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THREE REMARKS ON SYMMETRIC PRODUCTS AND SYMMETRIC MAPS

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The first remark establishes that the homotopy type of a certain space related to the m-fold symmetric product SP^mS^n of the n-sphere is that of an n^{th} suspension space. Remark two generalizes a well-known adjunction formula for SP^2S^n due to Steenrod to a filtration of length m of SP^mS^n. The final remark provides a group-theoretic construction of G-maps $f\colon (S^n)^m \to S^n$ where $G \subset S(m)$ acts on $(S^n)^m$ by permutation of its factors.

1. Joins. The join $X * Y$ of X and Y is the quotient space $X \times Y \times I/\sim$ where $(x, y, 0) \sim (x, y', 0)$ and $(x, y, 1) \sim (x', y, 1)$ for all $x, x' \in X$ and all $y, y' \in Y$. Let (D, S) denote the unit disc and sphere in euclidean n-space R^n with its usual inner product

$$\langle x, y \rangle = \Sigma x_i y_i .$$

For any decomposition $R^n = W_1 \oplus W_2$ of R^n into the direct sum of a k-dimensional subspace W_1 and its orthogonal complement $W_2 = W_1^\perp$, let $D_i = D \cap W_i$ and $S_i = S \cap W_i$, $i = 1, 2$, be the associated discs and spheres. As well known the map $f\colon D_1 * S_2 \to D$ given by

$$f\,[sx, y, t] = s\sqrt{1-t}\,x + \sqrt{t}\,y$$

defines a homeomorphism of pairs

$$(1) \qquad (D_1 * S_2,\ S_1 * S_2) \cong (D, S) .$$

Give $V_i = R^n$, $i = 1, \cdots, n$, its usual inner product $\langle\ ,\ \rangle_i$. Then the formula $\langle x, y \rangle = \Sigma \langle x_i, y_i \rangle_i$ for $x = (x_1, \cdots, x_n)$, $y = (y_1, \cdots, y_n)$ in $V = V_1 \times \cdots \times V_m \cong R^{nm}$ coincides with the usual inner product on R^{nm}. So we may apply the preceding remarks to the diagonal subspace $W_1 = \{v \in V \mid v_1 = v_2 = \cdots = v_m\}$ and its orthogonal complement $W_2 = \{v \in V \mid \Sigma v_i = 0\} = W_1^\perp$. The full symmetric group $S(m)$ acts on V by permutation of its factors V_i. For any subgroup H of $S(m)$ D_i and S_i are H-spaces and f an H-map inducing another homeomorphism of pairs

$$(2) \qquad (D_1/H * S_2/H,\ S_1/H * S_2/H) \cong (D/H, S/H) .$$

As H acts trivially on W_1, $D_1/H = \bar{D}_1$ and $S_1/H = \bar{S}_1$ are again the disc and sphere. Moreover, for subgroups $H_1 \subset H_2$ of $S(m)$ it is easily checked that the quotient map $p\colon D/H_1 \to D/H_2$ corresponds via

(2) to the join map

$$id * p_2 \colon \bar{D}_1 * (S_2/H_1) \longrightarrow \bar{D}_1 * (S_2/H_2) .$$

Recall from [7] the definition of the spaces $X^n_{m,l}$ which appear in the geometry of the symmetric product $SP^m S^n$. Let $h_\tau \colon (D^n)^m \to (D^n)^m$ be the permutation homeomorphism defined by $\tau \in S(m)$, and set

$$A^n_{m,l} = (D^n)^{m-l} \times (S^{n-1})^l \qquad \text{for } 0 \leqslant l \leqslant m .$$

Then

$$\widetilde{X}^n_{m,l} = \bigcup_{\tau \in S(m)} h_\tau(A^n_{m,l})$$

is an $S(m)$-subspace of $(D^n)^m$ and so $X^n_{m,l} = \widetilde{X}^n_{m,l}/S(m)$ is well defined. To identify the pairs $(D/S(m), S/S(m))$ and $(X^n_{m,0}, X^n_{m,1})$ we make the following change of norms: let $V' = V$ as sets but set

$$\| v \|' = \max_i \| v_i \|_i$$

where $\| v_i \|_i = \langle v_i \cdot v_i \rangle_i^{1/2}$. Then $x \to (\| x \| / \| x \|') \cdot x$ defines a norm preserving (non-linear) $S(m)$-homeomorphism $V \to V'$ establishing the desired result $(D/S(m), S/S(m)) \cong (X^n_{m,0}, X^n_{m,1})$. Thus

