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ENUMERATION OF UP-DOWN PERMUTATIONS BY NUMBER OF RISES

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It is well known that $A(n)$, the number of up-down permutations of $\{1, 2, \cdots, n\}$ satisfies

$$\sum_{n=0}^{\in} A(2n)\frac{z^{2n}}{(2n)!} = \sec z ,$$

$$\sum_{n=0}^{\infty} A(2n+1)\frac{z^{2n+1}}{(2n+1)!} = \tan z .$$

In the present paper generating functions are obtained for the number of up-down permutations counting the number of rises among the "peaks".

1. If $(a_1, a_2, \cdots, a_n)$ denotes an arbitrary up-down permutation, then $(b_1, b_2, \cdots, b_n)$, where

$$b_i = n - a_i + 1 \qquad (i = 1, 2, \cdots, n) ,$$

is a down-up permutation and vice versa.

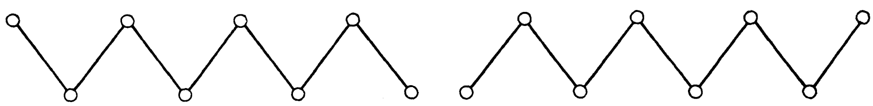

FIGURE 1

Thus, for $n > 1$, there is a one-to-one correspondence between up-down and down-up permutations so that it suffices to consider the former.

Let $A(n, r)$ denote the number of *up-down* permutations of $Z_n = \{1, 2, \cdots, n\}$ with r rises on the top line.

Let $C(n, r)$ denote the number of *down-up* permutations with r rises on the top line.

A *rise* is a pair of consecutive elements a, b with $a < b$. Also we agree to count a conventional rise on the left. For example

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have 3 and 2 rises, respectively.

It will be instructive first to derive the generating functions for $A(2n+1)$ and $A(2n)$. We have $A(1) = 1$ and

$$A(2n+1) = \sum_{k=0}^{n-1} \binom{2n}{2k+1} A(2k+1)A(2n-2k-1) \qquad (n > 0) . \tag{1.1}$$

Hence if we put

$$F(z) = \sum_{n=0}^{\infty} A(2n+1)\frac{z^{2n+1}}{(2n+1)!} ,$$

it follows from (1.1) that

$$F'(z) = 1 + F^2(z) .$$

Since $F(0) = 0$, we get $F(z) = \tan z$.

Next

$$A(2n) = \sum_{k=0}^{n-1} \binom{2n-1}{2k+1} A(2k+1)A(2n-2k-2) , \tag{1.2}$$

where $A(0) = 1$. Hence if

$$G(z) = \sum_{n=0}^{\infty} A(2n)\frac{z^{2n}}{(2n)!} ,$$

it follows from (1.2) that

$$G'(z) = F(z)G(z) .$$

Since $G(0) = 1$, this gives $G(z) = \sec z$. Thus we have proved that [1], [2, pp. 105–112]

$$\sum_{n=0}^{\infty} A(n)\frac{z^n}{n!} = \sec z + \tan z . \tag{1.3}$$

2. Turning next to $A(2n+1, r)$ we take

$$A(1, 0) = 1,\ A(1, r) = 0 \qquad (r > 0) .$$

Corresponding to (1.1) we have the recurrence

$$A(2n+1, r) = \sum_{k=0}^{n-1} \sum_{s=0}^{r} \binom{2k+1}{2n} A(2k+1, s)A^*(2n-2k-1, r-s) , \tag{2.1}$$

where

$$A^*(2n+1, r) = A(2n+1, r) \qquad (n > 0)$$

but

$$A^*(1, 0) = 0,\ A^*(1, 1) = 1 .$$

Put

$$A_{2n+1}(x) = \sum_r A(2n+1, r)x^r,\ A_1(x) = 1$$
$$A^*_{2n+1}(x) = A_{2n+1}(x) \qquad (n > 0),\ A^*_1(x) = x .$$

