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EQUIVARIANT EXTENSIONS OF MAPS

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This paper treats extension and retraction properties in the category $\mathscr{A}_p$ of compact metric spaces with periodic maps of a prime period p; the subspaces and maps in $\mathscr{A}_p$ are called equivariant subspaces and maps, respectively. The motivation of the paper is the following question: Let E be a Euclidean space and $a: E \times E \to E \times E$ be the involution $(x, y) \to (y, x)$, i.e., the symmetry with respect to the diagonal. Suppose that Z is a symmetric (i.e., equivariant) closed subset of $E \times E$ which is an absolute retract; that is, Z is a retract of $E \times E$. When does there exist a symmetric (i.e., equivariant) retraction $E \times E \to Z$?

This is an extension problem in the category $\mathscr{A}_p$. If X and Y are spaces in $\mathscr{A}_p$, A is a closed equivariant subspace of X and $f: A \to Y$ is an equivariant map, then the existence of an extension of f does not, in general, imply the existence of an equivariant extension. It is shown, however, that if A contains all the fixed points of the periodic map and $\dim(X-A) < \infty$, then a condition for the existence of an extension is also sufficient for the existence of an equivariant extension. In particular, it follows that a finite dimensional space X in $\mathscr{A}_p$ is an equivariant ANR (i.e., an absolute neighborhood retract in the category $\mathscr{A}_p$) if and only if it is an ANR and the fixed point set of the periodic map on X is an ANR. Generally speaking, the paper deals with the question of symmetry in extension and retraction problems.

1. Preliminaries. Suppose that a group G acts on spaces X and Y and that A is an equivariant subspace of X (i.e., A is stable under the action of G). One can then ask for conditions for the extistence of an equivariant extension of f; or for conditions under which the existence of an extension of f implies also the existence of an equivariant extension. A general theorem of this type is due to A. Gleason [6] and R. S. Palais [12, p. 19]:

TIETZE-GLEASON THEOREM. *Let G be an orthogonal group acting on a Euclidean space E by means of orthogonal transformations and let G act on a normal space X. Let A be a closed equivariant subset of X and let $f: A \to E$ be an equivariant map. Then there is an equivariant extension $g: X \to E$ of f.*

This theorem is proved by first extending the map f to some map $\bar{f}: X \to E$ which may not necessarily be equivariant: and then by averaging $\bar{f}$, using a Haar measure on G, to make it equivariant.

Two facts play a crucial role in this proof: one is that E is convex; and the other is that the action of G is linear. While the second condition is not necessarily restrictive (in view of results due to Mostow [11]; Copeland and de Groot [2]; Kister and Mann [8]; the action of G can be linearized), the first condition makes it impossible to apply a theorem of this type to our original problem (these two conditions are, in fact, related: by linearization of the map, the convexity of the space may be distorted).

In this paper we consider actions of $\boldsymbol{Z}_p$, the cyclic group of a prime order p. In order words, we consider the category $\mathscr{A}_p$ whose objects are periodic homeomorphisms $a\colon X \to X$ of a prime period p on a space X; i.e., $a^p = 1$. An object $a\colon X \to X$ in $\mathscr{A}_p$ will also be denoted by (X, a), or simply by X, if the periodic map a is known. A morphism in $\mathscr{A}_p$ from (X, a) to (Y, b) is a map $f\colon X \to Y$ consistent with the periodic maps a and b; it will be called an equivariant map. A subspace A of X is said to be equivariant if it is stable under a, i.e., if $aA \subset A$. If A is an equivariant subspace of X then the periodic map $A \to A$ defined by the restriction of $a\colon X \to X$ of A will sometimes be denoted by $a_A\colon A \to A$.

The set of the fixed points of a map $a\colon X \to X$ will be denoted by $F(a)$. If $a\colon X \to X$ is a periodic map of a prime period p then $F(a) = F(a^q)$, for evey $q = 1, \cdots p - 1$.

An example of a theorem which carries over to the category $\mathscr{A}_p$ in a way similar to that of the Tietze theorem is the Dugundji extension theorem. In the category $\mathscr{A}_p$ it can be stated as follows:

DUGUNDJI EQUIVARIANT EXTENSION THEOREM (in the category $\mathscr{A}_p$). *Let (X, a) be a space in $\mathscr{A}_p$ such that X is metrizable and let A be an equivariant closed subspace of X. Let L be a locally covex vector space with a linear periodic map $b\colon L \to L$ of period p and let Q be an equivariant convex subspace of L. Let $f\colon A \to Q$ be an equivariant map. Then f can be extended to an equivariant map $g\colon X \to Q$.*

The proof is the same as that of the Tietze-Gleason theorem. By the Dugundji extension theorem there exists an extension $\bar{f}\colon X \to Q$. We define an equivariant extension g by

$$g = \frac{1}{p} \sum_{i=1}^{p} b^{p-i} \circ \bar{f} \circ a^i .$$

In other words, this theorem says that (L, b_L) is an "absolute extensor" in this category of spaces. One can likewise introduce the definitions of "absolute neighborhood extensor", "absolute retract" and "absolute neighborhood retract" in the category $\mathscr{A}_p$ or in other similar categories.