$$(3) \qquad (\bar{D}_1 * (S_2/S(m)),\ \bar{S}_1 * (S_2/S(m))) \quad \text{and} \quad (X^n_{m,0}, X^n_{m,1})$$

are homeomorphic pairs. Moreover the canonical map $D^n \times X^n_{m-1,0} \to X^n_{m,0}$ is the quotient map $(D^n)^m/S(m-1) \to (D^n)^m/S(m)$ induced by the inclusion homomorphism $S(m-1) \to S(m)$ sending $S(m-1)$ onto the subgroup $\{e\} \times S(m-1)$ of $S(m)$ which acts on $(D^n)^m = D^n \times (D^n)^{m-1}$ by the identity on the first factor and by the usual symmetric action on the second factor. Combining the remark of the preceding paragraph with the above $S(m)$-homeomorphism $V \to V'$ we see that the canonical map $D^n \times X^n_{m-1,0} \to X^n_{m,0}$ corresponds to the join map

$$\bar{D}_1 * (S_2/S(m-1)) \longrightarrow \bar{D}_1 * (S_2/S(m)) .$$

Our first remark establishes a conjecture stated in [7].

PROPOSITION 1.1. *$X^n_{m,m-1}/X^n_{m-1,m-2}$ has the homotopy type of a space of the form $S^{n-1} * K$ for K a suitable finite CW complex. Hence $X^n_{m,m-1}/X^n_{m-1,m-2}$ has the homotopy type of an n^{th} suspension.*

Proof. Proposition 2.6 of [7] asserts the existence of a homotopy equivalence

$$X^n_{m,m-1}/X^n_{m-1,m-2} \sim EX^{n-1}_{m,1} \bigcup_{E\psi} C(E(X^{n-1}_{1,1} * X^{n-1}_{m-1,1}))$$

where ψ is given by the canonical maps

$$X_{1,1}^{n-1} \times X_{m-1,0}^{n-1} \longrightarrow X_{m,1}^{n-1}, \quad X_{1,0}^{n-1} \times X_{m-1,1}^{n-1} \longrightarrow X_{m,1}^{n-1} .$$

ψ is just the restriction of the canonical map

$$\psi' \colon X_{1,0}^{n-1} \times X_{m-1,0}^{n-1} \longrightarrow X_{m,0}^{n-1}$$

which, as we have already noted, can be identified with the join map $id * \bar{\psi}' \colon \bar{D}_1 * (S_2/S(m-1)) \longrightarrow \bar{D}_1 * (S_2/S(m))$. Under this identification the subspaces $X_{1,1}^{n-1} \times X_{m-1,0}^{n-1} \bigcup X_{1,0}^{n-1} \times X_{m-1,1}^{n-1}$ and $X_{m,1}^{n-1}$ correspond to $\bar{S}_1 * (S_2/S(m-1))$ and $\bar{S}_1 * (S_2/S(m))$, and so the map ψ corresponds to the join map $id * \bar{\psi} \colon \bar{S}_1 * (S_2/S(m-1)) \to \bar{S}_1 * (S_2/S(m))$ which is the restriction of $id * \bar{\psi}'$. Hence there is a homotopy equivalence

$$\begin{aligned} X_{m,m-1}^{n}/X_{m-1,m-2}^{n} &\sim E((E^{n-1}(S_2/S(m))) \bigcup_{E^{n-1}\bar{\psi}} C(E^{n-1}X_{m-1,1}^{n-1})) \\ &\sim E^{n}((S_2/S(m)) \bigcup_{\bar{\psi}} CX_{m-1,1}^{n-1} \end{aligned}$$

and the result is proved.

For p-fold cyclic products (p any prime) there is an analogous result to 1.1 whose proof differs only slightly from the preceding. For this let now $h_\tau \colon (D^n)^p \to (D^n)^p$ be the (cyclic) permutation homeomorphism defined by $\tau \in Z_p \subset S(p)$ and set $A_{p,l}^n = (D^n)^{p-l} \times (S^{n-1})^l$. Then as before

$$\widetilde{X}_{p,l}^n = \bigcup_{\tau \in Z_p} h_\tau(A_{p,l}^n)$$

is a Z_p-space and $X_{p,l}^n = \widetilde{X}_{p,l}^n/Z_p$ is well defined. Let $W_l^{n-1} \subset (S^{n-1})^l$ be the subspace $\{x \in (S^{n-1})^l \mid x_i = \text{basepoint for some } i\}$, $\widetilde{Z}_{p,l}^n = (D^n)^{p-l} \times W_l^{n-1}$ and $Z_{p,l}^n$ the image of $\widetilde{Z}_{p,l}^n$ under the canonical projection