Then (2.1) gives

$$A_{2n+1}(x) = \sum_{k=0}^{n-1} \binom{2k+1}{2n} A_{2k+1}(x) A^{*}_{2n-2k-1}(x) \qquad (n > 0) . \tag{2.2}$$

Hence if

$$A(z) = A(x, z) = \sum_{n=0}^{\infty} A_{2n+1}(x) \frac{z^{2n+1}}{(2n+1)!} , \tag{2.3}$$

it follows from (2.2) that

$$\begin{aligned} A'(z) &= \sum_{n=0}^{\infty} A_{2n+1}(x) \frac{z^{2n}}{(2n)!} \\ &= 1 + \sum_{k=0}^{\infty} A_{2k+1}(x) \frac{z^{2k+1}}{(2k+1)!} \sum_{n=1}^{\infty} A^{*}_{2n-1}(x) \frac{z^{2n-1}}{(2n-1)!} , \end{aligned}$$

so that

$$\begin{aligned} A'(z) &= 1 + A(z)(A(z) - (1-x)z) \\ &= 1 - (1-x)zA(z) + A^2(z) . \end{aligned} \tag{2.4}$$

If we put

$$A(z) = \frac{1}{U}\frac{dU}{dz}, \quad \frac{dA}{dz} = \frac{1}{U^2}\left(\frac{dU}{dz}\right)^2 - \frac{1}{U}\frac{d^2U}{dz^2} ,$$

(2.4) becomes

$$\frac{d^2U}{dz^2} + (1-x)z\frac{dU}{dz} + U = 0 . \tag{2.5}$$

It is clear that U is an even function of z. We accordingly put

$$U = \sum_{n=0}^{\infty} (-1)^n a_n(x) \frac{z^{2n}}{(2n)!} \qquad (a_0(x) = 1) .$$

Substituting in (2.5) we get

$$-a_{n+1}(x) + 2n(1-x)a_n(x) + a_n(x) = 0 ,$$

so that

$$a_{n+1}(x) = (1 + 2n(1-x))a_n(x) . \tag{2.6}$$

It follows at once from (2.6) that

$$a_n(x) = \prod_{k=0}^{n-1} (1 + 2k(1-x)) .$$

Hence

$$U = \sum_{n=0}^{\infty} (-1)^n \prod_{k=0}^{n=1} (1 + 2k(1-x)) \cdot \frac{z^{2n}}{(2n)!} , \tag{2.7}$$

and

$$(2.8) \qquad A(z) = \frac{\sum_{n=0}^{\infty} (-1)^n \prod_{k=0}^{n} (1 + 2k(1-x)) \cdot \dfrac{z^{2n+1}}{(2n+1)!}}{\sum_{n=0}^{\infty} (-1)^n \prod_{k=0}^{n-1} (1 + 2k(1-x)) \cdot \dfrac{z^{2n}}{(2n)!}} .$$

The first few coefficients are given by

$$A_1(x) = 1,\ A_3(x) = 2x,\ A_5(x) = 8x + 8x^2,\ A_7(x) = 48x + 176x^2 + 48x^3 .$$

It follows by induction from

$$A_{2n+1}(x) = \sum_{k=1}^{n-2} \binom{2n+1}{2k} A_{2k+1}(x) A_{2n-2k-1}(x) + 2n(1+x) A_{2n-1}(x) \qquad (n > 1)$$

that

$$(2.9) \qquad x^{n+1} A_{2n+1}\left(\frac{1}{x}\right) = A_{2n+1}(x) .$$

This implies

$$(2.10) \qquad A(2n+1, r) = A(2n+1, n-r+1) \qquad (1 \leqq r \leqq n) .$$

Also, using the fuller notation $A(x, z)$, we have

$$(2.11) \qquad x^{1/2} A\left(\frac{1}{x}, x^{1/2} z\right) = (x-1)z + A(z, x) .$$

