + 0 + \frac{\varepsilon}{3} < \varepsilon\,. \end{aligned}$$

Hence, (*) holds for h, f, and ε.

For $h \in C(E, \mathbf{T})$ and $\varepsilon > 0$, let $H(h, \varepsilon) = \{f \in C(E, G) \mid (*)$ fails for h, f, and $\varepsilon\}$. It is easy to show that $H(h, \varepsilon)$ is closed. By the preceding paragraph, $H(h, \varepsilon)$ is nowhere dense in $C(E, G)$. Let $\{h_n\}_{n=1}^{\infty}$ be dense in $C(E, \mathbf{T})$. Let $H = \bigcup_{n,k=1}^{\infty} H(h_n, 1/k)$. Then H is a first category set in the complete metric space $C(E, G)$. Also, we have $f \in C(E, G)\backslash H$ if and only if every $h \in C(E, \mathbf{T})$ can be uniformly approximated by $\gamma \circ f$'s $(\gamma \in \Delta)$, which by the second paragraph of the proof is true if and only if f is a homeomorphism and $f(E)$ is a $K_{0,\Delta}$-set.

THEOREM 3.3. *Theorem II holds when G is metrizable.*

Proof. Let U be a nonvoid open subset of G. Let E be the Cantor set. Let H be as in Theorem 3.2. The result follows from Theorem 3.2 since $C(E, U)$ is open in $C(E, G)$ and $C(E, G)\backslash H$ is dense in $C(E, G)$.

THEOREM 3.4. *Let G be locally connected. Then G is topologically isomorphic with $D \times \mathbf{R}^n \times K$, where D is discrete, n is a nonnegative integer, and K is a compact, connected, locally connected abelian group.*

Proof. Let C be the component of the identity in G. Then G is topologically isomorphic with $(G/C) \times C$. Since G/C is totally disconnected and locally connected, it is discrete. Since C is connected and locally connected, it is topologically isomorphic with $\mathbf{R}^n \times K$, where n is a nonnegative integer and K is compact, connected, and locally connected.

Proof of Theorem II. 3.5. By Theorem 3.4, we may suppose that $G = H \times K$, where H is locally connected and metrizable and K is compact, connected, and locally connected. We then have $X = Y \times F$, where Y and F are the character groups of H and K, respectively. Let U be a nonvoid open subset of G. We may suppose that $U = V \times W$, where V and W are nonvoid open subsets of H and K, respectively. We denote elements of X by (α, β), where $\alpha \in Y$ and $\beta \in F$. Let $\Gamma = \{\beta \in F \mid (\alpha, \beta) \in \Delta\}$.

Case 1. Γ is finite: There is a $\beta_0 \in \Gamma$ such that $\{(\alpha, \beta_0) \in \Delta\}^-$ is not compact in X. Let $\Delta_0 = \{\alpha \in Y \mid (\alpha, \beta_0) \in \Delta\}$. Then Δ_0^- is not compact in Y. Hence, by Theorem 3.3, V contains a K_{0,Δ_0}-set P homeomorphic to the Cantor set. Let $z \in W$. Then $P \times \{z\}$ is a $K_{0,\Delta}$-set in U homeomorphic to the Cantor set.

Case 2. Γ is infinite: Let $x \in V$. Let $\{(\alpha_m, \beta_m)\}_{m=1}^{\infty}$ be a sequence in Δ such that the β_m are distinct and such that $\alpha_m(x) \to s \in \mathbf{T}$ as $m \to \infty$. Let $\Delta_0 = \{\beta_m\}_{m=1}^{\infty}$. Since Δ_0 is infinite and K is compact and connected, W contains a K_{0,Δ_0}-set P homeomorphic to the Cantor set by Theorem I. Then $\{x\} \times P$ is a $K_{0,\Delta}$-set in U homeomorphic to the Cantor set.