$$(D^n)^{p-l} \times (S^{n-1})^l \longrightarrow X_{p,l}^n .$$

Then $Z_{p,p-1}^n \subset X_{p,p-1}^n$ and formula (3.3) of [8] asserts the existence of a homeomorphism

$$X_{p,p-1}^n/Z_{p,p-1}^n \cong EX_{p,1}^{n-1} \cup e^{np-p+1} . \tag{4}$$

The top cell e^{np-p+1} arises from the product $D^n \times (D^{n-1})^{p-1}$ and the attaching map of (4) $S^{n-1} \times (D^{n-1})^{p-1} \cup D^n \times \partial[(D^{n-1})^{p-1}] \to EX_{p,1}^{n-1}$ sends the contractible subspace

$$A = S^{n-1} \times \partial[(D^{n-1})^{p-1}] \cup \text{point} \times (D^{n-1})^{p-1}$$

to the basepoint of $EX_{p,1}^{n-1}$ and so factors as $(E\psi) \circ p$, where p is the collapsing homotopy equivalence

$$S^{n-1} * \partial\,[(D^{n-1})^{p-1}] \longrightarrow S^{n-1} * \partial\,[(D^{n-1})^{p-1}]/A$$

and ψ the canonical projections

$$S^{n-2} \times (D^{n-1})^{p-1} \longrightarrow X^{n-1}_{p,1}\,, \qquad D^{n-1} \times \partial\,[(D^{n-1})^{p-1}] \longrightarrow X^{n-1}_{p,1}$$

(see the proof of Prop. 2.6 of [7] with Z_p replacing $S(m)$). By the above $X^{n-1}_{p,1}$ is homeomorphic to the join $S_1 * (S_2/Z_p) - V$ is now $R^{np}-$ and the map ψ can be identified via this homeomorphism with the join map $id * p$: $S_1 * S_2 \to S_1 * (S_2/Z_p)$. Thus we obtain the analogous result to 1.1.

PROPOSITION 1.2. *For the p-fold cyclic product of spheres the space $X^n_{p,p-1}/Z^n_{p,p-1}$ has the homotopy type of a space of the form $S^{n-1} * K$ for K a suitable finite CW complex. Hence $X^n_{p,p-1}/Z^n_{p,p-1}$ has the homotopy type of an n^{th} suspension.*

Application of 1.2 was made in [8].

Consider again the symmetric product situation. Lemma 2.5 (iii) of [7] provides a homeomorphism

$$X^n_{m,l}/X^n_{m-1,l-1} \cong (X^n_{m,l+1}/X^n_{m-1,l}) \cup C(X^n_{m-l,1} * X^{n-1}_{l,1})\,.$$

For $l = m$ and $l = m - 1$ the spaces $X^n_{m,l}/X^n_{m-1,l-1}$ have now been shown to have the homotopy type of a space of the form $S^{n-1} * K$. It seems reasonable to expect the same to be true for the remaining values of l, $2 \leqslant l \leqslant m-2$.

2. Geometry of SP^mEX. Our second remark extends the Steenrod adjunction formula [3]

$$SP^2S^n \cong E(SP^2S^{n-1}) \cup e^{2n}$$

to higher symmetric products SP^mEX of suspension spaces. Let

$$\begin{aligned} I^n &= \{x \in R^n \mid 0 \leqslant x_i \leqslant 1\} \\ A_n &= \{x \in I^n \mid x_n = 1\} \\ T_n &= \{x \in I^n \mid x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n\} \\ p &= \{(1, 1, \cdots, 1)\} \subset I^n\,. \end{aligned}$$

For $i = 1, 2, \cdots, n-1$ define f_i: $I^n \to I^n$ by $f_i(t_1, \cdots, t_n) = (t'_1, \cdots, t'_n)$ where $t'_j = t_j$ if $j \neq i$ and $t'_i = t_{i+1} + t_i(1 - t_{i+1})$. One shows easily that the composite $g_n = f_1 \circ f_2 \circ \cdots \circ f_{n-1}$ defines a relative homeomorphism $(I^n, A_n) \cong (T_n, p)$. The map g_n is useful in studying the quotients A_{i+1}/A_i arising from a filtration

$$SP^m EX = A_m \supset A_{m-1} \supset \cdots \supset A_1 = E(SP^m X) \tag{5}$$

which we define as follows. For $x' = [x, t] \in EX$ call t the *height* of x'. As each element $[x'_1, \cdots, x'_m] \in SP^m EX$ has a representative with heights $t_1 \geqslant t_2 \geqslant \cdots \geqslant t_m$ we can set A_i to be the subset of $SP^m EX$ of all elements having representatives with at most i distinct heights. The A_i define a filtration (5) of $SP^m EX$.