3. Now we consider the case $A(2n, r)$. We take

$$A(0, 0) = 1,\ A(0, r) = 0 \qquad (r > 0) .$$

Corresponding to (1.2) we have the recurrence

$$(3.1) \qquad A(2n+2, r) = \sum_{k=0}^{n} \sum_{s} \binom{2n+1}{2k+1} A(2k+1, s) A^*(2n-2k, r-s) \qquad (n \geqq 0) ,$$

where

$$A^*(2n, r) = A(2n, r) \qquad (n > 0) ,$$

but

$$A^*(0, 0) = 0,\ A^*(0, 1) = 1 .$$

Now put

$$A_{2n}(x) = \sum_r A(2n, r)x^r,\ A_0(x) = 1\ ,$$
$$A^*_{2n}(x) = A_{2n}(x)(n > 0),\ A^*_0(x) = x\ .$$

Then (3.1) gives

$$A_{2n+2}(x) = \sum_{k=0}^{n} \binom{2n+1}{2k+1} A_{2k+1}(x)A^*_{2n-2k}(x) \qquad (n \geqq 0)\ . \tag{3.2}$$

Hence if

$$B(z) = B(x, z) = \sum_{n=0}^{\infty} A_{2n}(x)\frac{z^{2n}}{(2n)!}\ , \tag{3.3}$$

we have

$$B'(z) = A(z)(B(z) - 1 + x)\ .$$

Replacing $A(z)$ by U'/U, we get

$$UB' + U'B = (1 - x)U'\ . \tag{3.4}$$

Since $B(0) = 1$, $U(0) = 1$, it follows from (3.4) that

$$UB = x + (1 - x)U\ .$$

Therefore

$$B(z) = 1 - x + \frac{x}{U}\ . \tag{3.5}$$

The first few coefficients are

$$A_0(x) = A_2(x) = x,\ A_4(x) = 3x + 2x^2,\ A_6(x) = 15x + 38x^2 + 8x^3\ .$$

4. We turn now to $C(2n, r)$. We take

$$C(0, 0) = 1,\ C(0, r) = 0 \qquad (r > 0)\ .$$

We have the recurrence

$$C(2n + 2, r) = \sum_{k=0}^{n}\sum_{s} \binom{2n+1}{2k} C(2k, s)A^*(2n - 2k + 1, r - s)\ , \tag{4.1}$$

where $A^*(2k + 1, s)$ has the same meaning as in §2.

Thus, if

$$C_{2n}(x) = \sum_r C(2n, r)x^r\ ,$$

we get

$$C_{2n+2}(x) = \sum_{k=0}^{n} \binom{2n+1}{2k} C_{2k}(x)A^*_{2n-2k+1}(x)\ . \tag{4.2}$$

Put

$$C(z) = C(x, z) = \sum_{n=0}^{\infty} C_{2n}(x)\frac{z^{2n}}{(2n)!} . \tag{4.3}$$

Then it follows from (4.2) that

$$C'(z) = C(z)(A(z) - (1 - x)z) , \tag{4.4}$$

so that

$$\frac{C'(z)}{C(z)} = \frac{U'}{U} - (1 - x)z .$$

Since $C(0) = 1$, this yields

$$C(z) = \frac{1}{U}e^{-1/2(1-x)z^2} . \tag{4.5}$$

The first few coefficients are

$C_0(x) = 1$, $C_2(x) = x$, $C_4(x) = 2x + 3x^2$, $C_6(x) = 8x + 38x^2 + 15x^3$.

