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ESSENTIAL PRODUCTS OF NONSINGULAR RINGS

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By an essential product of two rings is meant a subdirect product which contains an essential right ideal of the direct product. The aim of this paper is to investigate the utility of this concept in the study of nonsingular rings. The first section derives some basic properties of essential products and develops some criteria for recognizing essential products. In the second section, a study of the socles of nonsingular modules leads to a theorem that any nonsingular ring is an essential product of a ring with essential socle and a ring with zero socle. The third section is devoted to a theorem which tells when an essential product can be a splitting ring, i.e., a ring such that the singular submodule of any right module is a direct summand. In the final section, this theorem is used to construct two examples of splitting rings of types previously unknown.

In this paper all rings are associative with identity, and all modules are unital. We also require that a subring of a ring have the same identity as the ring. Unless otherwise noted, all modules are right modules.

Inasmuch as we use singular and nonsingular modules throughout this paper, we recall the relevant definitions here. Given a ring R, we use $\mathscr{S}(R)$ to denote the collection of essential right ideals of R; then the *singular submodule* of a right R-module A is the set $Z_R(A) = \{a \in A \mid aI = 0 \text{ for some } I \in \mathscr{S}(R)\}$. The module A is said to be *singular* [*nonsingular*] provided $Z_R(A) = A[Z_R(A) = 0]$. The singular submodule of R_R is a two-sided ideal of R, called the *right singular ideal* of R and denoted $Z_r(R)$; R is a *right nonsingular ring* when $Z_r(R) = 0$.

1. Essential products.

Given two right nonsingular rings R_1 and R_2, we define an *essential product* of R_1 and R_2 to be any subdirect product R of R_1 and R_2 which contains an essential right ideal of $R_1 \times R_2$. [Recall that for R to be a subdirect product of R_1 and R_2, R must be a subring of $R_1 \times R_2$ such that the projections $R \to R_1$ and $R \to R_2$ are both surjective.] The aim of this section is to consider the relationships among singular and nonsingular modules over R_1, R_2, and R, and to establish criteria for judging which rings are essential products.

N.B.-For the first three propositions in this section, we assume that R is an essential product of two right nonsingular rings R_1 and

R_2. We define $E_1 \subset R_1$ and $E_2 \subset R_2$ by the conditions $E_1 \times 0 = R \cap (R_1 \times 0)$ and $0 \times E_2 = R \cap (0 \times R_2)$. Inasmuch as $E_1 \times 0$ is a two-sided ideal of R and R is a subdirect product of R_1 and R_2, E_1 must be a two-sided ideal of R_1. Likewise, E_2 is a two-sided ideal of R_2, hence $E_1 \times E_2$ is a two-sided ideal of $R_1 \times R_2$ as well as a two-sided ideal of R. Note that the induced ring homomorphisms $R/(E_1 \times 0) \to R_2$ and $R/(0 \times E_2) \to R_1$ are both isomorphisms, from which we conclude that the induced ring homomorphisms

$$R/(E_1 \times E_2) \longrightarrow R_2/E_2 \quad \text{and} \quad R/(E_1 \times E_2) \longrightarrow R_1/E_1$$

are also isomorphisms.

Since R is an essential product, it must contain some essential right ideal I of $R_1 \times R_2$. Noting that $I = I_1 \times I_2$ for some $I_1 \in \mathscr{S}(R_1)$ and $I_2 \in \mathscr{S}(R_2)$, we infer from $I_1 \times 0 \leqq E_1 \times 0$ that $E_1 \in \mathscr{S}(R_1)$; likewise $E_2 \in \mathscr{S}(R_2)$. Consequently $E_1 \times E_2 \in \mathscr{S}(R_1 \times R_2)$.

PROPOSITION 1. *Let* $T = R_1 \times R_2$ *and* $E = E_1 \times E_2$.

(a) $\mathscr{S}(T) = \{K \leqq T_T \mid K \cap R \in \mathscr{S}(R)\}$.

(b) $\mathscr{S}(R) = \{J \leqq R_R \mid JE \in \mathscr{S}(T)\}$.

(c) $Z_T(A) = Z_R(A)$ *for all* A_T.

(d) $Z_r(R) = Z_R(T) = 0$.

Proof. (a) Suppose that $K \in \mathscr{S}(T)$ and $A \leqq R_R$ such that $A \cap (K \cap R) = 0$. Then $AE \cap K = 0$, hence from $K \in \mathscr{S}(T)$ we obtain $AE = 0$. Since $E \in \mathscr{S}(T)$ it follows that $A \leqq Z_r(T) = 0$, and so $K \cap R \in \mathscr{S}(R)$.