Returning to our original problem of the existence of an equivariant retraction, let us now state it as follows (in the case of compact metric spaces):

Question I. Let Q be a Hilbert cube and let $a: Q \to Q$ be a periodic map of a prime period p such that a is linear with respect to the linear structure on Q. Let $Z \subset Q$ be an equivariant closed subspace of Q which is a retract of Q. When does there exist an equivariant retraction $Q \to Z$?

First, it is known that if X is any separable metric space with a peiodic map $a: X \to X$ of period p then there exists an equivariant embedding of X in a Hilbert cube with a linear, even a distance preserving, map of period p. Such an embedding is known as a linearization of (X, a) (see [2], Theorem II). We choose any embedding $X \subset Q$ and then define an equivariant embedding $X \to Q^p$ by

$$x \mapsto (x, ax, \cdots, a^{p-1}x) .$$

Thus the periodic map becomes a cyclic permutation of the coordinates of Q^p (in fact, our original case was of the involution $E \times E \to E \times E$ of this form). Similarly, if $\dim X < \infty$, then X can be equivariantly embedded in a finite-dimensional cube I^n with an isometric periodic map.

Returning to Question I, let us assume therefore that there exists an equivariant retraction $r: Q \to Z$. Consider the fixed point sets $F(a)$ and $F(a_Z)$ of the map a on Q and Z, respectively. Then r defines a retraction of $F(a)$ to $F(a_Z)$. But since a is linear, $F(a)$ is a compact convex subset of Q and hence an absolute retract (it is, in fact, homeomorphic to Q or to a finite-dimensional cube: see [7] and [9]). Therefore the fixed point set $F(a_Z)$ of a would have to be an absolute retract; and this need not necessarily be the case, since there is the following example due to E. E. Floyd [4].

Floyd's example. There exists a 5-dimensional compact contractible polyhedron Z with an involution $a: Z \to Z$ whose fixed point set $F(a_Z)$ is not contractible; in fact, $H_1(F(a_Z)) \neq 0$.

Similarily, one can construct an example of a compact AR Z with an involution $a: Z \to Z$ such that $F(a_Z)$ is not an ANR.

Thus, in Question I, the condition that both Z and $F(a_Z)$ be AR's is necessary for Z to be an equivariant retract of Q; similarly, the condition that Z and $F_a(Z)$ be ANR's is necessary for Z to be an equivariant neighborhood retract of Q. Consequently, the question arises as to whether these conditions are also sufficient.

Let us specify our questions as follows:

Questions. Let Q be a Hilbert cube with a linear periodic map $a: Q \rightarrow Q$ of period p and let Z be an equivariant closed subspace of Q.

Question I'. Suppose that both Z and the fixed point set $F(a_Z)$ of a are AR's. Is Z an equivariant retract of (Q, a)?

Question I''. Suppose that both Z and $F(a_Z)$ are ANR's. Is Z an equivariant neighborhood retract of an equivariant neighborhood of Z in Q?

The main result of this paper is to show that if p is prime and the dimension of Z in finite then the answer to Questions I' and I'' is affirmative. In fact, the following theorem will be proved:

THEOREM 1.1. *Let X be a compact metric space with a periodic map $a: X \rightarrow X$ of period p and let A be an equivariant closed subspace of X containing all the fixed points, of a and such that* $\dim (X - A) < \infty$. *Let Y be a compact metric space with a periodic map $b: Y \rightarrow Y$ of period p and let $f: A \rightarrow Y$ be an equivariant map. Then:*

(i) *If Y is an AR, then there exists an equivariant extension $g: X \rightarrow Y$ of f over X;*

(ii) *If Y is an ANR, then there exists an equivariant extension $g: U \rightarrow Y$ of f over an equivariant neighborhood U of A in X.*

We can now use Theorem (1.1) to answer our questions in the finite-dimensional case; in fact, we use part (i) to answer Question I' and part (ii) to answer Question I''. Let us consider, for instance, the case (i) and I'. Since $\dim Z < \infty$, there is an equivariant embedding of (Z, a_Z) in an n-cube I^n: that is, an equivariant homeomorphism of Z onto an equivariant subspace Z' of I^n with a periodic map $b: I^n \rightarrow I^n$ (which can even be assumed isometric). Let us apply Theorem (1.1) to $X = I^n$, $A = Z' \cup F(b) = Y$ and $f = 1_A$ (the identity map $A \rightarrow A$). By Theorem (1.1), there exists an equivariant retraction $r: I^n \rightarrow A$. Since $F(b_{Z'})$ is an AR, there exists a retraction $q: F(b) \rightarrow F(b_{Z'})$. The retraction q defines a retraction $q': A \rightarrow Z'$ by extending via the identity $Z' \rightarrow Z'$. The composition $q' \circ r$ is an equivariant retraction of I^n to Z'. Now, since I^n can be embedded as an equivariant retract of Q, it follows that Z is an equivariant retract of Q.

Similarly, part (ii) of Theorem (1.1) yields an affirmative answer to Question I$''$.

The proof Theorem (1.1) uses the classical method of replacing $X - A$ by the nerve of a covering adjusted to the equivariant category.

It works, however, only under the assumption $\dim(X - A) < \infty$. It is an open question whether this finite-dimensional assumption in Theorem (1.1) is essential.

The main results of this paper have been announced in [6].

