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A BOUNDARY FOR THE ALGEBRAS OF BOUNDED HOLOMORPHIC FUNCTIONS

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Let (X, A) be a ringed space and let D be a domain in X. Let $B = B(D) = \{f \in A(D);\ ||f||_D < \infty\}$. A minimal boundary for B is defined as a unique smallest subset of $\bar{D}$ such that every function in B attains its supremum near the set. The followings are shown: If X is locally compact, D is relatively compact, and B separates the points of D then there exists a minimal boundary. Under the same assumptions, the natural projection of the Silov boundary $\partial_{\hat{B}}$ into X is the minimal boundary. If A is a maximum modulus algebra and the set of frontier points for A is the minimal boundary, then any holomorphic function which is bounded near the minimal boundary must be bounded. Finally, if D is the spectrum of B (with the compact open topology), then the topological boundary of D is the set of frontier points for B.

Introduction. Let (X, A) be a ringed space; a subsheaf of rings with identity of the sheaf of germs of continuous functions on a Hausdorff space X. Let $\Gamma(U, A)$ be the set of all sections of A over U, U is an open subset of X. Let $A(U) = \{f \in C(U): f(x) = \phi(x)(x) = {}_xf(x),\ x \in U\}$, where $\phi \in \Gamma(U, A)$ and ${}_xf$ is the germ of f at x. A function f in $A(U)$ is called A-holomorphic or holomorphic. Let $B(U) = \{f \in A(U): f$ is bounded on $U\}$. Then $B(U)$ is an algebra (over $\boldsymbol{C}$) with identity.

Let D be an open subset of X and let $\bar{D}$ be the closure of D in X. For $\varDelta \subset \bar{D}$ let $N(\varDelta)$ be the filter base of open neighborhoods of $\varDelta$ in X and denote $N_0(\varDelta)$ be the trace of $N(\varDelta)$ on D.

DEFINITION. For $f \in A(D)$, define $\mathrm{cl}\, t_f(\varDelta) = \{\bigcap \mathrm{cl}\, f(W): W \in N_0(\varDelta)\}$, where $\mathrm{cl}\, f(W)$ is the closure of $f(W)$ in the Riemann sphere $\boldsymbol{C} \cup (\infty\}$, the cluster set of f at $\varDelta$, and write $\mathrm{cl}\, t_f(x)$ for $\mathrm{cl}\, t_f(\{x\})$. Define $M_f(\varDelta) = \sup |\mathrm{cl}\, t_f(\varDelta)| \in [0, \infty]$, and write $M_f(x)$ for $M_f(\{x\})$.

Let $B = B(D)$. Denote B_s for B with the topology of supremum norm on D and B_c for B with the topology of uniform convergence on compact subsets of D (c.o. topology). Then B_s is a Banach algebra. Let $S(B_s)$ be the space of nonzero complex homomorphisms of B_s onto $\boldsymbol{C}$ and $S(B_c)$ be the space of nonzero continuous complex homomorphisms of B_c onto $\boldsymbol{C}$. Then $S(B_s) \supset S(B_c)$, for, if $h \in S(B_c)$ then there exists a compact subset K_h of D such that $|h(f)| \leqq ||f||_{K_h}$ for all $f \in B$, which implies $|h(f)| \leqq ||f||_D$ for all $f \in B$, so that $h \in S(B_s)$. Endow $S(B_s)$ with the weakest topology for which each $\hat{f}$ is continuous, where $\hat{f}$ is the Gelfand representation of f on $S(B_s)$ such that $\hat{f}(h) = h(f)$ for all $h \in S(B_s)$. Then $S(B_s)$ is compact. Equip $S(B_c)$ with the relative topology of $S(B_s)$. For $x \in D$ define $h_x(f) = f(x)$ for all $f \in B$ then $h_x \in S(B_s)$, moreover $h_x \in S(B_c)$, since $|h_x(f)| = |f(x)| \leqq \|f\|_K$ for all $f \in B$, where K is a compact subset containing $\{x\}$. Now if B separates the points of D then it separates strongly the points of D (in the sense of [8]), since B contains constant functions. If D is locally compact and B separates the points then the natural embedding ρ of D into $S(B_s)$ is a homeomorphism (See Cor. 3.2.5 of Rickart [8]). Henceforth, we denote ρ for this homeomorphism. Let π be a continuous mapping from $S(B_s)$ into X such that $\pi \mid \rho D$ is the inverse mapping of ρ, so that $\pi \mid \rho D$ is a homeomorphism of ρD onto D.

