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EXTENSIONS OF INEQUALITIES OF THE LAGUERRE AND TURÁN TYPE

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It is shown that

$$\sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!}\binom{2k}{j} F^{(n+j)}(z)F^{(n+2k-j)}(z) \geqq 0 \,,$$

for $-\infty < z < \infty, n \geqq 1$ **and** $k \geqq 0$, **where** $F(z)$ **is an entire function of a special type. For** $k = 1$ **this simply is the well known Laguerre inequality**

$$(F^{(n+1)}(z))^2 - F^{(n)}(z)F^{(n+2)}(z) \geqq 0$$

$-\infty < z < \infty, n \geqq 0$. **From these inequalities we obtain the inequalities**

$$\sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!}\binom{2k}{j} u_{n+j}(x)u_{n+2k-j}(x) \geqq 0$$

which hold for such values of x, **for which the functions** $u_n = u_n(x)$ **have a generating function of the type**

$$\sum_{n=0}^{\infty} u_n \frac{z^n}{n!} = F(z) \,.$$

1. Introduction. Consider the entire function of the form

$$F(z) = Ce^{-\alpha z^2+\beta z}z^r \prod_m (1 - z/z_m)e^{z/z_m} \tag{1.1}$$

where $\alpha \geqq 0$, C, β and all z_m are real, and $\sum_m z_m^{-2}$ is convergent. These functions have been studied by Laguerre (see Borel [1; pp. 32-47]), Pólya and Schur [9], and others. Apart from constant factors, these functions are the only ones which are limits of polynomials with real zeros only.

In § 2 we show, for integers $n \geqq 0$, that

$$|F^{(n)}(x + iy)|^2 = \sum_{k=0}^{\infty} L_k(F^{(n)}; x)y^{2k} \tag{1.2}$$

where

$$L_k(F^{(n)}; x) = \sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!}\binom{2k}{j} F^{(n+j)}(x)F^{(n+2k-j)}(x) \tag{1.3}$$

and $F^{(n)}(z)$ denotes the nth derivative of $F(z) = F^{(0)}(z)$. Furthermore, we show that

$$L_k(F^{(n)}; x) \geqq 0 \tag{1.4}$$

for each $n \geqq 0$, $k = 0, 1, 2, \cdots$, and all real x. Patrick [8] obtained inequality (1.4) for polynomials with real zeros only. For $k = 1$, (1.4) is the well known and useful Laguerre inequality.

$$(1.5) \quad (F^{(n+1)}(z))^2 - F^{(n)}(z)F^{(n+2)}(z) \geqq 0, \quad -\infty < z < \infty, n \geqq 0 .$$

Finally, in § 3, we use (1.4) to obtain the inequalities

$$(1.6) \qquad \sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!} \binom{2k}{j} u_{n+j}(x) u_{n+2k-j}(x) \geqq 0$$

which holds for such values of x, for which the functions $u_n = u_n(x)$ have a generating function of the type

$$(1.7) \qquad \sum_{n=0}^{\infty} u_n \frac{z^n}{n!} = F(z)$$

where $F(z)$ is of the form (1.1).

For $k = 1$, (1.6) is the interesting and much studied Turán inequality

$$(1.8) \qquad (u_{n+1}(x))^2 - u_n(x)u_{n+2}(x) \geqq 0, n \geqq 0 .$$

In 1948 Szegö [13] called attention to (1.8) for Legendre polynomials. Since that time the inequality has been studied for other special functions by Mukherjee and Nanjundiah [6]; Nanjundiah [7]; Szász [11], [12]; Thiruvenkatachar and Nanjundiah [14]; Venkatachaliengar and Rao [15], and others. These authors have shown that the functions which satisfy (1.8) include the Legendre, Tchebychef, ultraspherical, Hermite and Laguerre polynomials and the Bessel and modified Bessel functions. A summary of these results are contained in Skovgaard [10]. Also in [10], Skovgaard uses the Laguerre inequality (1.5) to obtain the Turán inequality for sequences $u_n = u_n(x)$, $n = 0, 1, 2, \cdots$ satisfying (1.7). He shows that for such sequences the Turán inequality is a special case of the Laguerre inequality. Then he is able to easily prove the inequality (1.8) for most of the above mentioned functions. These results served as motivation for the work which resulted in this paper.