4. Proof of Theorem III.

LEMMA 4.1. *Let k be an integer greater than one. Let G be the product of infinitely many copies of $\mathbf{T}_{(k)}$. Let Δ be an infinite subset of X and suppose there is an integer a greater than one such that all elements of Δ have order a and that whenever γ_1 and γ_2 are distinct elements of Δ, then $\gamma_1\gamma_2^{-1}$ has order a. Then for every sequence $\Delta_0 = \{\gamma_1, \gamma_2, \cdots\}$ of distinct elements of Δ, the sequence $\{\gamma_j(x)\}_{j=1}^{\infty}$ is equidistributed in $\mathbf{T}_{(a)}$ for m-almost all $x \in G$.*

Proof. For $r \in \{1, 2, \cdots, a-1\}$ and $n \in \mathbf{N}$, let $f_{nr}(x) = 1/n\, \Sigma_{j=1}^{n} \gamma_j^r(x)$. By our hypothesis on Δ, $\gamma_j \neq \gamma_l$ implies that $(\gamma_j\gamma_l^{-1})^r \neq 1$. Also, since G is compact, $\int_G \gamma(x)dm(x) = 0$ when $\gamma \neq 1$. Hence we have

$$\int_G |f_{nr}|^2 dm = n^{-2}\int_G \Sigma_{j,l=1}^{n} \gamma_j^r(x)\overline{\gamma_l^r(x)}dm(x) = n^{-1}\,.$$

We thus have $\Sigma_{n=1}^{\infty} \|f_{n^2,r}\|_2^2 < \infty$ and hence $f_{n^2,r}(x) \to 0$ as $n \to \infty$ for m-almost all $x \in G$. Suppose that $f_{n^2,r}(x) \to 0$ as $n \to \infty$ for all $x \notin A_r$, where $m(A_r) = 0$. The device used in the proof of Lemma 2.2 yields $f_{nr}(x) \to 0$ as $n \to \infty$ for $x \notin A_r$. Let $A = \cup_{r=1}^{a-1} A_r$. Then $m(A) = 0$ and for $x \notin A$ we have for all $r \in \{1, 2, \cdots, a-1\}$ that $f_{nr}(x) \to 0$ as $n \to \infty$. Therefore, by 1.5(a), $\{\gamma_j(x)\}_{j=1}^{\infty}$ is equidistributed in $\mathbf{T}_{(a)}$ for all $x \notin A$.

LEMMA 4.2. *Let k, G, Δ, and a be as in Lemma 4.1. Let $V_1, \cdots, V_n$ be nonempty open subsets of G. Then there are $x_j \in V_j (1 \leqq j \leqq n)$ such that $\{x_1, \cdots, x_n\}$ is a $K_{a,\Delta}$-set.*

Proof. For a positive integer q, $y_1, \cdots, y_q \in \mathbf{T}_{(a)}$, and $w_j \in V_j (1 \leqq j \leqq q)$, let $\Delta(y_1, \cdots, y_q, w_1, \cdots, w_q) = \{\gamma \in \Delta \mid \gamma(w_j) = y_j, 1 \leqq j \leqq q\}$. By Lemma 4.1, there is an $x_1 \in V_1$ such that for all $y_1 \in \mathbf{T}_{(a)}$, $\Delta(y_1, x_1)$ is infinite.

Let $r \in \{1, 2, \cdots, n-1\}$ and suppose that $x_j \in V_j (1 \leqq j \leqq r)$ have been found with the property that for all $y_1, \cdots, y_r \in \mathbf{T}_{(a)}$, $\Delta(y_1, \cdots, y_r, x_1, \cdots, x_r)$ is infinite. Fixing $(y_1, \cdots, y_r) \in \mathbf{T}_{(a)}^r$ and applying Lemma 4.1 with $\Delta(y_1, \cdots, y_r, x_1, \cdots, x_r)$ in place of Δ, we find that m-almost all $x \in V_{r+1}$ have the property that for all $y_{r+1} \in \mathbf{T}_{(a)}$, $\Delta(y_1, \cdots, y_{r+1}, x_1, \cdots, x_r, x)$ is infinite. Hence, m-almost all $x \in V_{r+1}$ have the property that for all $y_1, \cdots, y_{r+1} \in \mathbf{T}_{(a)}$, $\Delta(y_1, \cdots, y_{r+1}, x_1, \cdots, x_r, x)$ is infinite. In particular, an $x_{r+1} \in V_{r+1}$ with this property exists.

Hence, by induction, there are $x_j \in V_j (1 \leqq j \leqq n)$ such that for all $y_1, \cdots, y_n \in \mathbf{T}_{(a)}$, $\Delta(y_1, \cdots, y_n, x_1, \cdots, x_n)$ is infinite and, in particular, nonvoid. Hence, $\{x_1, \cdots, x_n\}$ is a $K_{a,\Delta}$-set.