For any partition $\pi = [i_1: i_2: \cdots: i_q]$ of m let $A_{q\pi} \subset A_q$ be the set of all points having representatives with heights $t_1 \geqslant t_2 \geqslant \cdots \geqslant t_q$, i_1 of the m coordinates at height t_1, i_2 of them at height t_2, etc. Set $Y_1 = C(SP^{i_1}X)$, $Y_j = (SP^{i_j}X) \times I$ for $2 \leqslant j < q$, $Y_q = \tilde{C}(SP^{i_q}X)$ and $Y = Y_1 \times \cdots \times Y_q$ where

$$C(Z) = Z \times I/Z \times \{1\} \qquad \text{and} \qquad \tilde{C}(Z) = Z \times I/Z \times \{0\} .$$

Set

$$\partial C(Z) = \{[z, t] \in C(Z) \mid t = 0\},\ \partial \tilde{C}(Z) = \{[z, t] \in \tilde{C}(Z) \mid t = 1\}$$

and

$$\partial(SP^k X \times I) = SP^k X \times \{0, 1\} .$$

This defines ∂Y_i for $i = 1, \cdots, q$. Finally set

$$\partial Y = \bigcup_{i=1}^{q} Y_1 \times \cdots \times \partial Y_i \times \cdots \times Y_q .$$

Clearly ∂ is just a kind of boundary operator for cones and related spaces.

PROPOSITION 2.1. *The map $Y \to SP^m EX$ given by*

$$\begin{aligned}&([x_1, t_1], (x_2, t_2), \cdots, (x_{q-1}, t_{q-1}), [x_q, t_q])\\ &\longrightarrow [[x_1, t''_1], [x_2, t''_2], \cdots, [x_q, t''_q]]\end{aligned}$$

where t''_j is the j^{th} coordinate of $g_q(t_1, \cdots, t_q)$, induces a relative homeomorphism $(Y, \partial Y) \cong (A_{q\pi}, A_{q-1})$ for each $2 \leqslant q < m$ and each partition $\pi = [i_1: \cdots: i_q]$ of m.

The proof is straighforward.

To obtain an expression for A_q/A_{q-1} first observe that

$$A_q/A_{q-1} = \bigvee_{\pi} (A_{q\pi}/A_{q-1}) ,$$

the wedge taken over all partitions of m. Thus by 2.1 it suffices to consider for each π the corresponding quotient $Y/\partial Y$.

PROPOSITION 2.2. *For $\pi = [i_1: \cdots : i_q]$ and $Y = Y_1 \times \cdots \times Y_q$ as above the space $Y/\partial Y$ has the same homotopy type as the space*

$$E^q(SP^{i_1}X \wedge SP^{i_q}X) \vee E^q\left(SP^{i_1}X \wedge \prod_{j=2}^{q-1} SP^{i_j}X \wedge SP^{i_q}X\right).$$

Proof. Let B be a space with $\partial B = \phi$. Then the obvious quotient map $Q = CA \times B \times I^{q-2} \times CA' \to P = EA \times B \times S^{q-2} \times EA'$ induces a relative homeomorphism $(Q, \partial Q) = (P, P')$ where

$$P' = EA \times B \times S^{q-2} \vee EA \times B \times EA' \vee B \times S^{q-2} \times EA'.$$

So $P/P' = B \times (EA \wedge S^{q-2} \wedge EA')/B \times$ point and the latter quotient has the homotopy type of the wedge $E^q(A \wedge A') \vee E^q(A \wedge B \wedge A')$ [5]. Therefore 2.2 is obtained by setting $A = SP^{i_1}X$, $B = \prod_{j=2}^{q-1} SP^{i_j}X$ and $A' = SP^{i_q}X$.