We shall now show that $U = U(x, z)$ satisfies the functional equation

$$U\left(\frac{1}{x}, x^{1/2}z\right)e^{-1/2(1-x)z^2} = U(x, z) , \tag{4.6}$$

or

$$\sum_0^{\infty}(- 1)^n a_n\left(\frac{1}{x}\right)\frac{x^n z^{2n}}{(2n)!} \sum_0^{\infty} (- 1)^k \frac{(1 - x)^k z^{2k}}{2^k \cdot k!} = \sum_0^{\infty}(- 1)^n a_n(x)\frac{z^{2n}}{(2n)!} .$$

This is equivalent to

$$\sum_{k=0}^{n} \frac{(2n)!}{(2k)!(n - k)!} \left(\frac{1 - x}{2}\right)^{n-k} a_k\left(\frac{1}{x}\right) = a_n(x) . \tag{4.7}$$

The left hand side of (4.7) is equal to

$$\frac{(2n)!}{n!} \left(\frac{1 - x}{2}\right)^n \sum_{k=0}^{n} (- 1)^k \frac{(- n)_k}{k!\left(\frac{1}{2}\right)_k}\left(\frac{x}{2(1 - x)}\right)^k \prod_{j=0}^{k-1}\left(1 + 2j\left(1 - \frac{1}{x}\right)\right)$$

$$= \frac{(2n)!}{n!} \left(\frac{1 - x}{2}\right)^n \sum_{k=0}^{n} \frac{(- n)_k}{k!\left(\frac{1}{2}\right)_k}\left(\frac{x}{2(k - 1)}\right)_k$$

$$= \frac{(2n)!}{n!} \left(\frac{1 - x}{2}\right)^n \frac{\left(\frac{1}{2(1 - x)}\right)_n}{\left(\frac{1}{2}\right)_n} = 2^n(1 - x)^n\left(\frac{1}{2(1 - x)}\right)_n = a_n(x) ,$$

by Vandermonde's theorem.

It evidently follows from (3.5), (4.5), and (4.6) that

$$C_{2n}(x) = x^{n+1}A_{2n}\left(\frac{1}{x}\right) \qquad (n > 0) \tag{4.8}$$

and therefore

$$C(2n, r) = A(2n, n - r + 1) \qquad (1 \leqq r \leqq n)\,. \tag{4.9}$$

5. Finally we consider $C(2n + 1, r)$. We now take

$$C(1, 1) = 1,\ C(1, r) = 0 \qquad (r \neq 1).$$

We have the recurrence

$$C(2n + 1, r) = \sum_{k=0}^{n}\sum_{s}\binom{2n}{2k}C(2k)A^*(2n - 2k, r - s)\,. \tag{5.1}$$

Thus, if

$$C_{2n+1}(x) = \sum_{r} C(2n + 1, r)x^r\,,$$

it follows that

$$C_{2n+1}(x) = \sum_{k=0}^{n}\binom{2n}{2k}C_{2k}(x)A^*_{2n-2k}(x)\,. \tag{5.2}$$

Put

$$D(z) = D(x, z) = \sum_{n=0}^{\infty}C_{2n+1}(x)\frac{z^{2n+1}}{(2n + 1)!}\,.$$

Then, by (5.2),

$$D'(z) = C(z)(B(z) - 1 + x)\,. \tag{5.3}$$

It follows that

$$\begin{aligned} D'(z) &= \frac{x}{U^2(x, z)}e^{-1/2(1-x)z^2} \\ &= \frac{x}{U(x, z)\,U(x^{-1}, x^{1/2}z)} \\ &= xC(x, z)C(x^{-1}, x^{12}z)\,. \end{aligned}$$

Therefore

$$C_{2n+1}(x) = \sum_{k=0}^{n}\binom{2n}{2k}x^{n-k+1}C_{2k}(x)C_{2n-2k}(x^{-1})\,. \tag{5.4}$$

It follows from (5.4) that

$$x^{n+2}C_{2n+1}(x^{-1}) = C_{2n+1}(x) ,$$

so that

(5.5) $$C(2n + 1, r) = C(2n + 1, n - r + 2) \quad (1 \leqq r \leqq n + 1) .$$

The first few values of $C_{2n+1} + (x)$ are given by

$$C_1(x) = x,\ C_3(x) = x + x^2,\ C_5(x) = 3x + 10x^2 + 3x^3,\ C_7(x) \\ = 15x + 121x^2 + 121x^3 + 15x^4 .$$

Note that $C_{2n+1}(x)$ is of degree $n + 1$.