THEOREM 4.3. *Let k, G, Δ, and a be as in Lemma* 4.1. *Let G be metrizable. Let U be a nonvoid open subset of G. Then U contains a $K_{a,\Delta}$-set homeomorphic to the Cantor set.*

Proof. Repeat the proof of [4, (41.5), part III], choosing all characters in Δ and using Lemma 4.2 whenever [4] uses [4, (41.4)].

REMARK 4.4. We now proceed to reduce Theorem III to the case described in Theorem 4.3.

LEMMA 4.5. *Let k be an integer greater than one. Let X be the weak direct product of infinitely many copies of $\mathbf{T}_{(k)}$. Let Δ be an infinite subset of X. Then there exist an integer $a \geqq 2$ and an infinite subset Γ of Δ with the property that whenever γ_1 and γ_2 are distinct elements of Γ, then $\gamma_1\gamma_2^{-1}$ has order exactly a.*

Proof. We remark that this result is trivial if k is prime. (Take $a = k$ and $\Gamma = \Delta$.)

Let $b_0 = k$ and $\Delta_0 = \Delta$. Let $\gamma_1 \in \Delta_0$. Let $\Gamma_1 = \{\gamma_1\alpha^{-1} \mid \alpha \in \Delta_0\}$. Since Γ_1 is infinite, there is an integer b_1, $2 \leqq b_1 \leqq b_0$, such that Γ_1 contains infinitely many elements of order b_1. Let $\Delta_1 = \{\alpha \in \Delta_0 \mid \gamma_1\alpha^{-1}$ has order $b_1\}$. Suppose that $n \in \mathbf{N}$ and that $\gamma_1, \cdots, \gamma_n, \Gamma_1, \cdots, \Gamma_n, b_1 \cdots, b_n$ and $\Delta_1, \cdots, \Delta_n$ have been found such that for $1 \leqq j \leqq n$ we have (i) $\gamma_j \in \Delta_{j-1}$, $\Gamma_j = \{\gamma_j\alpha^{-1} \mid \alpha \in \Delta_{j-1}\}$, Γ_j has infinitely many elements of order b_j, $2 \leqq b_j \leqq b_{j-1}$, and $\Delta_j = \{\alpha \in \Delta_{j-1} \mid \gamma_j\alpha^{-1}$ has order $b_j\}$. Observe that from (i) it follows that (ii) for $1 \leqq j \leqq n$, we have $\gamma_j \notin \Delta_j$ so Δ_j is a proper infinite subset of Δ_{j-1} and the γ_j are distinct.

Let $\gamma_{n+1} \in \Delta_n$. Let $\Gamma_{n+1} = \{\gamma_{n+1}\alpha^{-1} \mid \alpha \in \Delta_n\}$. Since Γ_{n+1} is infinite, there is an integer b_{n+1} with $2 \leqq b_{n+1} \leqq b_n$ such that Γ_{n+1} contains infinitely many elements of order b_{n+1}. Let $\Delta_{n+1} = \{\alpha \in \Delta_n \mid \gamma_{n+1}\alpha^{-1}$ has order $b_{n+1}\}$. Thus, we can define $\gamma_n, \Gamma_n, \Delta_n$, and b_n for all $n \in \mathbf{N}$ in such a way that properties (i) hold for all n. Since $\{b_n\}$ is a monotone nonincreasing sequence of integers greater than one, there exist positive integers r and a such that $b_n = a$ for all $n > r$. Let $\Gamma = \{\gamma_{r+n} \mid n \in \mathbf{N}\}$. We show that Γ and a are as demanded. Let n_1 and $n_2 \in \mathbf{N}$ with $n_1 > n_2$. Then, by construction of the Δ_n, we have $\gamma_{r+n_1} \in$

$\varDelta_{r+n_1-1} \subset \varDelta_{r+n_2}$ so $\gamma_{r+n_2}\gamma_{r+n_1}^{-1}$ has order $b_{r+n_2} = a$.