The prototype of these phenomena is the following: Let D be a relatively compact domain in C^n and $B = B(D)$. Set $S = S(B_s)$. With the coordinate function $z_1, z_2, \cdots, z_n$ in B, define $\pi: S \to C^n$ by $\pi(h) = (\hat{z}_1(h), \cdots, \hat{z}_n(h)), h \in S(\pi(S)$ is the joint spectrum of $z_1, z_2, \cdots, z_n$. Then π is continuous and it is a homeomorphism on ρD. Moreover $\pi s({}_cB) \subset D$ and $\pi S \supset \bar{D}$.

A minimal boundary.

PROPOSITION 1.
(i) $M_f(\Delta) = \lim_{N_0(\Delta)} \sup \{|f(W)|: W \in N_0(\Delta)\}$, where $\Delta \subset \bar{D}$. *For* $x \in D$, $M_f(x) = f(x)$. $\|f\| = \sup_{x \in D} |f(x)| = M_f(D) = M_f(\bar{D})$.
(ii) *The function* $M_f(\cdot): \bar{D} \to [0, \infty]$ *is upper semi-continuous.*
(iii) *For a closed subset* $\Delta \subset \bar{D}$, *there exists a point* $p \in \Delta$ *such that* $M_f(\Delta) = M_f(p)$.
(iv) $M_{fg}(\Delta) \leqq M_f(\Delta) \cdot M_g(\Delta)$, *where* $\Delta \subset \bar{D}$.

Proof. For (i), (ii), and (iii), see Quigley [5]. (iv) is trivial.

DEFINITION 2. Let $H \subset A(D)$. We call a subset Γ of $\bar{D}$ an H-set if Γ is closed in $\bar{D}$ and $\|f\| = M_f(D) = M_f(\Gamma)$ for all $f \in H$. An H-set is minimal if it properly contains no H-set. Denote Γ_H for a minimal H-set.

If $H = B = B(D)$, Γ_B is a minimal B-set.

PROPOSITION 2. *If D is relatively compact then a minimal H-set exists for every* $H \subset A(D)$.

Proof. See Quigley [5].

PROPOSITION 3. *Let X be locally compact and B separate the points of D. Let π be a continuous mapping from $S(B_s)$ into X such that $\pi \circ \rho$ is the identity mapping on D. Let* cl ρD *be the closure of ρD in $S(B_s)$. Then* $\pi(\mathrm{cl}\, \rho D) = \bar{D}$ *and* $\pi(\mathrm{cl}\, \rho D - \rho D) = \bar{D} - D$.

Proof. Since $\operatorname{cl}\rho D$ is compact and $\pi(\operatorname{cl}\rho D) \supseteqq D$, $\pi(\operatorname{cl}\rho D) \supseteqq \bar{D}$. Let $h \in \operatorname{cl}\rho D$ then for any net $\{h_n\} \subset \rho D$ which converges to h, $\{\pi(h_n)\}$ converges to $\pi(h)$, since π is continuous. Since $\{\pi(h_n)\} \subset D$, $\pi(h) \in \bar{D}$. So $\pi(\operatorname{cl}\rho D) \subseteqq \bar{D}$. Hence $\pi(\operatorname{cl}\rho D) = \bar{D}$.

Let $h \in \operatorname{cl}\rho D - \rho D$ and assume that $\pi(h) \in D$. Take any $f \in B$. Since f is continuous, we may choose, for arbitrary $\varepsilon' > 0$, a neighborhood U of $\pi(h)$; $U = \{x \in D: |f_i(x) - f_i(\pi(h))| < \varepsilon, i = 1, 2, \cdots, n\}$, such that $y \in U$ implies $|f(y) - f(\pi(h))| < \varepsilon'$. Again, since $\hat{f}$ is continuous on $S(B_s)$ and $h \in \operatorname{cl}\rho D$, there is $y_0 \in D$ with $\rho(y_0) \in N = \{\varphi \in S(B_s): |\hat{f}_i(\varphi) - \hat{f}_i(h)| < \varepsilon, i = 1, 2, \cdots, n\}$ such that $|\hat{f}(h) - \hat{f}(\rho(y_0))| < \varepsilon'$. Note that $y_0 \in U = \pi \mid \rho D(N)$, so $|f(y_0) - f(\pi(h))| < \varepsilon'$. Also $f(y_0) = \hat{f}(\rho(y_0))$ and $f(\pi(h)) = \hat{f}(\rho(\pi(h))$, so it follows that $|\hat{f}(h) - \hat{f}(\rho(\pi(h)))| < 2\varepsilon'$. Since ε' is arbitrary, we have $\hat{f}(h) = \hat{f}(\rho(\pi(h)))$ foy every $f \in B$. Hence $h = \rho(\pi(h)) \in \rho D$, which is absurd. Hence $\pi(\operatorname{cl}\rho D - \rho D) = \bar{D} - D$.