2. Linearization. We summarize some results on linearization of periodic maps (see [2]). Given a space Z, we denoted by $c(p, Z)$ the periodic map of the p-fold Cartesian product Z^p defined by $(z_1, \cdots, z_p) \to (z_p, z_1, \cdots, z_{p-1})$ i.e., $c(p, Z)$ is a cyclic permutation of the coordinates.

(2.1). If Z is a vector space then $(Z^p, c(p, Z))$ is a vector space with the periodic map $c(p, Z)$ being linear with respect to the product vector space structure.

(2.2). If Z is a metric space then $c(p, Z)$ is isometric (i.e., distance preserving) with respect to the product metric in Z^p.

(2.3). If (X, a) is an object in $\mathscr{A}_p$ and there is an embedding $h: X \to Z$ of X in a space Z, then there is an equivariant embedding of (X, a) in $(Z^p, c(p, Z))$ defined by $x \mapsto (hx, h(ax), \cdots, h(a^{p-1}x))$.

In particular

(2.4). If $(X, a)$is an object of $\mathscr{A}_p$ such that X is a compact metric space, then there is an equivariant embedding of (X, a) in (Q, c) where Q is a Hilbert cube with an isometric map $c: Q \to Q$ of period p. If $\dim X < \infty$, then there is an equivariant embedding of (X, a) in (I^n, c), where I^n is a finite-dimensional cube with an isometric periodic map $c: I^n \to I^n$ of period p.

Let us also note that if (V, a) is a vector space with a linear periodic map $a: V \to V$ of period p and Z is an equivariant convex subset of V then the equivariant embedding $h: x \mapsto (hx, h(ax), \cdots, h(a^{p-1}x))$ carries Z onto a convex subset of $(V^p, c(p, V))$. In particular, an n-cube (I^n, a) with a linear periodic map $a: I^n \to I^n$ can be equivariantly embedded as a convex subset of (Q, c), where Q is a Hilbert cube with a linear periodic map $c: Q \to Q$. Using the Dugundji equivariant extension theorem we obtain the following corollary:

(2.5) If (I^n, a) is an n-cube with a linear periodic may $a: I^n \to I^n$ of period p then (I^n, a) can be equivariantly embedded as an equivariant retract of a Hilbert cube (Q, c) with a periodic map $c: Q \to Q$.

3. Retracts and extensors in the category $\mathscr{A}_p$. We summarize the definitions and main properties of retracts and extensors in the category $\mathscr{A}_p$ of spaces with $\boldsymbol{Z}_p$-actions (compare Palais [12], p. 25) which are usually called $\boldsymbol{Z}_p$-retracts and $\boldsymbol{Z}_p$-extensors. Since the prime integer p and the group $\boldsymbol{Z}_p$ is fixed throughout the paper (except where the results are specialized to the case $p = 2$), we shall simply call them equivariant retracts and equivariant extensors.

DEFINITION 3.1. An object (Y, b) of $\mathscr{A}_p$ is said to be an equivariant absolute extensor (abbreviated to EAE) if given an object (X, a) of $\mathscr{A}_p$ such that X is a metric space, given a closed equivariant subspace A of X and given an equivariant map $f: A \to Y$, there is an equivariant extension $g: X \to Y$ of f.

An object (Y, b) of $\mathscr{A}_p$ is said to be equivariant absolute neighborhood extensor (EANR) if given (X, a), A and f as above, there is an equivariant extension $g: U \to Y$ of f over some equivariant neighborhood U of A in X.

DEFINITION 3.2. An object (X, a) of $\mathscr{A}_p$ is said to be an equivaraiant absolute retract (abbreviated to ERA) if X is a metric space and for any equivariant imbedding $h: (X, a) \to (Y, b)$ in an object (Y, b) of $\mathscr{A}_p$ such that Y is a metric space and hX is closed in Y, the image hX in an equivariant retract of (Y, b).

An object (X, a) of $\mathscr{A}_p$, where X is a metric space, is said to be an equivariant absolute neighborhood retract (EANR) if given $h: (X, a) \to (Y, b)$ as above, the image hX is an equivariant neighborhood retract of (Y, b).

The following theorems are proved in the same way as in the topological category:

THEOREM 3.3. *An equivariant retract of an EAE is an EAE; an equivariant neighborhood retract of an EANE is an EANE.*

THEOREM 3.4. *A Hilbert cube (Q, c) with a linear periodic map $c: Q \to Q$ of a prime period p is an EAE.*

This is, in fact, a particular case of the Dugundji extension theorem in the category $\mathscr{A}_p$ (§ 1).

THEOREM 3.5. *Let (X, a) be an object of $\mathscr{A}_p$ such that X is a compact metric space. Then the following conditions are equivalent:*
(i) *(X, a) is an EAE.*
(ii) *(X, a) is an EAR.*
(iii) *(X, a) can be equivariantly embedded as an equivariant retract of (Q, c), where Q is a Hilbert cube with an isometric periodic map $c: Q \to Q$ of period p.*

Similarly, the following conditions are equivalent:
(iN) *(X, a) is an EANE.*

(iiN) *(X, a) is an EANR.*

(iiiN) *(N, a) can be equivariantly embedded as an equivariant neighborhood retract of (Q, c), where (Q, c) is as above.*

Moreover, if $\dim X < \infty$, *then the Hilbert cube (Q, c) can be replaced by a finite-dimensional cube I^n with an isometric involution.*

Theorem (3.5) is proved by using the linearization embeddings (§2).

