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GENERALIZED SYLOW TOWER GROUPS, II

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A well-known result of P. Hall shows that finite solvable groups may be characterized by a permutability requirement on Sylow subgroups. The notion of a generalized Sylow tower group (*GSTG*) arises when this permutability condition on Sylow subgroups is replaced by a suitable normalizer condition. In an earlier papar, one of the authors showed that the nilpotent length of a *GSTG* cannot exceed the number of distinct primes which divide the order of the group. The present investigation utilizes the 'type' of a *GSTG* to obtain improved bounds for the nilpotent length of a *GSTG*. It is also shown that a *GSTG* with nilpotent length n possesses a Hall subgroup of nilpotent length n which is a Sylow tower group.

Let G be a finite group with order $p_1^{a_1} \cdots p_r^{a_r}$, where $p_1, \cdots, p_r$ are distinct primes and $a_1, \cdots, a_r$ are positive integers. For each integer i, $1 \leqslant i \leqslant r$, let G_i denote a Sylow p_i-subgroup of G. The collection of subgroups $\mathscr{S} = \{G_1, \cdots, G_r\}$ is then called a complete set of Sylow subgroups of G. If the elements of $\mathscr{S}$ are pairwise permutable as subgroups (that is, if $G_iG_j = G_jG_i$ holds for all i and j) then $\mathscr{S}$ will be called a Sylow basis for G. The notion of a generalized Sylow tower group arises when the permutability condition for a Sylow basis is replaced by a normalizer condition. Thus, we say that a finite group G is a generalized Sylow tower group (*GSTG*) if and only if some complete set of Sylow subgroups $\mathscr{S}$ of G satisfies the normalizer condition (N): if G_i and G_j are distinct elements of $\mathscr{S}$, at least one of these subgroups normalizes the other. It should be noted that not every complete set of Sylow subgroups of a *GSTG* need satisfy condition (N).

A well-known result of P. Hall states that a finite group is solvable if and only if the group possesses a Sylow basis. If a complete set of Sylow subgroups $\mathscr{S}$ of a group G satisfies condition (N) then any two elements of $\mathscr{S}$ are permutable as subgroups and $\mathscr{S}$ is a Sylow basis for G. Consequently every generalized Sylow tower group must be solvable.

A finite group G is called a Sylow tower group (*STG*) if every nontrivial epimorphic image of G possesses a nontrivial normal Sylow subgroup. Equivalently, the group G is a *STG* if the prime divisors $p_1, \cdots, p_r$ of the order of G can be labelled in such a way that a Sylow p_i-subgroup of G normalizes a Sylow p_j-subgroup of G whenever $i > j$. It is clear from this definition that a Sylow tower group is necessarily a generalized Sylow tower group. An example of a $GSTG$ which is not a Sylow tower group was given in ([1]; p. 638).

In order to handle generalized Sylow tower groups it will be necessary to introduce the 'type' of a $GSTG$. Suppose G is some given $GSTG$ and let $\mathscr{T}$ be a Sylow basis for G. Since any two Sylow bases for G are conjugate ([3]; p. 665), $\mathscr{T}$ satisfies the normalizer condition (N). Let R be a relation on the set of all primes with the property that either pRq or qRp (or both) holds for any primes p and q. If the Sylow p_i-subgroup of G in $\mathscr{T}$ normalizes the Sylow p_j-subgroup of G in $\mathscr{T}$ whenever p_iRp_j holds, then G will be called a $GSTG$ of type R. It follows directly from the conjugacy of Sylow bases that the type of a $GSTG$ is independent of the choice of a Sylow basis. It should be noted that a group can be a $GSTG$ of more than one type.

It was shown in [1] that the class of all generalized Sylow tower groups of a given type R is a formation. In addition, any subgroup of a $GSTG$ of type R was shown to be a $GSTG$ of type R. We list the main results about $GSTG$'s in [1] for easy reference.

THEOREM 1.7 [1]. *If G is a $GSTG$ then the nilpotent length of G does not exceed the number of distinct prime divisors of the order of G.*

THEOREM 1.8 [1]. *If G is a $GSTG$ and the nilpotent length of G is equal to the number of distinct prime divisors of the order of G then G is a Sylow tower group.*

All groups mentioned are assumed to be finite. The following notations will be used. For a group G

$c(G)$ denotes the set of distinct prime divisors of the order of G

$\pi(G)$ denotes the number of distinct prime divisors of the order of G

$\ell(G)$ denotes the nilpotent (Fitting) length of G.

If H is a subgroup of G then $N_G(H)$ means the normalizer of H in G and $C_G(H)$ means the centralizer of H in G.