LEMMA 4.6. *Let k be an integer greater than one. Let I be an infinite index set and let $X = \prod_{\iota \in I}^{*} G_\iota$, where each G_ι is a copy of $\mathbf{T}_{(k)}$. Let $\varDelta$ be an infinite subset of X. Then there exist an integer $a \geqq 2$ and an infinite subset $\varDelta_0$ of $\varDelta$ and a finite (possibly empty) subset I_0 of I such that projection of $\varDelta_0$ onto $Y = \prod_{\iota \in I \backslash I_0}^{*} G_\iota$ gives an infinite subset $\tilde{\varDelta}_0$ of Y consisting solely of elements of order a and such that whenever γ_1 and γ_2 are distinct elements of $\tilde{\varDelta}_0$, $\gamma_1\gamma_2^{-1}$ has order a.*

Proof. By Lemma 4.5, there exist an integer $a_1 \geqq 2$ and an infinite subset $\varGamma_1$ of $\varDelta$ such that whenever γ_1 and γ_2 are distinct elements of $\varGamma_1$, $\gamma_1\gamma_2^{-1}$ has order a_1. Let $\tilde{\varGamma}_1$ be an infinite subset of $\varGamma_1$ consisting of elements all of the same order b_1. It is clear that $b_1 \geqq a_1$. (If γ_1 and γ_2 are distinct elements of $\tilde{\varGamma}_1$, then $\gamma_1\gamma_2^{-1}$ has order at most b_1. But $\gamma_1\gamma_2^{-1}$ has order a_1.) If $b_1 = a_1$, we are done. (Take $I_0 = \varnothing$, $\varDelta_0 = \tilde{\varGamma}_1$, and $a = a_1$.) Suppose $b_1 > a_1$. Let $\tilde{\gamma}_1 \in \tilde{\varGamma}_1$. There is a finite subset I_1 of I such that the ιth coordinate of $\tilde{\gamma}_1$ is the identity of G_ι for $\iota \notin I_1$. Let $X_1 = \prod_{\iota \in I \backslash I_1}^{*} G_\iota$. Since I_1 is finite and $\tilde{\varGamma}_1$ is infinite, projection of $\tilde{\varGamma}_1$ onto X_1 (denoted by π_1) gives an infinite subset $\varDelta_1$ of X_1 consisting of elements of order at most a_1. (For $\alpha \in \tilde{\varGamma}_1$, order of $\pi_1(\alpha)$ in X_1 = order of $\pi_1(\alpha\tilde{\gamma}_1^{-1})$ in $X_1 \leqq a_1$.) Applying Lemma 4.5 to X_1 and $\varDelta_1$ we get an integer a_2 with $2 \leqq a_2 \leqq a_1$ and an infinite subset $\varGamma_2$ of $\varDelta_1$ such that whenever γ_1 and γ_2 are distinct elements of $\varGamma_2$, then $\gamma_1\gamma_2^{-1}$ has order a_2. Let $\tilde{\varGamma}_2$ be an infinite subset of $\varGamma_2$ consisting of elements all of the same order b_2. Then we have $a_2 \leqq b_2 \leqq a_1 < b_1$. If $a_2 = b_2$, we are done. (Take $I_0 = I_1$, $a = a_2$, $Y = X_1$, and $\varDelta_0 = \{\alpha \in \varDelta \mid \pi_1(\alpha) \in \tilde{\varGamma}_2\}$ Suppose $a_2 < b_2 \leqq a_1 < b_1$. Pick $\tilde{\gamma}_2 \in \tilde{\varGamma}_2$; let $I_2 = \{\iota \in I \backslash I_1 \mid$ ι th coordinate of $\tilde{\gamma}_2$ is not the identity of $G_\iota\}$; project $\tilde{\varGamma}_2$ onto $X_2 = \prod_{\iota \in I \backslash (I_1 \cup I_2)}^{*} G_\iota$; $\cdots$ etc. We must eventually have $b_n = a_n$ for some n (otherwise, $\{b_n\}$ would be an infinite strictly decreasing sequence of positive integers). For that n, we have a finite subset $I_0 = I_1 \cup \cdots \cup I_{n-1}$ of I and an infinite subset $\tilde{\varGamma}_n$ of $Y = \prod_{\iota \in I \backslash I_0}^{*} G_\iota$ such that all elements of $\tilde{\varGamma}_n$ have order $a_n = b_n$ and such that whenever γ_1 and γ_2 are distinct elements of $\tilde{\varGamma}_n$, $\gamma_1\gamma_2^{-1}$ has order a_n. Let $\varDelta_0 = \{\alpha \in \varDelta \mid \pi(\alpha) \in \tilde{\varGamma}_n\}$, where π is the projection of X onto Y.