COROLLARY 3.6. *The following objects of $\mathscr{A}_p$ are equivariant absolute retracts:*

(1) *A Hilbert cube (Q, c) with a linear periodic map $c: Q \to Q$.*

(2) *An n-cube (I^n, c) with a linear periodic map $c: Q \to Q$.*

4. Equivariant coverings and replacement by polyhedra. In this section we describe the classical constructions due to Kuratowski [10] and Dugundji [2] which are used in extending maps. We adjust them spaces with periodic maps, but we restrict ourselves to compact metric spaces.

The following notation will be used: diam S is the diameter of a subset S of a metric space X; $B(x, \varepsilon)$ is the open ball in X of center x and radius ε; and Conv S is the convex hull of a subset S of a linear space L.

Let α be a collection of subsets of X. If $U \in \alpha$, then $St_\alpha U$ is the union of the members of α which meet U. We say that $\operatorname{Ord} \alpha \leqq n$ if every collection of $n+1$ members of α has an empty intersection. If X is an object of $\mathscr{A}_p$ with a periodic map $a: X \to X$, let $a\alpha = \{a^q U \mid U \in \alpha, q = 1, \cdots, p-1\}$; the collection α is said to be equivariant if $a\alpha = \alpha$.

COVERING LEMMA 4.1. *Let (X, a) be an object of $\mathscr{A}_p$ such that X is a compact metric space and let A be an equivariant closed subspace of X containing the fixed point set $F(a)$ of a. Let α be an equivariant open cover of $X - A$. Then there exists an equivariant countable open cover β of $X - A$ which is a refinement of α and satisfies the following conditions:*

(i) $\lim_{U \in \beta} (\operatorname{diam} U) = 0$

(ii) *If* $U \in \beta$ *then* $\operatorname{Cl} U \subset X - A$.

(iii) *Every neighborhood of A in X contains all but a finite number of elements of β.*

(iv) *For every $U \in \beta$, the sets $St_\beta U$, $a(St_\beta U)$, $\cdots$, $a^{p-1}(St_\beta U)$ are mutually disjoint.*

(v) *If* $\dim (X - A) \leqq n$ *then* $\operatorname{Ord} \beta \leqq p(n+1)$.

Proof. We can assume that d is an equivariant distance function on X, i.e., that $a: X \to X$ is isometric. Let $A_0 = X$,

$$A_i = \left\{x \in X \mid d(x, A) < \frac{1}{2^i}\right\}, \qquad i = 1, 2, \cdots$$

$$C_i = \mathrm{Cl}(A_i) - A_{i+1}, \qquad i = 0, 1, \cdots$$

The sets C_i are compact. Since p is prime, the group Z_p acts freely on $X - A$, i.e., for every $x \in X - A$, the orbit $\{x, ax, \cdots, a^{p-1}x\}$ consists of p distinct points. It follows that, for each $i = 0, 1, \cdots$, there is a positive number η_i such that

$$d(x, a^p x) \geqq \eta_i \quad \text{for every } q = 1, \cdots, p-1 \text{ and } x \in C_i . \tag{4.2}$$

For each $i = 0, 1, \cdots$, there is a finite open cover γ_i of C_i by open balls in X with centers in C_i and radii $r_i > 0$ such that γ_i is a refinement of α and

$$r_i \leqq 2^{-i-3} \tag{4.3}$$

$$r_{i+1} \leqq \frac{1}{2} r_i \tag{4.4}$$

$$6\gamma_i \leqq \eta_{i+1} . \tag{4.5}$$

Let $\beta_i = \gamma_i \cup a(\gamma_i) \cup \cdots \cup a^{p-1}(\gamma_i)$ and $\beta = \beta_0 \cup \beta_1 \cup \cdots$. Then β is an equivariant countable open cover of $X - A$ and is a refinement of α since α is equivariant. Conditions (i), (ii) and (iii) follow directly from the construction of β.

Let us verify condition (iv). Observe that by (4.3) and (4.4), and since the map a is isometric, every member of β is an open ball contained in $A_{j-1} - \mathrm{Cl}(A_{j+1})$, for some $j = 1, 2, \cdots$. Thus if a member V of β meets a member U of β_i then $V \in \beta_{i-1} \cup \beta_i \cup \beta_{i+1}$ and, by (4.3) and (4.4), U and V are open balls of radii $\leqq r_{i-1} \leqq 2^{-i-2}$.

Since the map $a: X \to X$ is isometric, to prove condition (iv) it suffices to show that $(St_\beta U) \cap (a^q(St_\beta U)) = \varnothing$ for each $U \in \beta$ and $q = 1, \cdots, p-1$. Let $U \in \beta$ and suppose that $U \in \beta_i$. Then the center x of the open ball U is in C_i. By the remark above, for every $y \in St_\beta U$ we have $d(x, y) < r_i + 2r_{i-1} < 3r_{i-1}$. If we suppose that $(St_\beta U) \cap (a^q(St_\beta U)) \neq \varnothing$ for some $q = 1, \cdots, p-1$, then, since the map a is is isometric, it would follow that $d(x, a^q x) < 6r_{i-1} \leqq \eta_i$, which contradicts to (4.5) and (4.2).