As an illustration of the preceding analysis let us return to the Steenrod formula for the symmetric square of a sphere. The space Y is just $CX \times \tilde{C}X$ and the map $CX \times \tilde{C}X \to SP^2EX$ is

$$([x_1, t_1], [x_2, t_2]) \longmapsto [[x_1, t_2 + t_1(1 - t_2)], [x_2, t_2]].$$

The subspace $X \times \tilde{C}X \cup CX \times X$ (given by $t_1 = 0$ or $t_2 = 1$) is mapped to ESP^2X. It is well known that there are homeomorphisms

$$X \times \tilde{C}X \cup CX \times X \cong X * X$$

and

$$CX \times \tilde{C}X \cong C(X \times \tilde{C}X \cup CX \times X).$$

Hence we obtain the adjunction formula $SP^2EX \cong ESP^2X \cup C(X * X)$ extending the Steenrod result from spheres to suspensions.

REMARK. For $X = S^{n-1}$ 2.2 can be used to recompute Nakaoka's results [4] on the integral cohomology of SP^mS^n for low values of m.

3. Group theoretic construction of symmetric maps.

Let $H \subset G \subset S(m)$ be subgroups of the symmetric group $S(m)$ and let $S(G/H)$ be the symmetric group on the set of right cosets G/H. Define a homomorphism $\alpha: G \to S(G/H)$ by $\alpha(g)(Hg_1) = Hg_1g^{-1}$. Kernel of α is just the normal subgroup $B = \bigcap_{g \in G} gHg^{-1}$ and so there is an injection $G/B \to S(G/H)$. Let A denote the image of α and $|G/H|$ the cardinality of G/H.

PROPOSITION 3.1. *If $v: X^{|G/H|} \to X$ and $w: X^m \to X$ are A and H-maps respectively, then $F: X^m \to X$ given by*

$$F(x) = v(w(g_1 \cdot x), w(g_2 \cdot x), \cdots, w(g_l \cdot x))$$

for $g_1, \cdots, g_l$ *a complete set of coset representatives in* G/H, *is a* G-*map*.

Proof. As w is an H-map we have for any $g \in G$ and any

$$1 \leqslant i \leqslant l = |G/H|$$

the existence of an $h \in H$ and a unique $1 \leqslant j \leqslant l$ such that

$$w(g_i \cdot (g \cdot x)) = w(h \cdot (g_j \cdot x)) = w(g_j \cdot x) ,$$

where h arises from the coset equality $Hg_ig = Hg_j$. Hence there exists an element $\sigma \in S(G/H)$ in $A =$ image (α) satisfying

$$\begin{aligned} F(g \cdot x) &= v(w(g_1 \cdot (g \cdot x)), \cdots, w(g_l \cdot (g \cdot x))) \\ &= v(w(g_{\sigma(1)} \cdot x), \cdots, w(g_{\sigma(l)} \cdot x)) \\ &= v(\sigma \cdot (w(g_1 \cdot x), \cdots, w(g_l \cdot x))) \\ &= v(w(g_1 \cdot x), \cdots, w(g_l \cdot x)) = F(x) . \end{aligned}$$

The result follows.

To compute the James number of F when $X = S^n$ note that the degree of the composite $S^n \xrightarrow{\Delta} (S^n)^m \xrightarrow{F} S^n$ (Δ the diagonal map) equals the product $\deg(v \circ \Delta) \cdot \deg(w \circ \Delta)$, since $F \circ \Delta = v \circ \Delta \circ w \circ \Delta$ as maps. Therefore the James number of F is easily computed from those of v and w via the Künneth formula.

Applications. Let $n = 2t + 1$ in the following four applications.

1. Let $H = \{id, (123), (132)\} \cong Z_3$ so $H \lhd S(3) = G$. Choose v: $(S^n)^{|G/H|} \to S^n$ to be an $S(2)$-map with $J_v = 2^{\phi(2t)}$ [2] and w: $(S^n)^3 \to S^n$ to be an H-map with $J_w = 3^t$ [8]. Then $J_F = 2^{\phi(2t)+1} \cdot 3^t$. However obstruction theory can improve this result as follows. From [7] we know that there exists a map $SP^mS^n \to S^n$ of James number N if and only if the composite $X^n_{m,m-1} \xrightarrow{\phi} S^n \xrightarrow{f_N} S^n$, is nullhomotopic where $\deg f_N = N$. Here ϕ arises from the geometry of SP^mS^n given in [7, § 2]. As $X^n_{2,1} \subset X^n_{3,2}$ the obstructions to extending an $S(2)$-map g_1: $(S^n)^2 \to S^n$ to an $S(3)$-map g: $(S^n)^3 \to S^n$ lie in the groups $H^i(X^n_{3,2}, X^n_{2,1}; \pi_i S^n)$, which by Nakaoka [4] (see also [1], Lemma (4.3)) are 3-primary. Hence there exists an $S(3)$-map G: $(S^n)^3 \to S^n$ with $J_G = 2^{\phi(2t)} \cdot 3^r$ for some r. As the set of all possible James numbers of $S(m)$-maps forms an ideal [1], there must also exist an $S(3)$-map G': $(S^n)^3 \to S^n$ with $J_{G'} = 2^{\phi(2t)} \cdot 3^t$ and so we recover the main result

of [7].