THEOREM 1. *Let X be locally compact and D be relatively compact in X. If $B(D)$ separates the points of D, then the minimal B-set Γ_B is unique.*

Proof. Let Γ_1 and Γ_2 be minimal B-sets, and let $p \in \Gamma_1$ be an arbitrary point of Γ_1. We will show that every neighborhood of p intersects Γ_2 so that $p \in \Gamma_2$. So $\Gamma_1 \subset \Gamma_2$. The same argument shows that $\Gamma_2 \subset \Gamma_1$.

Let $p \in \Gamma_1$. Let W be any neighborhood of p in $\bar{D}$ and let $\varphi \in \operatorname{cl}\rho D$ such that $\pi(\varphi) = p$. Take a neighborhood N of φ in $S(B_s) = S$ such that $N \subset \pi^{-1}(W)$; $N = \{h \in S: |\hat{f}_i(h) - \hat{f}_i(\varphi)| < \varepsilon, i = 1, 2, \cdots, n\}$. Put $U = \{x \in D: |f_i(x) - a_i| < \varepsilon, i = 1, 2, \cdots, n\}$, where $a_i = \hat{f}_i(\varphi)$. Then $U = \pi(N) \cap D \subset \pi(N)$. Let $V = \{x \in \bar{D}: M_{f_i - a_i}(x) < \varepsilon/2, i = 1, 2, \cdots, n\}$. Since $M_{f_i - a_i}(x) = |f_i(x) - a_i|$ for $x \in D$, $V \cap D = U$. And, since $M_{f_i - a_i}$ is upper semicontinuous, V is open in $\bar{D}$ and it is easy to see that $M_{f_i - a_i}(p) = 0$, so V is an open neighborhood of p. Note that $M_{f_i}(p) = |a_i|$. Now, since $M_{f_i - a_i}(x) < \varepsilon/2$ in V, we may choose a neighborhood G of p in $\bar{D}$ such that $|(f_i - a_i)(x)| < \varepsilon$ for all $x \in G \cap D$ and $G \subset \pi N$. Then $V \subset G \subseteqq \pi N \subset W$.

Since $\Gamma_1 - V$ is closed in $\bar{D}$ and it is a proper subset of Γ_1, it is not a B-set. Hence there exists $g \in B(D)$ such that $M_g(\Gamma_1 - V) < M_g(\Gamma_1) = \|g\|$. So $M_g(\Gamma_1 - V)\|g\|^{-1} < 1$. Choose m large enough such that $\{M_g(\Gamma_1 - V)\|g\|^{-1}\}^m < \varepsilon(1 + \sum_1^n \|f_i - a_i\|)^{-1} = \delta$, and set $f = g^m$. Then $M_f(\Gamma_1 - V) = M_{g^m}(\Gamma - V) \leqq \{M_g(\Gamma - V)\}^m < \delta\|g\|^m = \delta\|f\|$. If $x \in V$ then $M_{f_i - a_i}(x) < \varepsilon/2$ so that

$$M_{f_i - 2}M_f(x) = M_{f_i - a_i}(x)M_f(x) < \frac{\varepsilon}{2}M_f(\bar{D}) = \frac{\varepsilon}{2}\|f\| .$$