Now let $K \leqq T_T$ and assume that $K \cap R \in \mathscr{S}(R)$. If $A \leqq T_T$ and $A \cap K = 0$, then from $(A \cap R) \cap (K \cap R) = 0$ we obtain $A \cap R = 0$, hence $A \cap E = 0$. Thus $A = 0$ and so $K \in \mathscr{S}(T)$.

(b) If $J \leqq R_R$ and $JE \in \mathscr{S}(T)$, then $JE \in \mathscr{S}(R)$ by (a), whence $J \in \mathscr{S}(R)$.

Now consider any $J \in \mathscr{S}(R)$. Inasmuch as $E \in \mathscr{S}(T)$ and $Z_r(T) = 0$, the left annihilator of E in T is zero. In particular, it follows that every nonzero element of J has a nonzero right multiple in JE. Thus JE is an essential R-submodule of J, hence $JE \in \mathscr{S}(R)$, and then $JE \in \mathscr{S}(T)$ by (a).

(c) follows directly from (a) and (b).

(d) According to (c), $Z_R(T) = 0$, and then $Z_r(R) = 0$ also.

Letting Q_1 and Q_2 denote the respective maximal right quotient rings of R_1 and R_2, then $Q_1 \times Q_2$ is the maximal right quotient ring of $R_1 \times R_2$. Inasmuch as $Z_r(R_1 \times R_2) = 0$, [4, Theorem 1 + 2, p. 69] says that $Q_1 \times Q_2$ is a right self-injective ring.

PROPOSITION 2. $Q_1 \times Q_2$ *is also the maximal right quotient ring of* R.

Proof. Set $T = R_1 \times R_2$, $E = E_1 \times E_2$, and $Q = Q_1 \times Q_2$. Note that $T \cap Z_T(Q) = Z_r(T) = 0$, from which we obtain $Z_T(Q) = 0$.

We first show that Q is a right quotient ring of R, i.e., that Q_R is a rational extension of R_R. (See [4, pp. 58, 64].) Inasmuch as $Z_r(R) = 0$, [4, Proposition 5, p. 59] says that it suffices to prove that Q_R is an essential extension of R_R. Thus consider any $A \leqq Q_R$ such that $A \cap R = 0$. Then $AE \cap E = 0$. Since E is an essential right ideal of T it must be an essential T-submodule of Q, so that we obtain $AE = 0$ and $A \leqq Z_T(Q) = 0$. Therefore Q is a right quotient ring of R, hence we many assume that Q is a subring of the maximal right quotient ring P of R. The injectivity of Q_Q implies that $P_Q = Q \oplus B$ for some B. Then from $R \cap B = 0$ we infer that $B = 0$ and $P = Q$.

In view of Proposition 2, we may refer again to [4, Theorem 1 + 2, p. 69] and conclude that $(Q_1 \times Q_2)_R$ is an injective hull for R_R. Now we obtain from [7, Proposition 1, p. 427] the following alternate description of the singular submodule of a right R-module A: $Z_R(A) = \cap \{\ker f \mid f \in \mathrm{Hom}_R(A, Q_1 \times Q_2)\}$. In particular, A is singular if and only if $\mathrm{Hom}_R(A, Q_1 \times Q_2) = 0$, from which we conclude that any extension of a singular module by a singular module is singular. The corresponding property for nonsingular modules is a consequence of the observation that A is nonsingular if and only if $\mathrm{Hom}_R(R/I, A) = 0$ for all $I \in \mathscr{S}(R)$.

According to Proposition 1, any nonsingular right $R_1 \times R_2$-module is also a nonsingular right R-module. We can view this as saying that the direct sum of a nonsingular right R_1-module with a nonsingular right R_2-module gives a nonsingular right R-module. The converse is false unless R is actually the direct product of R_1 and R_2, but according to the next proposition we can at least realize any nonsingular right R-module as an extension of a nonsingular right R_1-module by a nonsingular right R_2-module.

PROPOSITION 3. (a) *A right R-module A is nonsingular if and only if it has a submodule A' such that A' is a nonsingular right R_1-module and A/A' is a nonsingular right R_2-module.*

(b) *A right R-module C is singular if and only if it has a submodule C' such that C' is a singular right R_1-module and C/C' is a singular right R_2-module.*

Proof. (a) If A has such a submodule A', then according to Proposition 1, A' and A/A' are both nonsingular R-modules, hence A must be nonsingular.

Now assume that A is nonsingular. In view of the discussion above, the intersection of the kernels of the homomorphisms from A into $Q_1 \times Q_2$ must be zero. Thus we may assume that A is a submodule of some direct product B of copies of $Q_1 \times Q_2$. Note that $B = B_1 \oplus B_2$, where B_i is a direct product of copies of Q_i; since Q_i is the maximal right quotient ring of R_i, B_i is a nonsingular right R_i-module. Consequently, $A' = A \cap B_1$ is a nonsingular right R_1-module, and A/A' is a nonsingular right R_2-module.