If p_i is a prime, G_i will denote a Sylow p_i-subgroup of G.

The following lemma will be used in several of our arguments.

LEMMA 1. *If G is a nontrivial $GSTG$ then at least one of the following holds:*

(1) *G contains a nontrivial normal Sylow subgroup P with $C_G(P) \subseteq P$*

(2) *G contains nontrivial normal subgroups having relatively prime orders.*

Proof. Let G be a $GSTG$ and suppose that $\mathscr{S} = \{G_1, \cdots, G_n\}$ is a Sylow basis for G. Since a $GSTG$ is necessarily solvable, G possesses a nontrivial minimal normal subgroup M with order a power of some prime q. Let Q denote the maximum normal q-subgroup of G and suppose that G_1 is the Sylow q-subgroup of G belonging to $\mathscr{S}$. Since $\mathscr{S}$ satisfies the normalizer condition (N), either

$$G_1 \subseteqq N_G(G_k) \quad \text{or} \quad G_k \subseteqq N_G(G_1)$$

must hold for each integer k, $1 \leqslant k \leqslant n$. We distinguish two cases.

First suppose that $G_1 \subseteqq N_G(G_k)$ holds for some k, $1 < k \leqslant n$. Since Q is normal in G and has order prime to the order of G_k, G_k centralizes Q. Hence $C = C_G(Q)$ is not a q-group. Set $Q_0 = Q \cap C_G(Q)$ and consider the factor group C/Q_0. Since C/Q_0 is a nontrivial solvable group, C/Q_0 contains a nontrivial minimal normal subgroup L/Q_0 with order a power of some prime p. Let T/Q_0 be the maximum normal p-subgroup of C/Q_0. Since Q is the maximum normal q-subgroup of G it follows that $p \neq q$. If N is a Sylow p-subgroup of T then T is the direct product of N and Q_0. The normality of T in G then implies the normality of N in G. Therefore G has nontrivial normal subgroups with relatively prime orders.

Now suppose that $G_1 \not\subseteqq N_G(G_k)$ holds for all integers k, $1 < k \leqslant n$. Since $\mathscr{S}$ satisfies the normalizer condition (N), $G_k \subseteqq G_G(G_1)$ must then hold for all integers k. Thus G_1 is a normal Sylow q-subgroup of G. Then $G_1 \cap C_G(G_1)$ is a normal Sylow q-subgroup of $C_G(G_1)$ and $C_G(G_1)$ has a normal q-complement W. If W is nontrivial then W is a normal subgroup of G having q'-order and (2) holds. If W is trivial then $G_1 \cap C_G(G_1) = C_G(G_1)$. In this case $C_G(G_1) \subseteqq G_1$ and (1) holds.

THEOREM S. *Let G be a $GSTG$. If $H_1, \cdots, H_n$ are pairwise permutable Hall subgroups of G with $G = H_1 \cdots H_n$, then the nilpotent length of G does not exceed the sum of the nilpotent lengths of $H_1, \cdots, H_n$.*

Proof. (By induction on the order of G.) Since the product $H_2 \cdots H_n$ is a Hall subgroup of G permutable with H_1 we may assume that $n = 2$. First suppose that G possesses nontrivial normal subgroups A and B with $A \cap B = 1$. Then G is isomorphic to a subgroup of the direct of G/A and G/B. Hence

$$\ell(G) = \max\{\ell(G/A), \ell(G/B)\}$$

and it suffices to show that

$$\ell(G/A) \leqslant \ell(H_1) + \ell(H_2) \quad \text{and} \quad \ell(G/B) \leqslant \ell(H_1) + \ell(H_2) .$$

Since G/A is the product of the permutable Hall subgroups H_1A/A and H_2A/A of G/A, the induction hypothesis gives

$$\ell(G/A) \leqslant \ell(H_1A/A) + \ell(H_2A/A) .$$

Since H_1A/A and H_2A/A are epimorphic images of H_1 and H_2 (respectively), we know that $\ell(H_iA/A) \leqslant \ell(H_i)$ for $i = 1, 2$. Therefore $\ell(G/A) \leqslant \ell(H_1) + \ell(H_2)$. The same argument applied to G/B will show $\ell(G/B) \leqslant \ell(H_1) + \ell(H_2)$. This verifies the theorem in this case.