If $x \in \Gamma_1 - V$ then $M_f(x) \leqq M_f(\Gamma_1 - V) < \delta\|f\|$, so that again

$$M_{f_i - a_i} M_f(x) < \frac{\varepsilon}{2} \|f\| \,.$$

Since Γ_1 is a B-set it follows that $M_{f_i-a_i}M_f(\bar{D}) < (\varepsilon/2)M_f(\bar{D}) = (\varepsilon/2)\|f\|$. Let q be any point of Γ_2 with $M_f(q) = M_f(\bar{D}) = M_f(D) = \|f\|$. Then $M_{f_i-a_i}(q)M_f(q) < (\varepsilon/2)\|f\|$. Hence $M_{f_i-a_i}(q) < \varepsilon/2$ and this is true for all $i = 1, 2, \cdots, n$. Thus $q \in V$, so $V \cap \Gamma_2 \neq \varnothing$. Hence $W \cap \Gamma_2 \neq \phi$. Since Γ_2 is closed, $p \in \Gamma_2$. The proof is complete.

We call the unique minimal B-set the minimal boundary for B.

Note. Let Γ_B be a minimal boundary for B then $x \in \Gamma_B$ if and only if for every neighborhood U of x there exists $f \in B$ such that $\|f\| = M_f(U) > M_f(\bar{D} - U)$.

THEOREM 2. *Let X be locally compact and D be relatively compact in X. We assume that B separates the points of D. Then $\pi\partial\hat{B}$ is a minimal boundary.*

Proof. Since $M_f(\bar{D}) = \|f\|_D = \|\hat{f}\|\rho_D = \|\hat{f}\|_S$ for all $f \in B$, we have $\partial_{\hat{B}} \subset \mathrm{cl}\,\rho D$. Let $x \in \pi\partial_{\hat{B}}$ then there exists $h \in \partial_{\hat{B}}$ such that $x = \pi h$. Now, $h \in \partial_{\hat{B}}$ implies that for arbitrary neighborhood N of h in $S = S(B_s)$ there exists $\hat{f} \in \hat{B}$ such that $\|\hat{f}\|_S = \|\hat{f}\|_N > \|\hat{f}\|_{S-N}$. Since $S - N \supset \rho D - N \cap \rho D$, we have $\|\hat{f}\|_{S-N} \geqq \|\hat{f}\|_{\rho D - N \cap \rho D}$. So $\|\hat{f}\|_{\rho D} = \|\hat{f}\|_S > \|\hat{f}\|_{S-N} \geqq \|\hat{f}\|_{\rho D - N \cap \rho D}$. Hence it follows that $\|\hat{f}\|_{\rho D} = \|\hat{f}\|_{N \cap \rho D} > \|\hat{f}\|_{\rho D - N \cap \rho D}$. This is equivalent to $\|f\|_D = \|f\|_{\pi(N \cap \rho D)} > \|f\|_{D - \pi(N \cap \rho D)}$. Since $\pi(N \cap \rho D)$ is a trace of a neighborhood of $x = \pi h$ on D and a trace of any neighborhood of x on D can be written as such a form, $x = \pi h$ belongs to a minimal boundary Γ_B. So $\pi\partial_{\hat{B}} \subset \Gamma_B$. On the other hand, if W is any open set containing $\pi\partial_{\hat{B}}$, then by the continuity of π, there exists an open set G in S containing $\partial_{\hat{B}}$ such that $\pi(G) \subseteqq W$ and hence $\pi(G \cap \rho D) \subseteqq W \cap D$. For any $f \in B$, we have

$$\|f\|_{W \cap D} \geqq \|\hat{f}\|_{G \cap \rho D} = \|\hat{f}\|_{G \cap \mathrm{cl}\,\rho D} = \|\hat{f}\|_{\partial\hat{B}} = \|f\|_D \,.$$

If follows that $M_f(\pi\partial_{\hat{B}}) = \|f\|_D$ for all $f \in B$. Since $\pi\partial_{\hat{B}}$ is closed, it is a B-set. Thus $\pi\partial_{\hat{B}}$ is a minimal boundary.

For instance: Let D be the unit open disc in $\boldsymbol{C}$ and let $B(D) = H^\infty$. Define a natural continuous mapping π of S into the closed unit disc $\bar{D}$ by $\pi(h) = h(z)$, $h \in S$ and z is the coordinate function. Then the minimal boundary Γ_B is the unit circle and the Šilov boundary $\partial_{\hat{B}}$ on S is a proper closed subset of $\mathrm{cl}\,\rho D - \rho D$ which is totally disconnected. The image of $\partial_{\hat{B}}$ under π is the unit circle.