(b) If C has such a submodule C', then according to Proposition 1, C' and C/C' are both singular R-modules, whence C must be singular.

Conversely, assume that C is singular. Clearly $C' = C(E_1 \times 0)$ is a right R_1-module [because $C'(0 \times E_2) = 0$] and C/C' is a right R_2-module. Inasmuch as C' and C/C' are singular R-modules, Proposition 1 says that C' is a singular R_1-module and C/C' is a singular R_2-module.

N. B.-We now drop the a priori assumption that R is an essential product of R_1 and R_2, in order to find conditions under which R can be such an essential product.

THEOREM 4. *Let R_1 and R_2 be right nonsingular rings, R any subring of $R_1 \times R_2$. Then the following conditions are equivalent:*

(a) *R is an essential product of R_1 and R_2.*

(b) *There exist two-sided ideals $E_1 \in \mathscr{S}(R_1)$ and $E_2 \in \mathscr{S}(R_2)$, and a ring isomorphism ϕ: $R_1/E_1 \to R_2/E_2$, such that*

$$R = \{(x, y) \in R_1 \times R_2 \mid \phi(x + E_1) = y + E_2\} .$$

(c) *R is a subdirect product of R_1 and R_2, $R/[R \cap (R_1 \times 0)]$ and $R/[R \cap (0 \times R_2)]$ are both nonsingular right R-modules, and*

$$[R \cap (R_1 \times 0)] + [R \cap (0 \times R_2)] \in \mathscr{S}(R) .$$

Proof. Set $H_1 = R \cap (R_1 \times 0)$ and $H_2 = R \cap (0 \times R_2)$, both of which are two-sided ideals of R.

(a) $\Rightarrow$ (c): By definition, R is a subdirect product of R_1 and R_2. Set $E_1 \times 0 = H_1$ and $0 \times E_2 = H_2$: then $H_1 + H_2 = E_1 \times E_2$, which in the discussion prior to Proposition 1 is shown to belong to $\mathscr{S}(R_1 \times R_2)$. Quoting part (a) of Proposition 1, we see that $H_1 + H_2 \in \mathscr{S}(R)$. As for the modules $(R/H_1)_R$ and $(R/H_2)_R$, they are both isomorphic to submodules of $(R_1 \times R_2)_R$, which is nonsingular by Proposition 1.

(c) $\Rightarrow$ (b): Since R is a subdirect product of R_1 and R_2, it follows as in the discussion before Proposition 1 that there exist two-sided ideals E_1 in R_1 and E_2 in R_2 such that $E_1 \times 0 = H_1$ and $0 \times E_2 = H_2$, and that the induced maps $f\colon R/(E_1 \times E_2) \to R_1/E_1$ and $g\colon R/(E_1 \times E_2) \to R_2/E_2$ are ring isomorphisms. Setting $\phi = gf^{-1}$, we check that

$$R = \{(x, y) \in R_1 \times R_2 \mid \phi(x + E_1) = y + E_2\}\ .$$

Inasmuch as $(R/H_1)_R$ is nonsingular and $H_1 + H_2 \in \mathscr{S}(R)$, no nonzero element of R/H_1 is annihilated on the right by $H_1 + H_2$. Noting that the isomorphism $R/H_1 \to R_2$ [induced by the projection $R \to R_2$] carries $(H_1 + H_2)/H_1$ onto E_2, we infer that the left annihilator of E_2 in R_2 is zero. Since E_2 is a two-sided ideal of R_2, it follows that every nonzero element of R_2 has a nonzero right multiple in E_2, whence $E_2 \in \mathscr{S}(R_2)$. Likewise, $E_1 \in \mathscr{S}(R_1)$.

(b) $\Rightarrow$ (a): It is easy to check that under these hypotheses, R is a subdirect product of R_1 and R_2. Also, R contains $E_1 \times E_2$, which is an essential right ideal of $R_1 \times R_2$, hence R is an essential product.

Theorem 4 may be thought of as characterizing "external" essential products. As an immediate consequence of the equivalence of (a) and (c), we also get the following characterization of "internal" essential products.

COROLLARY 5. *Let R_1 and R_2 be right nonsingular rings. Then a ring R is isomorphic to an essential product of R_1 and R_2 if and only if there exist two-sided ideals H_1 and H_2 in R such that*

(a) $H_1 \cap H_2 = 0$.

(b) $R/H_1 \cong R_2$ *and* $R/H_2 \cong R_1$.

(c) $(R/H_1)_R$ *and* $(R/H_2)_R$ *are both nonsingular.*

(d) $H_1 + H_2 \in \mathscr{S}(R)$.