THEOREM 4.7. *Let k be an integer greater than one. Let $G = \prod_{\iota \in I} G_\iota$, where each G_ι is a copy of $\mathbf{T}_{(k)}$ and I is infinite. Let $\varDelta$ be an infinite subset of X. Then there is an integer a greater than one such that every neighborhood of the identity of G contains a $K_{a,\varDelta}$-set homeomorphic to the Cantor set.*

Proof. We may suppose that Δ is countable. We identify X with $\prod^*_{\iota \in I} G_\iota$. Let a, I_0, Y, and $\tilde{\Delta}_0$ be as in Lemma 4.6. Let $I_1 = \{\iota \in I \backslash I_0 \mid$ some $\gamma \in \tilde{\Delta}_0$ has ιth coordinate different from the identity of $G_\iota\}$. Plainly I_1 is countably infinite. Let $I_2 = I \backslash (I_0 \cup I_1)$. Let $G_j = \prod_{\iota \in I_j} G_\iota$, and let G_j have character group X_j, $j = 0, 1, 2$. Since I_1 is countable, G_1 is metrizable. Since I_0 is finite, G_0 is finite. Let Γ_0 be the image of the projection of $\tilde{\Delta}_0$ onto X_1. We may suppose that our neighborhood of the identity of G has the form $U = \{e_0\} \times V_1 \times V_2$, where e_0 is the identity of G_0 and V_j is open in G_j, $j = 1, 2$. Applying Theorem 4.3 to k, G_1, Γ_0, and a, we find a subset P_1 of V_1 homeomorphic to the Cantor set which is a K_{a,Γ_0}-set. Let $P = \{e_0\} \times P_1 \times \{e_2\}$, where e_2 is the identity of G_2. Then P is a $K_{a,\Delta}$-set in U homeomorphic to the Cantor set.

Proof of Theorem III. 4.8. If G is a compact torsion group, then there are integers $r_1, \cdots, r_q$ greater than one and disjoint infinite index sets $I_1, \cdots, I_q$ and there is a finite abelian group F such that G is topologically isomorphic to $F \times G_1 \times \cdots \times G_q$, where $G_j = \prod_{\iota \in I_j} K_\iota$ and each K_ι is a copy of $\mathbf{T}_{(r_j)}$ when $\iota \in I_j$ $(1 \leqq j \leqq q)$. Let G_j have character group X_j $(1 \leqq j \leqq q)$. Then for some j_0, the image Γ of the projection of Δ onto X_{j_0} is infinite. Let a be as in Theorem 4.7 applied to G_{j_0}, X_{j_0}, and Γ. Let U be a neighborhood of the identity of G. We will prove that U contains a $K_{a,\Delta}$-set homeomorphic to the Cantor set. Clearly, this will establish Theorem III. We may suppose that U has the form $\{e_F\} \times U_1 \times \cdots \times U_q$, where e_F is the identity of F and U_j is a neighborhood of the identity e_j of G_j $(1 \leqq j \leqq q)$. By Theorem 4.7, U_{j_0} contains a $K_{a,\Gamma}$-set P_{j_0} homeomorphic to the Cantor set. Let

$$P = \{e_F\} \times \{e_1\} \times \cdots \times \{e_{j_0-1}\} \times P_{j_0} \times \{e_{j_0+1}\} \times \cdots \times \{e_q\} .$$

Then P is a $K_{a,\Delta}$-set in U homeomorphic to the Cantor set.

5. Examples.

5.1. The hypothesis that $\bar{\Delta}$ is not compact is necessary in Theorem II. If $\bar{\Delta}$ is compact, then there is a nonempty open $U \subset G$ which contains no $K_{0,\Delta}$-set and no $K_{a,\Delta}$-set for any integer $a \geqq 2$. Indeed, let $U = \{x \in G: |\gamma(x) - 1| < 1$ for all $\gamma \in \bar{\Delta}\}$. Then U is an open neighborhood of the identity in G and $\operatorname{Re} \gamma(x) > 0$ for all $x \in U$ and all $\gamma \in \Delta$. Hence, the function -1 cannot be matched within 1 on any nonvoid subset of U by any $\gamma \in \Delta$, nor can the function ω_a (where ω_a is an ath root of unity with $\operatorname{Re} \omega_a < 0$) be matched on any nonvoid subset of U by any $\gamma \in \Delta$ for any integer $a \geqq 2$. Hence, no subset of U is a $K_{0,\Delta}$-set or a $K_{a,\Delta}$-set.