6. A number of special values can be obtained. It follows first from

$$A_{2n+1}(x) = \sum_{k=1}^{n-2} \binom{2n}{2k+1} A_{2k+1}(x) + 2n(1 + x)A_{2n-1}(x) \qquad (n > 1)$$

and

$$x \mid A_{2k+1}(x) \qquad (k > 0)$$

that

$$A'_{2n+1}(0) = 2nA'_{2n-1}(0) .$$

This yields

(6.1) $$A(2n + 1, 1) = 2nA(2n - 1, 1) = 2^n n! .$$

Next, it follows from

$$A_{2n+2}(x) = (2n + 1)A_{2n}(x) + \sum_{k=1}^{n-1} \binom{2n+1}{2k+1} A_{2k+1}(x)A_{2n-2k}(x) + xA_{2n+1}(x)$$

and

$$x \mid A_{2k}(x) \qquad (k > 0)$$

that

$$A'_{2n+2}(0) = (2n + 1)A'_{2n}(0) .$$

This gives

(6.2) $$A(2n, 1) = (2n - 1)(2n - 3) \cdots 3.1 .$$

It follows from

$$C_{2n+2}(x) = A_{2n+1}(x) + \sum_{k=1}^{n-1} \binom{2n+1}{2k} C_{2k}(x)A_{2n-2k+1}(x) \\ + (2n + 1)xC_{2n}(x)$$

and

$$x \mid C_{2k}(x) \qquad (k > 0)$$

that

$$C'_{2n+2}(x) = A'_{2n+1}(0) \qquad (n > 0) .$$

Hence

$$C(2n + 2, 1) = 2^n n! . \tag{6.3}$$

Finally, from

$$C_{2n+1}(x) = A_{2n}(x) + \sum_{k=1}^{n-1} \binom{2n}{2k} C_{2k}(x) A_{2n-2k}(x) + x C_{2n}(x)$$

and

$$x \mid C_{2k+1}(x) \qquad (k \geqq 0) ,$$

we get

$$C'_{2n+1}(0) = A'_{2n}(0) ,$$

so that

$$C(2n + 1, 1) = (2n - 1)(2n - 3) \cdots 3.1 . \tag{6.4}$$

In view of (4.9),

$$C(2n, r) = A(2n, n - r + 1) \qquad (1 \leqq r \leqq n) ,$$

we have also

$$A(2n + 2, n + 1) = 2^n n! , \tag{6.5}$$

$$C(2n, n) = (2n - 1)(2n - 3) \cdots 3.1 . \tag{6.6}$$

7. We remark that the differential equation

$$U'' + (1 - x) z U' + U = 0 \tag{7.1}$$

has the second solution $W = UV$, where

$$V' = e^{-1/2(1-x)z^2} U^{-2} . \tag{7.2}$$

Thus, by (5.3), we have

$$D'(z) = x V'(z) ,$$

so that

$$D(z) = x V(z) . \tag{7.3}$$

Since W is an odd function of z we may put

$$W(z) = = \sum_{0}^{\infty} (-1)^n b_n(x) \frac{z^{2n+1}}{(2n+1)!} .$$

It follows from the differential equation that

$$b_{n+1}(x) = (1 + (2n+1)(1-x))b_n(x) ,$$

so that

$$b_n(x) = \prod_{k=0}^{n-1} (1 + (2k+1)(1-x)) . \tag{7.4}$$

Finally we have

$$D(z) = x \frac{\sum_{n=0}^{\infty} (-1)^n \prod_{k=0}^{n-1} (1 + (2k+1)(1-x)) \cdot \frac{z^{2n+1}}{(2n+1)!}}{\sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^{n-1} (1 + 2k(1-x)) \cdot \frac{z^{2n}}{(2n)!}} . \tag{7.5}$$

We may, if we prefer, express both U and V as hypergeometric functions of the type ${}_1F_1$.

References

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