2. Socles and antisocles. The purpose of this section is to prove that any right nonsingular ring is isomorphic to an essential product of a ring with essential socle and a ring with zero socle. To this end we first develop some results about socles of nonsingular modules over an arbitrary ring R.

For any simple right R-module R/M, there are only two choices for the submodule $Z_R(R/M)$, hence R/M must be either singular or nonsingular. In case it is nonsingular, then $M \notin \mathscr{S}(R)$, whence $M \cap I = 0$ for some nonzero right ideal I of R. By the maximality of M, $R = M \oplus I$. Thus we see that every simple right R-module is either singular or projective. Consequently, every nonsingular semisimple right R-module is projective.

PROPOSITION 6. *Let* $J = \operatorname{soc}(R_R)$. *Then* $\operatorname{soc}(A) = AJ$ *for any nonsingular right R-module A.*

Proof. The module AJ is clearly semisimple, hence $AJ \leqq \operatorname{soc}(A)$.

On the other hand, any simple submodule B of A is projective and thus is isomorphic to eR for some idempotent $e \in R$, whence $B = BeR \leqq AJ$.

PROPOSITION 7. *Let A be any nonsingular right R-module, $J = \operatorname{soc}(R_R)$.*

(a) *$B = \{a \in A \mid aJ = 0\}$ is the largest submodule of A with zero socle.*

(b) *A/B has essential socle.*

(c) *A/B is nonsingular.*

Proof. (a) is immediate from Proposition 6.

(b) If N/B is any nonzero submodule of A/B, then since $N \nleqq B$ we must have $NJ \neq 0$. We infer from the semisimplicity of NJ that $NJ \cap B = 0$, whence $(NJ + B)/B \neq 0$ and thus $[N/B] \cap [(AJ + B)/B] \neq 0$. Therefore $(AJ + B)/B$ is an essential submodule of A/B. Inasmuch as $(AJ + B)/B$ is semisimple, we conclude that A/B has essential socle.

(c) Noting that $AJ \cap B = 0$, we see that $(AJ + B)/B \cong AJ$, which is a nonsingular module. Thus A/B is an essential extension of a nonsingular module, hence A/B must be nonsingular.

According to Proposition 7, a nonsingular right R-module A always has a largest submodule with zero socle; Professor Kaplansky has suggested the name *antisocle* for this submodule of A. We now proceed in a similar manner to show that A also has a largest submodule with essential socle. We refrain from introducing a name for this submodule, since in the presence of a suitable notion of closure it is describable simply as the closure of $\operatorname{soc}(A)$.

PROPOSITION 8. *Let A be any nonsingular right R-module. Set $J = \operatorname{soc}(R_R)$, and let H be the left annihilator of J in R.*

(a) *$C = \{a \in A \mid aH = 0\}$ is the largest submodule of A with essential socle.*

(b) *A/C is nonsingular.*

(c) *A/C has zero socle.*

Proof. Inasmuch as J is a two-sided ideal of R, H is a two-sided ideal also. Note that $J + H \in \mathscr{S}(R)$: For if $x \in R \backslash H$, then $xJ \neq 0$ and thus x has a nonzero right multiple in $J + H$.

(a) will follow immediately from the following claim: A nonsingular right R-module B has essential socle if and only if $BH = 0$.

First suppose that B has essential socle, i.e., that BJ is essential in B. Inasmuch as $BH \cap BJ$ is semisimple, we obtain $BH \cap BJ = (BH \cap BJ)J \leqq BHJ = 0$, from which $BH = 0$ follows. Conversely, if

$BH = 0$, then since $J + H \in \mathscr{S}(R)$ we must have $xJ \neq 0$ for all nonzero $x \in B$, from which we infer that BJ is essential in B.

(b) Inasmuch as $AHJ = 0$, we see that AH has zero socle, whence $AH \cap \operatorname{soc}(C) = 0$ and thus $AH \cap C = 0$. In particular, we find that $(AH + C)/C \cong AH$, which is a nonsingular module.

If N/C is any nonzero submodule of A/C, then since $N \nleqq C$ we must have $NH \neq 0$. Since $NH \cap C \leqq AH \cap C = 0$, it follows that $(NH + C)/C \neq 0$, whence $[N/C] \cap [(AH + C)/C] \neq 0$. Thus A/C is an essential extension of the nonsingular module $(AH + C)/C$, so A/C must be nonsingular.

(c) According to (a), $\operatorname{soc}(A) \leqq C$. Inasmuch as the nonsingularity of A/C implies that $\operatorname{soc}(A/C)$ is projective, we conclude that $\operatorname{soc}(A/C) = 0$.