Suppose now that $\dim(X - A) \leqq n$. Then the open cover β has an open refinement ω of order $\leqq n + 1$. Since C_i is compact, there is a finite subcollection ω_i of ω which covers C_i. Let $\beta'_i = \omega_i \cup a(\omega_i) \cup \cdots \cup a^{p-1}(\omega_i)$ and $\beta' = \beta'_0 \cup \beta'_1 \cup \cdots$. Then β' is an equivariant countable open refinement of β which satisfies the conditions corresponding to (i), (ii) and (iii). Condition (iv) for β' follows from the fact that it holds for β and that β' is a refinement of β.

Let us verify condition (v). Suppose that σ is a subcollection of β' containing p distinct elements whose intersection is nonempty. For each $W \in \sigma$, there is an integer q, $0 \leqq q < p$, such that $a^q W \in \omega$; let $q(W)$ denote the smallest integer with this property. Then this defines a map $q\colon \sigma \to \{0, \cdots, p-1\}$. Note that $\operatorname{Card} q^{-1}(j) \leqq n+1$, for each $j = 0, \cdots, p-1$. For if $W_0, \cdots, W_r$ are distinct elements of $q^{-1}(j)$, then $a^j(W_0), \cdots, a^j(W_r) \in \omega$ and $a^j(W_0) \cap \cdots \cap a^j(W_r) \neq \varnothing$ and, consequently, $r \leqq n$, since $\operatorname{Ord} \omega \leqq n+1$. Since $\sigma = q^{-1}(0) \cup \cdots \cup q^{-1}(p-1)$, it follows that $\operatorname{Card} \sigma \leqq p \cdot (n+1)$.

This completes the proof. Let us observe that conditions (i), (ii) and (iii) imply the following lemma.

LEMMA 4.6. *If β is a cover constructed in Lemma* (4.1), *then for every $x \in A$ and for every neighborhood V of x in X there exists a neighborhood W of x in X such that if $U \in \beta$ and $U \cap W \neq \varnothing$ then $U \subset V$.*

(4.7) LEMMA (*Replacement by polyhedra*). *Let (X, a) be an object of $\mathscr{A}_p$ such that X is a compact metric space and let A be an equivariant closed subspace of X containing the fixed point set $F(a)$ of a. Then there exists an object (Z, c) in $\mathscr{A}_p$ such that Z is a Hausdorff space with a periodic map $c\colon Z \to Z$ and:*

(i) *Z contains A as an equivariant subspace.*

(ii) *$Z - A$ has a countable locally finite triangulation K, $|K| = Z - A$, such that the map c is simplicial and free on K.*

(iii) *There is an equivariant map of pairs: $(X, X-A) \to (Z, |K|)$ which is the identity on A.*

(iv) *There is an equivariant retraction $r^0\colon A \cup |K^0| \to A$.*

(v) *If $\dim (X - A) \leqq n$ then $\dim K \leqq p \cdot (n+1) - 1$.*

Proof. Let β be an equivariant open over of $X - A$ satisfying the conditions of the Covering Lemma (4.1). Let K be the nerve of β and Z be the disjoint set sum of A and $|K|$. Given a member U of β, we shall also be denoting by U the vertex of K corresponding to U; and $St_L U$ will denote the open star of the vertex U in the complex K, while $St_\beta U$ will, as before, denote the union of the members of β intersecting U.

For a subset S of X, let $\hat{S}$ denote the union of $A \cap S$ and of the open stars of the vertices corresponding to the members of β which are contained in S; i.e., $\hat{S} = (A \cap S) \cup (\cup [St_K U \mid U \subset S])$. The space Z is topologized by means of the subbasis consisting of all the open subsets of $|K|$ and all the sets of the form $\hat{U}$, where U is an open subset of X.

Before we proceed with the rest of the proof, we shall establish the following lemma:

LEMMA 4.8. *For every $x \in A$ and every neighborhood V of x in X, there is a neighborhood O_V of x in Z such that if $y \in O_V \cap (Z - A)$ and s is an open simplex of K containing y then all the vertices of s (as members of the cover β) are contained in V.*

Proof. Given a neighborhood V, we choose a neighborhood W of x according to Lemma (4.6). Let $O_V = \hat{W}$. Then if $y \in O_V \cap (Z - A)$ and s is the carrier of y in K, some vertex U of s is contained in W; and all the other vertices are contained in V since they meet $U \subset W$.

Continuation of the proof of (4.7). The fact that Z is Hausdorff follows readily from lemma (4.8). Since the cover β is equivariant, it follows that $a: X \to X$ defines a periodic map on K which, together with the map a, define a periodic map $c: Z \to Z$ of period p. The continuity of c follows from the fact that β is equivariant; and condition (iv) of (4.1) implies that the map c is free on K. Thus conditions (i) and (ii) hold.

The map $\mu: (X, X - A) \to (Z, Z - A)$ of condition (iii) is defined by the identity on A and a canonical map $X - A \to |K|$ of $X - A$ into the space of the nerve of the covering β which can be described as follows: if $x \in X - A$, then the barycentric coordinates of μx with respect to the vertex U of K is

$$\frac{d(x, X - U)}{\sum_{V \in \beta} d(x, X - U)} .$$

Since β is equivariant and the map a is isometric with respect to d, it follows that $\mu: X \to Z$ is equivariant. The continuity of μ follows easily from the definition of the topology, just as in [2].