Next, we have a question that whether a function f with $M_f(\Gamma_B) < \infty$ is bounded.

PROPOSITION 4. *Suppose* $A = A(D)$ *and* $B = B(D)$ *have the unique minimal boundaries* Γ_A *and* Γ_B *respectively. If* $\Gamma_A \neq \Gamma_B$ *then there exists a function* $f \in A$ *which is bounded near* Γ_B (i.e., $M_f(\Gamma_B) < \infty$), *but not in* B.

Proof. In general, $\Gamma_A \supset \Gamma_B$. Take $x \in \Gamma_A - \Gamma_B$ and choose a neighborhood U of x in $\bar{D}$ such that $M_f(U) = \|f\| > M_f(\bar{D} - U)$ and $U \cap \Gamma_B = \phi$. Then $M_f(\Gamma_B) < \infty$ but $f \notin B$.

DEFINITION. A point $x \in \bar{D}$ is a frontier point of D for $H \subset A(D)$ if for each compact subset K of D with $x \notin K$ there exists $f \in H$ such that $M_f(x) > \|f\|_K$. Let F_H be the set of all frontier points of D for H. Denote F_A for $A(D)$ and F_B for $B(D)$ respectively.

We introduce a generalized form of a theorem in Bochner and Martin [2] (see Theorem 1, Ch. V) as follows:

PROPOSITION 5. *Let* X *be locally compact,* D *be a subset of* X *which is countable at* ∞, *and let* $\bar{D} - D$ *be first countable. Let* $A = A(D)$ *be a maximum modulus algebra and c.o. complete. Then* $x \in F_A$ *if and only if there is a function* $f \in A$ *such that* $M_f(x) = \infty$. *In fact, there is a single function* f *such that* $M_f(x) = \infty$ *for all* $x \in F_A$.

Proof. Use the analogous argument as in Bochner and Martin [2].

THEOREM 3. *Let* X *be locally compact,* D *be countable at* ∞, *and* $\bar{D} - D$ *be first countable. Let* A *be a maximum modulus algebra and c.o. complete. Suppose* Γ_B *is a minimal boundary and* $F_A = \Gamma_B$ *then every function* $f \in A$ *with* $M_f(\Gamma_B) < \infty$ *belongs to* B.

Proof. Assume that f is unbounded then there exists a sequence $\{x_n\} \subset D$ such that $|f(x_n)| \to \infty$ and $n \to \infty$. Let $x_n \to x$ then by Proposition 5, $x \in F_A$ and so $x \in \Gamma_B$. Thus $\infty = M_f(x) \leqq M_f(\Gamma_B) < \infty$, which is absurd. Hence $f \in B$.

We observe that $h \in S(B_s) - S(B_c)$ if and only if for any compact subset K of D there exists $f \in B$ (f may depend on K) such that $|h(f)| > \|f\|_K$.

THEOREM 4. *Let* X *be locally compact and* B *separate the points of* D. *Let* F_B *be the set of all frontier points for* B. *If* $\rho D = S(B_c)$ *then* $\bar{D} - D = F_B$.

Proof. Let bdry $S(B_c) = \mathrm{cl}\, S(B_c) - S(B_c)$. By Proposition 3, $\pi(\text{bdry } S(B_c)) = \text{bdry } D$. Now if $h \in \text{bdry } S(B_c)$, then for any compact subset K of D, there exists $f \in B$ such that $|h(f)| > ||f||_K$. We claim $M_f(\pi(h)) > ||f||_K$: Suppose $M_f(\pi(h)) = ||f||_K = r$, then there exists a net $\{x_n\} \subset D$ such that $||f(x_n)| - r| < 1/n$ as $x_n \to \pi(h)$. So $|f(x_n)| \to r$. Now, let $h_{x_n} \to h$. Since $\hat{f}$ is continuous, $\hat{f}(h_{x_n}) \to \hat{f}(h)$. So $f(x_n) \to h(f)$. In particular, $|f(x_n)| \to |h(f)|$. Then it follows that $|h(f)| = r = ||f||_K$. This is absurd. Hence $M_f(\pi(h)) > ||f||_K$. So bdry $D = F_B$.

Note. If D is a Stein manifold of bounded type then $\rho D = S(B_c)$ (see Kim [3]).

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