When R is presented as an essential product of rings R_1 and R_2, both R_1 and R_2 are factor rings of R. Thus we need to know that certain factor rings of R are nonsingular, for which reason we introduce the next proposition. The proof is routine, and may be found in [8, Proposition 1.11].

PROPOSITION 9. *Let H be a two-sided ideal of R such that $(R/H)_R$ is nonsingular.*

(a) $\mathscr{S}(R/H) = \{I/H \mid H \leqq I$ *and* $I \in \mathscr{S}(R)\}$.

(b) $Z_{R/H}(A) = Z_R(A)$ *for all* $A_{R/H}$.

(c) $Z_r(R/H) = 0$.

THEOREM 10. *Let R be any right nonsingular ring. Then R is isomorphic to an essential product of two right nonsingular rings R_1 and R_2 such that R_1 has essential right socle and R_2 has zero right socle.*

Proof. Let $J = \operatorname{soc}(R_R)$, H the left annihilator of J in R, K the left annihilator of H in R. Since J is a two-sided ideal, so are H and K. The modules $(R/H)_R$ and $(R/K)_R$ are nonsingular according to Propositions 7 and 8, hence by Proposition 9, $R_1 = R/H$ and $R_2 = R/K$ are both right nonsingular rings.

Referring to Propositions 7 and 8 again, we see that H_R has zero socle while K_R has essential socle, hence $H \cap K = 0$. We must also note that $H + K \in \mathscr{S}(R)$: For if $X \in R \backslash K$, then $xH \neq 0$ and so x has a nonzero right multiple in $H + K$. According to Corollary 5, we obtain that R is isomorphic to an essential product of R_1 and R_2.

Inasmuch as $(R/H)_R$ has essential socle by Proposition 7, $(R/H)_{R/H}$ must have essential socle. Similarly, $(R/K)_R$ has zero socle by Proposition 8, whence $(R/K)_{R/K}$ has zero socle.

3. **Splitting rings.** This section is concerned with the question

of when an essential product can be a *splitting ring*, by which we mean a ring R such that for every right R-module A, $Z_R(A)$ is a direct summand of A. As is noted in [2, Proposition 1.12], R is a splitting ring if and only if $\mathrm{Ext}^1_R(A, C) = 0$ for all nonsingular A_R and all singular C_R.

N.B.-Throughout this section, we assume that R is an essential product of two right nonsingular rings R_1 and R_2. Setting $E_1 \times 0 = R \cap (R_1 \times 0)$ and $0 \times E_2 = R \cap (0 \times R_2)$, we recall from §1 that each E_i is a two-sided ideal in $\mathscr{S}(R_i)$. According to Proposition 1, we also have $E_1 \times E_2 \in \mathscr{S}(R)$. Setting $H_1 = E_1 \times 0$ and $H_2 = 0 \times E_2$, we recall that $R/H_1 \cong R_2$ and $R/H_2 \cong R_1$, while from Theorem 4 we obtain that R/H_1 and R/H_2 are nonsingular right R-modules.

LEMMA 11. *Let T be a splitting ring, H a two-sided ideal in $\mathscr{S}(T)$, A any nonsingular right T-module.*

(a) $\mathrm{Tor}^T_1(A, T/H) = 0$.

(b) *A/AH is a projective right (T/H)-module.*

Proof. (a) Choosing a divisible abelian group D which contains $\mathrm{Tor}^T_1(A, T/H)$, we note that $\mathrm{Hom}_Z(T/H, D)$ is a two-sided T-module. As a right T-module, $\mathrm{Hom}_Z(T/H, D)$ is singular because $H \in \mathscr{S}(T)$, whence $\mathrm{Ext}^1_T(A, \mathrm{Hom}_Z(T/H, D)) = 0$. According to [1, Proposition 5.1, p. 120], we obtain $\mathrm{Hom}_Z(\mathrm{Tor}^T_1(A, T/H), D) = 0$, and thus

$$\mathrm{Tor}^T_1(A, T/H) = 0 .$$

(b) Choose an exact sequence $S: 0 \to K \to F \to A/AH \to 0$, where F is a free right (T/H)-module. Inasmuch as $KH = 0$ and $H \in \mathscr{S}(T)$, K is a singular right T-module. Thus $\mathrm{Ext}^1_T(A, K) = 0$, hence the natural map $p: A \to A/AH$ lifts to a map $f: A \to F$. Since $FH = 0$, f induces a map $g: A/AH \to F$ which splits S.