Now suppose that G possesses a unique minimal normal subgroup. Then Lemma 1 shows that G contains a nontrivial normal Sylow subgroup P with $C_G(P) \subseteq P$. Since the theorem is trivially true if $G = P$ we may assume this is not the case. Set

$$\bar{G} = G/P, \bar{H}_1 = H_1P/P, \bar{H}_2 = H_2P/P$$

and consider the nontrivial $GSTG$ $\bar{G}$. The induction hypothesis applied to $\bar{G} = \bar{H}_1\bar{H}_2$ gives $\ell(\bar{G}) \leqslant \ell(\bar{H}_1) + \ell(\bar{H}_2)$. We first observe that $\ell(G) = \ell(\bar{G}) + 1$. Since P is a nilpotent normal subgroup of G, P must lie in the Fitting subgroup F of G. If $P \neq F$ then F contains a nonidentity element of order prime to the order of P which belongs to the centralizer in G of P. This contradicts $C_G(P) \subseteq P$. Therefore $P = F$ and $\ell(G) = \ell(\bar{G}) + 1$.

Since H_1 and H_2 are Hall subgroups of G satisfying $G = H_1H_2$, the Sylow subgroup P must lie in H_1 or H_2. We may suppose that H_1 contains P. If $P = H_1$ then $\bar{G} = \bar{H}_2$ and so $\ell(\bar{G}) = \ell(\bar{H}_2) \leqslant \ell(H_2)$. Then $\ell(G) = \ell(\bar{G}) + 1 \leqslant \ell(H_2) + \ell(H_1)$, which is what we wanted to show. If $P \neq H_1$ then the argument used in the preceding paragraph can be repeated to show $\ell(H_1) = \ell(\bar{H}_1) + 1$. It follows from this that

$$\ell(G) = \ell(\bar{G}) + 1 \leqslant \ell(\bar{H}_1) + 1 + \ell(\bar{H}_2) \leqslant \ell(H_1) + \ell(H_2) .$$

This completes the argument.

It seems interesting to ask if this theorem has a converse, in the following sense. Does a $GSTG$ G necessarily possess pairwise permutable proper Hall subgroups $H_1, \cdots, H_n$ satisfying $G = H_1 \cdots H_n$ so that $\ell(G) = \ell(H_1) + \cdots + \ell(H_n)$ holds? The answer is obviously no, since any nilpotent group is a $GSTG$. If we insist that the group G not be nilpotent, the answer to the question is still no. This can easily be verified using the example of an N-group which is not a Sylow tower group (see [1]; p. 638). We now mention some consequences of the theorem.

Let G be a $GSTG$ and suppose that $\mathscr{S} = \{G_1, \cdots, G_n\}$ is a Sylow basis for G. Since $G_1, \cdots, G_n$ are pairwise permutable Sylow subgroups of G satisfying $G = G_1 \cdots G_n$, Theorem S gives

$$\ell(G) \leqslant \ell(G_1) + \cdots + \ell(G_n) = \pi(G) .$$

Consequently Theorem 1.7 [1] follows from Theorem S.

Now we show how the type of a $GSTG$ can be used to improve the bound on the nilpotent length of a $GSTG$ given by Theorem 1.7 [1]. It will be helpful to first introduce some terminology. Let R be a relation on the set of all primes and let σ denote some given set of primes. Then σ will be called a complete R-symmetric set provided both pRq and qRp hold for all primes p and q belonging to σ. If σ contains a single prime then σ is (trivially) a complete R-symmetric set. It is clear from this that any set of primes can be written as a union of complete R-symmetric subsets. The set σ will be called an R-cyclic set if σ contains distinct primes p, q, and w such that pRq, qRw, and wRp hold.

COROLLARY 1. *Let G be a $GSTG$ of type R. If $\sigma_1, \cdots, \sigma_d$ are complete R-symmetric subsets of $c(G)$ such that the union of the σ_i is $c(G)$ then $\ell(G) \leqslant d$.*

Proof. Let $\mathscr{S} = \{G_1, \cdots, G_n\}$ be a Sylow basis for G. For each i, $1 \leqslant i \leqslant d$, define the subgroup H_i of G to be the product of all Sylow p_k-subgroups G_k for which $p_k \in \sigma_i$. Since pRq and qRp hold for all distinct primes p and q from σ_i, each H_i is seen to be a nilpotent Hall σ_i-subgroup of G. Since the union of the σ_i's is $c(G)$, clearly $G = H_1 \cdots H_d$. The theorem then shows that

$$\ell(G) \leqslant \ell(H_1) + \cdots + \ell(H_d) = d .$$

COROLLARY 2. *Let G be a $GSTG$ of type R. If $\sigma_1, \cdots, \sigma_d$ are disjoint R-cyclic subsets of $c(G)$ then $\ell(G) \leqslant \pi(G) - d$.*