Some other papers containing results related to the Turán type expressions or inequalities are Carlitz [2]; A. E. Danese [3], [4]; Karlin and Szegö [5]; and Webster [16].

2. "Extended" Laguerre inequalities. We now prove the following theorem concerning entire functions of the form (1.1).

THEOREM 1. *Let $F(z) = F(x + iy)$ be an entire function of the* *form* (1.1) *and let* $F^{(n)}(z)$ *denote the* nth *derivative of* $F(z)$. *Then for any integer* $n \geqq 0$ *and any* $z = x + iy \neq \infty$

$$|F^{(n)}(x+iy)|^2 = \sum_{k=0}^{\infty} L_k(F^{(n)}; x) y^{2k} \tag{2.1}$$

where

$$L_k(F^{(n)}; x) = \sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!} \binom{2k}{j} F^{(n+j)}(x) F^{(n+2k-j)}(x) \ . \tag{2.2}$$

Furthermore

$$L_k(F^{(n)}; x) \geqq 0 \qquad \textit{for } k = 0, 1, 2, \cdots. \tag{2.3}$$

Proof. We first prove (2.1)-(2.2) for $n = 0$. Since $F^{(0)}(z) = F(z)$ is an entire function we can write for any z

$$F(z) = F(x+iy) = \sum_{k=0}^{\infty} A_k(F; x) y^k \tag{2.4}$$

and

$$F(\bar{z}) = F(x-iy) = \sum_{k=0}^{\infty} (-1)^k A_k(F; x) y^k \tag{2.5}$$

where

$$A_k(F; x) = \frac{i^k}{k!} F^{(k)}(x) \ , \qquad k = 0, 1, 2, \cdots. \tag{2.6}$$

Since C, α, β, and all z_m are real, $\overline{F(z)} = F(\bar{z})$ and so from (2.4) and (2.5) we have

$$|F(x+iy)|^2 = \sum_{m=0}^{\infty} C_m(F; x) y^m \tag{2.7}$$

where

$$\begin{aligned} C_m(F; x) = {} & (-1)^m A_0(F; x) A_m(F; x) + (-1)^{m-1} A_1(F; x) A_{m-1}(F; x) \\ & + \cdots + (-1) A_{m-1}(F; x) A_1(F; x) + A_m(F; x) A_0(F; x) \ . \end{aligned} \tag{2.8}$$

When m is odd, it follows from (2.6) and (2.8) that

$$\begin{aligned} & C_m(F; x) \\ & = \sum_{j=0}^{M} \left(\frac{i^j}{j!} \cdot \frac{(-1)^{m-j} i^{m-j}}{(m-j)!} + \frac{i^{m-j}}{(m-j)!} \cdot \frac{(-1)^j i^j}{j!} \right) F^{(j)}(x) F^{(m-j)}(x) \\ & = \sum_{j=0}^{M} \frac{(-1)^j i^m}{j(m-j)!} [(-1)^m + 1] F^{(j)}(x) F^{(m-j)}(x) = 0 \end{aligned} \tag{2.9}$$

where $M = [(m+1)/2]$. When m is even, say $m = 2k$, we have from (2.6) and (2.8)

$$C_{2k}(F; x) = \frac{(F^{(k)}(x))^2}{(k!)^2} + \sum_{j=0}^{k-1} 2\frac{(-1)^{j+k}}{j!(2k-j)!} F^{(j)}(x) F^{(2k-j)}(x) . \tag{2.10}$$

It follows easily that $L_k(F^{(n)}; x)$ in (2.2) with $n = 0$ is equal to $C_{2k}(F; x)$ in (2.10). From (2.7), (2.9), and (2.10) we have the desired result (2.1)-(2.2) for $n = 0$.

We now prove (2.1)-(2.2) for $n \geqq 1$. Skovgaard [10] shows that the derivative of a function of the form (1.1) is also of the form (1.1). Therefore, all derivatives $F^{(n)}(z)$ of $F(z)$ are of the form (1.1) and (2.1)-(2.2) for any $n \geqq 1$ follows from the proof of the $n = 0$ case by replacing $F(z)$ by $F^{(n)}(z)$.