A retraction $r^0: A \cup |K^0| \to A$ may be defined as follows. Let Λ^0 denote the set of orbits of the map c on K^0 and let $\varphi: \Lambda^0 \to K^0$ be any cross-section of the identification map $K^0 \to \Lambda^0$. Let $N^0 = \varphi(\Lambda^0)$. Since c acts freely on K^0, it follows that K^0 is the disjoint union $K^0 = N^0 \cup c(N^0) \cup \cdots \cup c^{p-1}(N^0)$; thus for each vertex V of K^0 there is a unique vertex U of N^0 and a unique integer j, $0 \leqq j < p$, such that $V = c^j U$. Given a vertex U of N^0, let $r^0 U$ denote any point of A such that $d(U, A) = d(U, r^0 U)$ (such a point exists since A is compact). If V is any vertex of K^0, choose $U \in N^0$ and an integer j as above such that $V = c^j U$ and define $r^0 V = a^j(r^0 U)$. Since the map a is isometric, we have $d(V, A) = d(V, r^0 V)$. Defining r^0 to be the identity on A, we obtain a retration $r^0\colon A \cup |K^0| \rightarrow A$. To prove the continuity of r^0, it suffices to consider the restriction $r^0|(A \cup N^0)$, since the sets $A \cup (a^j N^0)$ are closed and the intersection of any two of them is A. Thus let U be a vertex of N^0, let $z = r^0 U$ and let $B = B(z, \varepsilon)$ be an open ball with center z and radius ε. Let $V = B(z, \varepsilon/3)$ and let O_V be a corresponding neighborhood of x in Z satisfying the assertion of Lemma (4.8). Then $r^0((A \cup N^0) \cap O_V) \subset B$. Moreover, r^0 is equivariant by its definition.

Now, if $\dim(X - A) \leqq n$, then by (4.1), (v), $\operatorname{Ord} \beta \leqq p(n+1)$ and hence $\dim K \leqq p(n+1) - 1$. This proves condition (v).

REMARK. One can easily show that the space Z is, in fact, compact and metrizable.

5. **Proof of the extension Theorem (1.1).** By (2.4) we can assume that Y is an equivariant subspace of a Hilbert cube Q with an isometric periodic map $b\colon Q \rightarrow Q$ of period p such that the map $Y \rightarrow Y$ is the restriction of b.

We shall first prove case (ii) of (1.1). Suppose that Y is an ANR. Then there is an equivariant compact neighborhood C of Y in Q and a (not necessarily equivariant) retraction $r\colon C \rightarrow Y$. Let $\delta = d(Y, Q - C)$; then $\delta > 0$. By the uniform continuity of r there exists a function $\eta\colon R_+ \rightarrow R_+$ (R_+ = the set of positive numbers) below the diagonal ($\eta(\varepsilon) \leqq \varepsilon$) such that

(5.1) If $y \in C$ and $d(y, Y) \leqq \eta(\varepsilon)$ then $d(y, ry) \leqq \varepsilon$.

Consequently, $d(b^j y, b^j r y) \leqq \varepsilon$, for every $j = 0, \cdots, p-1$, since b is isometric.

Let $n = p\cdot(\dim(X - A) + 1) - 1$. Define a sequence of positive numbers $\varepsilon_0, \cdots, \varepsilon_n$ as follows:

$$(5.2) \qquad \varepsilon_n = \delta\ ; \quad \varepsilon_{m-1} = \frac{1}{2}\eta\left(\frac{\varepsilon_m}{4}\right) \quad \text{for} \quad 0 < m \leqq n\ .$$

By the uniform continuity of f there is a $\xi > 0$ such that if $x, x' \in A$ and $d(x, x') \leqq \xi$ then $d(fx, fx') \leqq \varepsilon_0$.

Let (Z, c) be a space with a periodic map $c\colon Z \rightarrow Z$ of period p, a triangulation K of $Z - A$, an equivarient map $\mu\colon X \rightarrow Z$ and an equivariant retraction $r^0\colon A \cup |K^0| \rightarrow A$ as provided by the Replacement Lemma (4.7). Let L be the subcomplex of K consisting of the simplices s of K such that $d(r^0 U, r^0 V) < \xi$ for all vertices U, V of s.

LEMMA 5.3. *$A \cup |L|$ is an equivariant neighborhood of A in Z.*

Proof. Let $z \in A$. By the countinuity of r^0 there is a neighborhood N of z in Z such that $d(r^0 U, z) < \xi/2$ for every $U \in (A \cup K^0) \cap N$. Thus $N \subset A \cup |L|$ since every simplex of K in N is in L.

We shall construct an extension of f over $A \cup |L|$. By induction, we construct a sequence of equivariant maps

$$h_m: A \cup |L^m| \longrightarrow Y$$

such that h_m extends h_{m-1} and that the following condition (5.4.m) holds:

(5.4.m) $$\operatorname{diam}(h_m s) \leqq \varepsilon_m \,, \quad \text{for each } m\text{-simplex } s \text{ of } L^m \,.$$

Define $h_0: A \cup |L^0| \to Y$ by $h_0 U = f r^0 U$, for each vertex U of L. Suppose that $h_{m-1}: A \cup |L^{m-1}| \to Y$ is defined so that condition (5.4.m–1) holds. Let $M = L^m - L^{m-1}$ be the set of the m-simplices of L; and let Λ denote the set of orbits of the simplices of M under the map c. Let $\varphi: \Lambda \to M$ be any cross-section of the identification map $M \to \Lambda$ and $N = \varphi(\Lambda)$. Since c acts freely on K, it follows that M is the disjoint union $M = N \cup (cN) \cup \cdots \cup (c^{p-1}N)$ and thus for each simplex t of M there is a unique simplex s of N and a unique integer j, $0 \leqq j < p$, such that $t = c^j s$.