5.2. The phrase "a translate of" is a necessary part of the conclusion of Theorem III, as is shown by the following example. Let $G = \mathbf{T}_{(2)} \times H$, where H is the product of infinitely many copies of $\mathbf{T}_{(3)}$. Write $X = \mathbf{Z}_2 \times Y$, where Y is the character group of H. Let $\Delta = \{1\} \times Y$. Let $U = \{-1\} \times H$. Then U is open in G and $\gamma(x) \in -\mathbf{T}_{(3)}$ for all $x \in U$ and all $\gamma \in \Delta$, so the constant function 1 cannot be matched on any subset of U by any $\gamma \in \Delta$. Hence, no subset of U is a $K_{a,\Delta}$-set for any integer $a \geqq 2$.

5.3. The hypothesis that G is a compact torsion group in Theorem III cannot be weakened to the hypothesis that G is compactly generated and contains a compact open torsion subgroup. For example, let H be an infinite compact torsion group and let $G = \mathbf{Z} \times H$. Take $\Delta = \mathbf{T} \times \{e\}$ (where e is the identity of the character group of H) and $U = \{0\} \times H$. Then $\gamma(x) = 1$ for all $x \in U$ and all $\gamma \in \Delta$. Hence, whenever $P \subset G$ is such that a translate of P is contained in U, we have γ constant on P. Therefore, no such totally disconnected P containing more than one point can be a $K_{a,\Delta}$-set for any integer $a \geqq 2$.

5.4. The hypothesis of local connectedness or something closely related to connectedness (cf. Theorem 2.1) in Theorems II and I respectively cannot be weakened to the hypothesis that G is not a torsion group. Indeed, there exist a compact metrizable group G which is not a torsion group and an infinite subset Δ of X such that G contains no $K_{0,\Delta}$-set. For example, let $G = \prod_{j=2}^{\infty} \mathbf{T}_{(2j)}$. Then, writing $X = \prod_{j=2}^{*\infty} \mathbf{Z}_{2j}$ and letting $\Delta = \{\gamma_2, \gamma_3, \cdots\}$ where γ_j has jth coordinate equal to j and the rest zero, we have $\gamma_j(x) = \pm 1$ for all $x \in G$ and all j, so every nonempty subset of G fails to be a $K_{0,\Delta}$-set.

Also, there exist a compact metrizable group G which is not a torsion group and an infinite subset Δ of X such that no subset of G containing more than one point is a $K_{a,\Delta}$-set for any integer $a \geqq 2$. Let $G = \prod_{j=1}^{\infty} \mathbf{T}_{(p_j)}$ where p_j is the jth prime. Write $X = \prod_{j=2}^{*\infty} \mathbf{Z}_{p_j}$ and let $\Delta = \{\gamma_1, \gamma_2, \cdots\}$ where γ_j has jth coordinate equal to 1 and the rest zero. Let P be a subset of G containing at least two points. Let $a \geqq 2$ be an integer. We will show that P is not a $K_{a,\Delta}$-set. Let p_k be a divisor of a. The open-closed sets in G form a basis for the topology of G, so there are two distinct $\mathbf{T}_{(p_k)}$-valued (and, hence, $\mathbf{T}_{(a)}$-valued) continuous functions, f_1 and f_2, on P both different from 1. If either f_i is matched on P by some γ_j, it must be matched by γ_k since no other γ_j attains values in $\mathbf{T}_{(p_k)}$ different from 1. Thus either f_1 or f_2 is a $\mathbf{T}_{(a)}$-valued continuous function not matched on P by any γ_j. Hence, P is not a $K_{a,\Delta}$-set.

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Received August 25, 1971 and in revised form April 25, 1972. The author was a National Science Foundation Graduate Fellow, 1969-1971. The results in this paper are taken from the author's doctoral dissertation, which was written at the University of Washington under the direction of Professor Edwin Hewitt.

UNIVERSITY OF WASHINGTON

Present address: California State College, Dominguez Hills, 90747