Next we prove (2.3) for $n = 0$. Using (1.1) we have

$$|F(z)|^2 = C^2 e^{-\alpha(z^2+\bar{z}^2)+\beta(z+\bar{z})}(z\bar{z})^r \prod_m (1 - z/z_m)(1 - \bar{z}/z_m) e^{z+\bar{z}/z_m} \tag{2.11}$$

or

$$\begin{aligned} &|F(x+iy)|^2 \\ &= C^2 e^{-2\alpha x^2 + 2\beta x} e^{2\alpha y^2} (x^2 + y^2)^r \prod_m [(1 - x/z_m)^2 + y^2/z_m] e^{2x/z_m} . \end{aligned} \tag{2.12}$$

Since e^{2x/z_m} is real and positive for all x we can write

$$\prod_m [(1 - x/z_m)^2 + y^2/z_m^2] e^{2x/z_m} = \sum_{k=0}^{\infty} C_k(x) y^{2k}$$

where $C_k(x) \geqq 0$ for $k = 0, 1, \cdots$, and all x. Also $\alpha \geqq 0$, $C^2 > 0$, $e^{-2\alpha x^2 + \beta x} > 0$ for all x, $(x^2 + y^2)^r$ is a polynomial in y^2 with positive coefficients that are functions of x, and $e^{2\alpha y^2}$ is an absolutely convergent series in y^2 with positive coefficients. Combining all of these facts with (2.12) we have for any $z = x + iy \neq \infty$.

$$|F(x+iy)|^2 = \sum_{k=0}^{\infty} F_k(x) y^{2k} \tag{2.13}$$

where

$$F_k(x) \geqq 0 \quad \text{for} \quad k = 0, 1, 2, \cdots.$$

But (2.1) and (2.13) are identical so we have $L_k(F; x) = F_k(x) \geqq 0$ for $k = 0, 1, 2, \cdots$, and (2.3) is proved for $n = 0$.

As in the proof of (2.1)-(2.2), the proof of (2.3) for $n \geqq 1$ follows immediately by replacing $F(z)$ by $F^{(n)}(z)$ in the above considerations.

For the sake of better understanding of the functions $L_k(F^{(n)}; x)$ we note that

$$L_0(F^{(n)}; x) = (F^{(n)}(x))^2 \tag{2.14}$$

$$L_1(F^{(n)}; x) = (F^{(n+1)}(x))^2 - F^{(n)}(x)F^{(n+2)}(x) \tag{2.15}$$

$$\begin{aligned} L_2(F^{(n)}; x) = \frac{(F^{(n+2)}(x))^2}{(2!)^2} - \frac{2F^{(n+1)}(x)F^{(n+3)}(x)}{1!\ 3!} \\ + \frac{2F^{(n)}(x)F^{(n+4)}(x)}{4!} \end{aligned} \tag{2.16}$$

and, in general,

$$\begin{aligned} L_k(F^{(n)}; x) = \frac{(F^{(n+k)}(x))^2}{(k!)^2} - \frac{2F^{(n+k-1)}(x)F^{(n+k+1)}(x)}{(k-1)!\ (k+1)!} \\ + \cdots + (-1)^k \frac{2F^{(n)}(x)F^{(n+2k)}(x)}{(2k)!}. \end{aligned} \tag{2.17}$$

We further note that (2.15) is the well-known Laguerre expression and that (2.3) with $k = 1$ the corresponding Laguerre inequality. For this reason we call (2.3) the "extended" Laguerre inequalities.

In [8] we show, among other things, that $0 \leqq L_j(F^{(n)}; x) \leqq L_j(F^{(n)}; 1)$, $j = 0, 1, 2, 3$, for $-1 \leqq x \leqq 1$ when $F(z)$ is restricted to the class of ultraspherical polynomials. We conjecture that a similar inequality holds for $j > 4$ over the same class of functions.

3. "Extended" Turán inequalities. Using Theorem 1 of § 2 and the results of Skovgaard [10] we can easily prove the following theorem.

THEOREM 2. *Let $u_n = u_n(x)$, $n = 0, 1, 2, \cdots$, be a system of real functions which for certain values of x have a generating function of the type*

$$\sum_{n=0}^{\infty} u_n \frac{z^n}{n!} = F(z) \tag{3.1}$$

where $F(z)$ is of the form (1.1). *Then for those values of x*

$$\sum_{j=0}^{2k} \frac{(-1)^{j+k}}{(2k)!} \binom{2k}{j} u_{n+j}(x) u_{n+2k-j}(x) \geqq 0 \quad \text{for } k = 0, 1, 2, \cdots . \tag{3.2}$$

Proof. As indicated by Skovgaard [10], we have for those values of x where $u_n = u_n(x)$ satisfies (3.1) that $u_n = F^{(n)}(0)$. That is for such x inequality (3.2) is a special case of "extended" Laguerre inequality (2.3).