By (5.4.m–1) we have for each simplex s of M

$$\operatorname{diam}(h_{m-1}(\dot{s})) \leqq 2\varepsilon_{m-1}$$

and $2\varepsilon_{m-1} \leqq \eta(\varepsilon_m/4) \leqq \varepsilon_m/4 \leqq \delta$. Therefore $\operatorname{Conv}(h_{m-1}(\dot{s})) \subset C$.

Let t be a closed m-simplex of M. Choose a simplex s of N and an integer j, $0 \leqq j < p$, such that $t = c^j s$; thus $t \in c^j N$. The map $h_{m-1}|\dot{s}: \dot{s} \to Y \subset C$, where $\dot{s}$ is the boundary of s, can be extended in $\operatorname{Conv}(h_{m-1}(\dot{s}))$ to a map $u: s \to C$. Then both u and $r \circ u: s \to Y$ extend $h_{m-1}|\dot{s}$. Note that if $x \in s$ then by (5.2),

$$d(ux, Y) \leqq 2\varepsilon_{m-1} = \eta\left(\frac{\varepsilon_m}{4}\right),$$

and by (5.1),

$$d(rux, ux) \leqq \frac{\varepsilon_m}{4} \,.$$

Therefore

(5.5) $$\operatorname{diam}((ru)s) \leqq \operatorname{diam}(us) + 2 \cdot \frac{\varepsilon_m}{4} \leqq \frac{\varepsilon_m}{4} + \frac{\varepsilon_m}{2} < \varepsilon_m \,.$$

Let $v^{(t)}: t \to Y$ be defined by

$$v^{(t)} = b^j \circ r \circ u \circ c^{p-j} . \tag{5.6}$$

Then the map $v^{(t)}$ agrees with h_{m-1} on $\dot{t}$, since h_{m-1} is equivariant. It follows that the maps $v^{(t)}$, $t \in M$, together with h_{m-1}, define an equivariant map

$$h_m: A \cup |L^{m-1}| \cup |M| = A \cup |L^m| \longrightarrow Y .$$

The fact that h_m satisfies the inductive condition (5.4.m) follows from the fact that $v^{(t)}$, $t \in M$, satisfies it by (5.5), (5.6) and since b is isometric.

This completes the inductive step of the construction of h_m. The maps h_m define a map $h: A \cup |L| \to Y$ by $h|(A \cup |L|) = h_m$. The map h is continuous on $|L|$ since it is defined simplicially there. Hence, it suffices to prove the continuity of h on A. Let $z \in A$ and let $B(hz, \varepsilon)$ be an open ε-ball in Y with center $fz = hz$. Let $\varepsilon_n = (1/2)\varepsilon$ and let positive numbers $\varepsilon_0, \cdots, \varepsilon_{n-1}$ be constructed as in (5.1). By the continuity of the maps

$$A \cup |K^0| \xrightarrow{r^0} A \xrightarrow{f} Y$$

there is a neighborhood G of z in Z such that $f(A \cap G) \subset B(fz, \varepsilon/2)$, $G \cap |K|$ is the union of open simplices of K, and for each simplex s of K in G, $fr^0(s^0) \subset B(fz, \varepsilon_0/2)$ (here s^0 denotes the set of the vertices of s). Then diam $(s^0) < \varepsilon_0$ and, by the construction of (5.1), it follows that diam $(hs) < \varepsilon/2$. Therefore $hG \subset B(hz, \varepsilon)$. This completes the proof in case (ii).

In case (i), when Y is an AR, there is retraction $r: Q \to Y$, i.e., we may take $C = Q$, which is convex. In this case the construction simplifies: we may take $\varepsilon_n = \infty$ which makes conditions (5.1) and (5.2) vacuous and $L = K$. By inductions we can define a map $h: A \cup |K| \to Y$ as before. The continuity of h must, however, be proved as in case (ii), by using the numbers defined in (5.1).

In either case, we have constructed a symmetric map $h: A \cup |L| \to Y$, where $A \cup |L|$ is a symmetric neighborhood of A in Z in case (ii) and $L = K$ in case (i). Define $g = h \circ \mu | \mu^{-1}(A \cup |L|): \mu^{-1}(A \cup |L|) \to Y$. Then h is a symmetric extension of f over the symmetric neighborhood $\mu^{-1}(A \cup |L|)$ of A in X which in case (i) is the whole of X.

This completes the proof.

6. Equivariant absolute retracts.

THEOREM 6.1. *Let (X, a) be an object of $\mathscr{A}_p$ such that X is a compact metric space with* $\dim X < \infty$. *Then:*

(i) *X is an EAR iff both X and the fixed point set $F(a)$ are AR's.*
(ii) *X is an EANR iff both X and the fixed point set $F(a)$ are ANR's.*

Proof. By (2.4) we can assume that (X, a) is equivariantly embedded in a finite-dimensional cube I^n with an isometric periodic map $a: I^n \to I^n$ of period p which we still denote by $a: I^n \to I^n$. Let $F = F(a_{I^n})$; then $F(a_X) = F \cap X$.