2. Let $H \subset S(4)$ be the subgroup generated by $\{(12), (34), (13)(24)\}$, so $|H| = 8$, $H \not\triangleleft G$ and

$$B = \bigcap_{g \in G} gHg^{-1} = \{id, (14)(23), (13)(24), (12)(34)\} \cong Z_2 \times Z_2 .$$

Hence $|B| = 4$ and $A = S(3)$. Apply 3.1 with v an $S(3)$-map with $J_v = 2^{\phi(2t)} \cdot 3^t$ and w the H-map $(S^n)^4 \xrightarrow{h \circ h^2} S^n$ where $h: (S^n)^2 \to S^n$ is an $S(2)$-map with $J_h = 2^{\phi(2t)}$. Clearly $J_w = 2^{2 \cdot \phi(2t)}$ and so we obtain an $S(4)$-map $F: (S^n)^4 \to S^n$ with $J_F = 2^{3\phi(2t)} \cdot 3^{t+1}$. Now an exactly analogous argument to that of (1) shows that the obstructions to extending an $S(3)$-map $(S^n)^3 \to S^n$ to an $S(4)$-map $(S^n)^4 \to S^n$ lie in the groups $H^i(X^n_{4,3}, X^n_{3,2}; \pi_i S^n)$, which again by Nakaoka are 2-primary. Thus there is an $S(4)$-map of James number $J = 2^r \cdot 2^{\phi(2t)} \cdot 3^t$ for some r. This as above implies the existence of an $S(4)$-map with James number $2^{3\phi(2t)} \cdot 3^t$. Note it is not difficult using K-theory to show that the James number of any $S(4)$-map $(S^n)^4 \to S^n$ must be a multiple of $2^{2t} \cdot 3^t$ (the first named author has improved this bound to $2^{\phi(2t)} \cdot 2^t \cdot 3^t$ via ad hoc considerations).

3. For $G = G^r$ the Sylow p-subgroup of $S(p^r)$ given by the r-fold Wreath product of $G^1 \cong Z_p$ with itself and $H = \prod_{k=1}^{p} G^{r-1} \triangleleft G = G^r$ (see [8, § 2]) we have $G/H \cong Z_p$. Let w be the composite

$$(S^n)^{p^r} \xrightarrow{\pi_1} (S^n)^{p^{r-1}} \xrightarrow{w_1} S^n$$

where w_1 is a G^{r-1}-map with James number J_{w_1} and π_1 is projection onto the first p^{r-1} factors of $(S^n)^{p^r}$; let $v: (S^n)^p \to S^n$ be a Z_p-map with James number J_v. Then $J_F = J_{w_1} \cdot J_v$ where F is given by 3.1. From a Z_p-map h with $J_h = p^t$ [2], this result plus induction on r provides a G^r-map h' with $J_{h'} = p^{rt}$. This iteration of 3.1 applied to the G^1-map h gives precisely the composite G^r-map $h \circ h^p \circ \cdots \circ h^{p^{r-1}}$: $(S^n)^{p^r} \to S^n$.

4. For $G = Z_{mn}$ and $H = Z_n \triangleleft G$ we have $A = Z_m$. In this situation 3.1 provides a G-map F with $J_F = J_w \cdot J_v$ where $w = w_1 \circ \pi_1$ is the composite of a Z_n-map w_1 and projection $\pi_1: X^m \to X^n$. Thus 3.1 provides the construction of the "best" cyclic map of order m from the "best" cyclic maps of prime-power orders occurring in the prime decomposition of m. The latter are studied in [6].

In conclusion we remark that if B is the trivial subgroup, 3.1 provides no useful information at all e.g. $G = S(m)$ for $m \geqslant 5$. Also the appearance of obstruction theory in applications 1 and 2 above indicate the limitations of 3.1. It would appear now from the results of [9] that the most natural approach to constructing $S(m)$-maps of minimal James number is via obstruction theory using [8] and Nakaoka's results relating the cohomology of SP^mS^n to that of iterated cyclic products of spheres.

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