THEOREM 12. *R is a splitting ring if and only if*

(a) *R_1 and R_2 are splitting rings.*

(b) *If L_i is any essential right R_i-submodule of E_i, then E_i/L_i is a direct summand of R_i/L_i.*

Proof. Assume that R is a splitting ring. If $H = H_1$ or $H = H_2$, and A is any right (R/H)-module, then $Z_{R/H}(A) = Z_R(A)$ by Proposition 9, hence $Z_{R/H}(A)$ is a summand of A. Therefore R/H_1 and R/H_2 are splitting rings; i.e., R_2 and R_1 are splitting rings.

Next consider any essential right R_1-submodule $L_1 \leqq E_1$. Noting that $L_1 \times 0$ is an essential right R-submodule of H_1, we see that $H_1/(L_1 \times 0)$ is a singular right R-module. Inasmuch as $(R/H_1)_R$ is nonsingular, the exact sequence

$$0 \longrightarrow H_1/(L_1 \times 0) \longrightarrow R/(L_1 \times 0) \longrightarrow R/H_1 \longrightarrow 0$$

must split. Tensoring with R/H_2, we obtain another split exact sequence

$$0 \longrightarrow (E_1 \times 0)/(L_1 \times 0) \longrightarrow R/(L_1 \times E_2) \longrightarrow R/(E_1 \times E_2) \longrightarrow 0 ,$$

from which we infer that E_1/L_1 is a summand of R_1/L_1. By symmetry, E_2/L_2 is also a summand of R_2/L_2 whenever L_2 is an essential right R_2-submodule of E_2.

Now suppose that (a) and (b) hold. Inasmuch as $E_1 \in \mathscr{S}(R_1)$ and $Z_r(R_1) = 0$, the left annihilator of E_1 in R_1 is zero. In particular, every nonzero element of E_1 has a nonzero right multiple in E_1^2, hence E_1^2 is an essential right R_1-submodule of E_1. According to (b), E_1/E_1^2 must be a summand of R_1/E_1^2, from which we infer that $E_1 = E_1^2$. Thus $H_1 = H_1^2$, and likewise $H_2 = H_2^2$.

We must show that $\mathrm{Ext}_R^1(A, C) = 0$ for any nonsingular A_R and any singular C_R. By Proposition 3, C has a submodule C' such that C' is a singular right R_1-module and C/C' is a singular right R_2-module. Since it suffices to prove that

$$\mathrm{Ext}_R^1(A, C') = 0 \quad \text{and} \quad \mathrm{Ext}_R^1(A, C/C') = 0 ,$$

we may thus assume that C is a singular module over either R_1 or R_2. In view of symmetry, we need only consider the case when C is a singular right R_1-module.

Proposition 3 also says that A has a submodule A' such that A' is a nonsingular right R_1-module and A/A' is a nonsingular right R_2-module. Inasmuch as it suffices to show that $\mathrm{Ext}_R^1(A', C) = 0$ and $\mathrm{Ext}_R^1(A/A', C) = 0$, we may thus assume that A is a nonsingular module over either R_1 or R_2.

Case I. A is a nonsingular right R_1-module.

Consider any exact sequence $S: 0 \to C \to B \to A \to 0$ of right R-modules. Since C and A are both R_1-modules, we infer from the relation $H_2 = H_2^2$ that $BH_2 = 0$ also; i.e., that S is an exact sequence of R_1-modules. Thus S must split, because R_1 is a splitting ring.

Case II. A is a nonsingular right R_2-module.

Inasmuch as R_2 is a splitting ring and $E_2 \in \mathscr{S}(R_2)$, Lemma 11 implies that $\mathrm{Tor}_1^{R_2}(A, R_2/E_2) = 0$. Equivalently,

$$\mathrm{Tor}_1^{R/H_1}(A, R/(H_1 \oplus H_2)) = 0 ,$$

from which we infer that $\mathrm{Tor}_1^R(A, R/(H_1 \oplus H_2)) = 0$. It follows that the map $A \otimes_R H_2 \to A \otimes_R (H_1 \oplus H_2) \to A$ is a monomorphism, hence $\mathrm{Tor}_1^R(A, R/H_2) = 0$.

Consider any exact sequence $S: 0 \to C \to B \to A \to 0$ of right R-modules. Inasmuch as $CH_2 = 0$ and $\mathrm{Tor}_1^R(A, R/H_2) = 0$, we obtain another exact sequence $S^*: 0 \to C \to B/BH_2 \to A/AH_2 \to 0$. An easy diagram chase shows that S will split if S^* does, so it suffices to prove that $\mathrm{Ext}_R^1(A/AH_2, C) = 0$. According to Lemma 11, A/AE_2 is a projective right (R_2/E_2)-module. Noting that $AH_1 = 0$, it follows that A/AH_2 is a projective right module over $R/(H_1 + H_2)$. Therefore to get $\mathrm{Ext}_R^1(A/AH_2, C) = 0$, it suffices to prove that