We shall prove case (ii); case (i) is just simpler and was done in § 1. If X is an EANR then there is an equivariant retraction $r: W \to X$ of an equivariant neighborhood W of X in I^n to X. The retraction r defines a retraction of $F \cap W$ to $F \cap X$. Since $a: I^n \to I^n$ is isometric, F is convex and compact, hence it is an AR (in fact, it is homeomorphic to a cube). Thus both X and $F \cap X$ are ANR's.

Suppose now that both X and $F \cap X$ are ANR's. Then by the Addition Theorem for ANR's ([1], p. 90), it follows that $F \cup X$ is an ANR. By the Equivariant Extension Theorem (1.1), the identity $F \cup X \to F \cup X$ can be extended to an equivariant retraction $r: U \to F \cup X$, where U is an equivariant neighborhood of $F \cup X$ in I^n. Since $F \cap X$ is an ANR, there is a neighborhood V of $F \cap X$ in F and a retraction $q: V \to F \cap X$. Note that $V \cup X$ is a neighborhood of X in $F \cup X$. Let $U_0 = r^{-1}(V \cap X)$. Then U_0 is an equivariant neighborhood of X in I^n and the map $U_0 \to X$ defined by $x \mapsto qrx$ is an equivariant retraction of U_0 to X.

Since the cube I^n with an isometric periodic map period p is an EAR (see (3.6)), it follows that X is an EANR.

We thus have an answer to our original question.

COROLLARY 6.2. *Let E be a Euclidean space, let F be the diagonal of $E \times E$ and let X be an equivariant compact subset of $E \times E$ (with respect to the involution $(x, y) \to (y, x)$). Then X is an equivariant retract of $E \times E$ if and only if X is a retract of $E \times E$ and $F \cap X$ is a retract of F.*

For, in this case, F is the fixed point set of the involution $E \times E \to E \times E$.

Just as in Theorem (1.1), it is an open question whether the finite-dimensional assumption in Theorem (6.1) is essential:

Question 6.3. Does there exist a space with an involution $a: X \times X$ such that both X and the fixed point set $F(a)$ are AR's but X is not an EAR?

More specifically, let Q be a Hilbert cube and consider the symmetry $Q \times Q \to Q \times Q$ with respect to the diagonal F of $Q \times Q$. Let

X be a symmetric subset of $Q \times Q$ such that X is a retract of $Q \times Q$ and $F \cap X$ is a retract of F. Does there exist a symmetric retraction of $Q \times Q$ to X?

7. **Equivariant homotopy.** As an application of the previous results, we prove in this section two equivariant homotopy extension theorems.

DEFINITION 7.1. If (X, a) is an object of $\mathscr{A}_p$ and I is the unit interval then an equivariant homotopy is an equivariant map $h: X \times I \to Y$ to an object (Y, b) of $\mathscr{A}_p$, where the periodic map on $X \times I$ is $a \times 1_I: X \times I \to X \times I$.

If A is an equivariant subspace of X then the maps a and $a \times 1_I$ define a periodic map $(a \times 1_I)_T: T \to T$, where $T = (X \times \{0\}) \cup (A \times I) \subset X \times I$; it is the restriction of $a \times 1_I$. The following lemma is an equivariant version of the Dowker lemma used in extending homotopies:

LEMMA 7.2. *Let (X, a) be an object of $\mathscr{A}_p$ such that X is a metric space, let A be a closed equivariant subspace of X and let $g: (T, (a \times 1_I)_T) \to (Y, b)$ be an equivariant map, where (Y, b) is an object $\mathscr{A}_p$. If g can be extended to an equivariant map $g': U \to Y$ of an equivariant neighborhood U of T in $X \times I$ then g can be extended to an equivariant map $h: X \times I \to Y$.*

Proof. Choose an equivariant neighborhood V of A in X such that $(\mathrm{Cl}\, V) \times I \subset U$ and an equivariant Urysohn function $u: X \to I$ which is 1 on A and 0 on $X - V$; for instance, u may be defined by using an equivariant distance function d on X with the usual formula:

$$ux = \frac{d(x, X - V)}{d(x, A) + d(x, X - V)} .$$

Then we define $h(x, t) = g'(x, (ux) \cdot t)$.

THEOREM 7.3. *Let (X, a) be an object of $\mathscr{A}_p$ such that X is a metric space, let (Y, b) be an EANE and let $f: X \to Y$ be an equivariant closed subspace of X. Then any equivariant homotopy of $f | A$ can be extended to an equivariant homotopy of f.*

This follows from (7.2) and (3.1). Similarly, (7.2) and (1.1) yield the following result:

THEOREM 7.4. *Let (X, a) be an object of $\mathscr{A}_p$ such that X is a compact metric finite-dimensional space, let A be a closed equivariant* *subspace of X containing all the fixed points of a, let (Y, b) be an object of* $\mathscr{A}_p$ *such that Y is a compact ANR, and let* $f: X \to Y$ *be an equivariant map. Then any equivariant homotopy of* $f|A$ *can be extended to an equivariant homotopy f.*

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