We note for $k = 1$ that (3.2) is the interesting Turán inequality

$$u_{n+1}^2(x) - u_n(x)u_{n+2}(x) \geqq 0 . \tag{3.3}$$

For this reason we call the inequalities (3.2) the "extended" Turán inequalities.

Skovgaard used (2.3) with $k = 1$ to obtain (3.2) with $k = 1$ for certain classical functions by indicating their generating functions of the type (1.1). These same generating functions, since they are of type (1.1), will, by Theorem 1, satisfy (2.3) for $k = 0, 1, 2, \cdots$. Therefore, by Theorem 2, the corresponding set of classical functions will satisfy the "extended" Turan inequalities (3.2). The following is a list of these classical functions:

(3.4) Ultraspherical polynomials

$$P_n^{(\lambda)}(x),\ -1 \leqq x \leqq 1,\ \lambda \geqq \frac{1}{2}\ ,$$

(3.5) pth derivative of $P_n^{(\lambda)}(x)$

$$D^p P_n^{(\lambda)}(x),\ p \leqq n,\ -1 \leqq x \leqq 1,\ \lambda \geqq \frac{1}{2} - p\ ,$$

(3.6) Laguerre polynomials

$$L_n^{(\alpha)}(x),\ -\infty < x < \infty,\ \alpha > -1\ ,$$

(3.7) pth derivative of $L_n^{(\alpha)}(x)$

$$D^p L_n^{(\alpha)}(x),\ p \leqq n,\ -\infty < x < \infty,\ \alpha \geqq -p\ ,$$

(3.8) Hermite polynomials

$$H_n(x),\ -\infty < x < \infty\ ,$$

(3.9) pth derivative of $H_n(x)$

$$D^p H_n(x),\ -\infty < x < \infty\ ,$$

(3.10) Sine and cosine functions

$$\begin{Bmatrix} \cos \\ \sin \end{Bmatrix} nx,\ -\infty < x < \infty\ ,$$

(3.11) Bessel functions

$$J_n(x),\ -\infty < x < \infty\ ,$$

(3.12) Derivative of Bessel function

$$J_n'(x),\ -\infty < x < \infty\ ,$$

(3.13) Tchebychef function of first kind

$$T_n(x), \ -1 \leqq x \leqq 1 ,$$

(3.14) Tchebychef function of second kind

$$U_n(x), \ -1 \leqq x \leqq 1 .$$

We point out that (3.4) and (3.5) includes Legendre polynomials and their derivatives.

Generating functions for (3.4), (3.6), (3.8), (3.10), (3.11), and (3.12) can be found in Skovgaard [10]. As in Skovgaard's paper we have analogously, since

$$D^p P_n^{(\lambda)}(x) = 2^p \frac{\Gamma(\lambda + p)}{\Gamma(\lambda)} P_{n-p}^{(\lambda+p)}(x), \ p \leqq n ,$$

that (3.5) satisfies (3.2) because (3.4) does. Also

$$D^p L_n^{(\alpha)}(x) = (-1)^p L_{n-p}^{(\alpha+p)}(x), \ p \leqq n ,$$

so (3.7) satisfies (3.2) because (3.6) does. Similarly

$$D^p H_n(x) = 2^p \frac{n!}{(n-p)!} H_{n-p}(x)$$

and (3.9) satisfies (3.2) because (3.8) does. Finally, (3.13) and (3.14) satisfy (3.2) because (3.10) does.

Using an inductive proof we showed that if $F^{(n)}(z)$ is of the form

$$F^{(n)}(z) = Ce^{\beta z} \tag{3.15}$$

then

$$L_k(F^{(n)}; z) = 0 \tag{3.16}$$

for $k = 1, 2, \cdots$. Also, (3.16) holds at multiple zeros of order $k + 1$ of $F^{(n)}(z)$. Consequently, given the assumptions of Theorem 2, equality in (3.2) holds for values of x for which $F^{(n)}(z)$ is of the form (3.15), or for which $z = 0$ is a zero of multiplicity $k + 1$ of $F^{(n)}(z)$.

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