$$\mathrm{Ext}_R^1(R/(H_1 + H_2), C) = 0 .$$

Thus consider any map $f: H_1 + H_2 \to C$. Noting that $CH_2 = 0$ and $H_2 = H_2^2$, we see that $H_2 \leqq \ker f$. Inasmuch as f induces an isomorphism of $[(H_1 + H_2)/H_2]/[(\ker f)/H_2]$ onto a submodule of C, we see that $[(H_1 + H_2)/H_2]/[(\ker f)/H_2]$ must be a singular right (R/H_2)-module. Since also $Z_r(R/H_2) = 0$, it follows easily that $(\ker f)/H_2$ is an essential submodule of $(H_1 + H_2)/H_2$. Considering that the isomorphism of R/H_2 onto R_1 carries $(H_1 + H_2)/H_2$ onto E_1, we obtain from (b) that

$$[(H_1 + H_2)/H_2]/[(\ker f)/H_2]$$

must be a direct summand of $[R/H_2]/[(\ker f)/H_2]$. Thus $(H_1 + H_2)/(\ker f)$ is a summand of $R/(\ker f)$, hence f extends to a map $R \to C$. Therefore $\mathrm{Ext}_R^1(R/(H_1 + H_2), C) = 0$.

The combination of Theorems 12 and 10 reduces the problem of characterizing splitting rings to characterizing those with either zero socle or essential socle. Since the splitting rings with essential socle have already been characterized [8, Corollary 5.4], only the case of zero socle remains.

4. Examples. In this section we indicate the usefulness of Theorem 12 in constructing splitting rings. The first example shows that essential products [as opposed to direct products] are definitely needed in the study of splitting rings, for this example is a splitting ring which cannot be decomposed into a direct product of a ring with essential socle and a ring with zero socle. The second example shows that a splitting ring with zero socle can have global dimension 2.

For both examples, we start with a left and right principal ideal domain C such that C is a simple ring but not a division ring, and such that every simple right C-module is injective. (Examples of such rings are constructed in [3].) In fact, all singular right C-modules are injective, which is proved in [8] in the discussion following Theorem 3.9. Thus C is certainly a splitting ring; however, it cannot be a factor in a nontrivial essential product because it has no nontrivial two-sided ideals. For this reason, we choose some maximal right ideal M of C and turn to the idealizer of M in C, that is, to the ring $I = \{c \in C \mid cM \leqq M\}$. In the following lemma, we establish the properties of I and M which are needed for our constructions.

LEMMA 13. (a) *I is a right nonsingular right Noetherian ring with zero right socle.*

(b) *I is a splitting ring.*

(c) *M is a two-sided ideal in $\mathscr{S}(I)$, and I/M is a division ring.*

(d) *If L is any essential submodule of M_I, then M/L is a direct summand of I/L.*

Proof. (a) It is pointed out in [6] that I is a right (and left) Noetherian integral domain in which M is a nontrivial two-sided ideal. Inasmuch as I is a domain it must be nonsingular; since it is not a division ring, we must also have $\operatorname{soc}(I_I) = 0$.

(b) This is [6, Lemma 2]. (See also [9].)

(c) We observe from the definition of I that I/M is isomorphic to the endomorphism ring of the simple module $(C/M)_C$.

(d) Inasmuch as LM is a nonzero right ideal of the right Ore domain C, we infer that $LM \in \mathscr{S}(C)$, and hence that M/LM is a singular right C-module. We have already noted that all singular right C-modules are injective; thus M/LM must be a direct summand of C/LM. Two applications of the modular law now show that, first, M/LM is a summand of I/LM, and second, that M/L is a summand of I/L.

EXAMPLE 1. Let R be the "matrix ring" $\begin{pmatrix} I/M & 0 \\ I/M & I \end{pmatrix}$. Then

(a) R is a right nonsingular splitting ring.

(b) $\operatorname{soc}(R_R)$ is neither zero nor essential in R.

(c) R is indecomposable as a ring.

(d) R cannot be the direct product of a ring with zero right socle and a ring with essential right socle.

Proof. (a) Set $R_1 = \begin{pmatrix} I/M & 0 \\ I/M & I/M \end{pmatrix}$, $E_1 = \begin{pmatrix} I/M & 0 \\ I/M & 0 \end{pmatrix}$, $R_2 = I$, $E_2 = M$. Each E_i is a two-sided ideal of R_i, and we may identify R_1/E_1 with R_2/E_2 in the obvious manner. Now R is isomorphic to the ring $R' = \{(x, y) \in R_1 \times R_2 \mid x + E_1 = y + E_2\}$, hence it suffices to show that R' is a right nonsingular splitting ring. We know from Lemma 13 that $Z_r(R_2) = 0$ and $E_2 \in \mathscr{S}(R_2)$, and it is easy to check that $E_1 \in \mathscr{S}(R_1)$. Inasmuch as I/M is a division ring, it follows easily that every element of $\mathscr{S}(R_1)$ contains E_1, from which we infer that $Z_r(R_1) = 0$. Now according to Theorem 4, R' is an essential product of R_1 and R_2. By Proposition 1, $Z_r(R') = 0$.

Since I/M is a division ring, [8, Theorem 2.15] says that all nonsingular right R_1-modules are projective, whence R_1 must be a splitting ring. [Alternatively, it can be shown that all singular right R_1-modules are injective. (See [8, Chapter 3].)] By Lemma 13, R_2 is a splitting ring too.

If L_1 is any essential right R_1-submodule of E_1, it is easy to check that $L_1 = E_1$, hence E_1/L_1 is certainly a summand of R_1/L_1. The corresponding property in R_2 is proved in Lemma 13, whence from Theorem 12 we see that R' is a splitting ring.

(b) Since $\begin{pmatrix} I/M & 0 \\ 0 & 0 \end{pmatrix}$ is a simple right ideal of R, $\operatorname{soc}(R_R) \neq 0$. By Lemma 13, we have $\operatorname{soc}(I_I) = 0$, from which we infer that $\begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix}$ is a nonzero right ideal of R which has no simple submodules. Thus $\operatorname{soc}(R_R)$ is not essential in R_R.

(c) Noting that I is a domain and I/M is a division ring, we see that neither I nor I/M has any nontrivial idempotents. It follows easily that the only nontrivial idempotents in R are of the form $\begin{pmatrix} 1 & 0 \\ x & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ x & 1 \end{pmatrix}$, neither of which is central in R. Thus R cannot be the direct product of two nonzero rings.

(d) is immediate from (b) and (c).

According to [2, Theorem 2.1], a commutative splitting ring has global dimension at most one, while [10, Theorem 2.2] shows that the right global dimension of a noncommutative splitting ring is at most 2. Examples have been constructed of splitting rings with right global dimension 2 ([6] and [8, Example 5.11]), but both of these examples have nonzero socle. Our next example shows that even with zero socle, a splitting ring can have right global dimension 2.

This example also disproves a conjecture in [5] concerning the *finitely generated splitting property* (FGSP). [A ring has FGSP provided the singular submodule of any finitely generated right module is a direct summand.] In the case of a right nonsingular ring with zero right socle, this conjecture reduces to the assertion that such a ring has FGSP if and only if all finitely generated nonsingular right modules are projective. Our example has FGSP because it is a splitting ring, but it has a finitely generated nonsingular right module which is not even flat.

EXAMPLE 2. Let $R = \{(x, y) \in I \times I \mid x + M = y + M\}$.

(a) R is a right nonsingular splitting ring.

(b) $\operatorname{soc}(R_R) = 0$.

(c) r.gl.dim. $(R) = GWD(R) = 2$.

(d) Not all finitely generated nonsingular right R-modules are flat.

Proof. (a) Inasmuch as $Z_r(I) = 0$ and M is a two-sided ideal in $\mathscr{S}(I)$, Theorem 4 shows that R is an essential product of I with itself. By Proposition 1, $Z_r(R) = 0$. Using Lemma 13 with Theorem 12, we see that R is a splitting ring.

(b) Observing that any simple submodule of $(M \times M)_R$ would have to be contained in $[\mathrm{soc}\,(I_I)] \times [\mathrm{soc}\,(I_I)]$, which is zero by Lemma 13, we obtain $(M \times M) \cap \mathrm{soc}\,(R_R) = 0$. Recalling from Proposition 1 that $M \times M \in \mathscr{S}(R)$, we see that $\mathrm{soc}\,(R_R) = 0$.

(c) Since R is a splitting ring, [10, Theorem 2.2] says that r.gl.dim. $(R) \leqq 2$. Inasmuch as $GWD(R) \leqq$ r.gl.dim. (R), it thus suffices to show that $GWD(R) > 1$.

Choose a nonzero $m \in M$ and set $A = (m, 0)R$. Noting that m is a non-zero-divisor in I, we see that the right annihilator of $(m, 0)$ in R is $0 \times M$, whence $A \cong R/(0 \times M)$. According to Lemma 13, M_I is finitely generated, hence $(0 \times M)_R$ is finitely generated and A_R is finitely presented. Observing that $0 \times M$ contains no nonzero idempotents, we see that A is not projective. Inasmuch as A is finitely presented, we conclude that A is a right ideal of R which is not flat, whence $GWD(R) > 1$.

(d) Since all right ideals of R are nonsingular, this is immediate from (c).

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