Proof. Let $\mathscr{S} = \{G_1, \cdots, G_n\}$ be a Sylow basis for G. For each integer i, $1 \leqslant i \leqslant d$, define the subgroup H_i of G to be the product of all Sylow p_k-subgroups G_k for which $p_k \in \sigma_i$. It is clear from the definition that the H_i are pairwise permutable Hall σ_i-subgroups of G. Since the product $H = H_1 \cdots H_d$ has a Hall complement in G, it suffices to show that $\ell(H) \leqslant \pi(H) - d$. Since the σ_i's are disjoint sets, this will follow if $\ell(H_i) \leqslant \pi(H_i) - 1$ holds for each i, $1 \leqslant i \leqslant d$. Let p, q, and w be distinct primes in σ_i satisfying pRq, qRp and wRp. Consider a Hall $\{p, q, w\}$-subgroup T of H_i. If T has a normal Sylow p-subgroup then pRq shows that T has a nilpotent Hall

$\{p, q\}$-subgroup. Then Corollary 1 shows $\ell(T) \leqslant \pi(T) - 1$. It now follows from Theorem S that $\ell(H_i) \leqslant \pi(H_i) - 1$. Consequently we may assume T has no nontrivial normal Sylow subgroup. By Lemma 1, T then has nontrivial normal subgroups A and B with $A \cap B = 1$. Since T/A and T/B are $GSTG$'s of type R, induction shows that $\ell(T/A) \leqslant 2$ and $\ell(T/B) \leqslant 2$. Using the fact that T is isomorphic to a subgroup of the direct product of T/A and T/B we obtain

$$\ell(T) \leqslant 2 = \pi(T) - 1 .$$

Theorem S applied to H_i now gives $\ell(H_i) \leqslant \pi(H_i) - 1$. Therefore we have shown that $\ell(H_i) \leqslant \pi(H_i) - 1$ holds for arbitrary i and the assertion follows.

The next consequence of the theorem is Theorem 1.8 [1].

COROLLARY 3. *Let G be GSTG with $\ell(G) = \pi(G)$. Then G is a Sylow tower group of exactly one type R, in the sense that the relation R is uniquely determined for pairs of primes p, q in $c(G)$.*

Proof. Let $\mathscr{S} = \{G_1, \cdots, G_n\}$ be Sylow basis for G. Define the relation R on the set of all primes as follows: R is reflexive and for distinct primes p and q, pRq holds if and only if either p or q does not divide the order of G or both p and q do divide the order of G and the Sylow p-subgroup of G belonging to $\mathscr{S}$ normalizes the Sylow q-subgroup of G belonging to $\mathscr{S}$. Clearly G is of type R. Since $\ell(G) = \pi(G)$, Corollary 1 shows that both pRq and qRp hold for no distinct primes $p, q \in c(G)$. In addition, Corollary 2 shows that pRq, qRw, and wRp hold for no distinct primes p, q, and w from $c(G)$. Since either pRq or qRp holds for any primes p, $q \in c(G)$, the restriction of R to $c(G)$ must be a linear order. Therefore G is a Sylow tower group of type R. Suppose that G is also a STG of type S and the restriction of S to $c(G)$ differs from the restriction of R to $c(G)$. Then G would necessarily have a nilpotent Hall $\{p, q\}$-subgroup for some distinct $p, q \in c(G)$. The conjugacy of Hall $\{p, q\}$-subgroups in G then implies that $\ell(G) \leqslant \pi(G) - 1$, a contradiction. Therefore G is a STG of exactly one type, in the sense mentioned.

We next give an example to show that the nilpotent length of a $GSTG$ cannot be found from the type alone. Let A be the holomorph of a cyclic group of order 7 and let B donote the Hall $\{7, 3\}$-subgroup of A. Define the group G_1 as the direct product of A and a symmetric group of degree 3 and define G_2 as the wreath product of B by a cyclic group of order 2. Both G_1 and G_2 are Sylow tower groups of type $7 < 3 < 2$ and no distinct Sylow subgroups of G_1 or G_2 centralize one another. Hence, for a given relation R on the set of all primes, G_1 is a $GSTG$ of type R if and only if G_2 is a $GSTG$ of type R. Yet the nilpotent length of G_1 is 2 and the nilpotent length of G_2 is 3.

THEOREM T. *Let G be a $GSTG$ with nilpotent length k. Then G contains a Hall subgroup which is a Sylow tower group and has nilpotent length k.*

Proof. By Theorem 1.8 [1] it is sufficient to show that G contains a Hall subgroup L with $\ell(L) = \pi(L) = k$. We proceed by induction on the order of G.

Suppose G contains a proper subgroup W with $\ell(W) = k$. The induction hypothesis then shows that W contains a Hall subgroup T with $\ell(T) = \pi(T) = k$. Choose a Hall subgroup L of G with $T \subseteq L$ and $c(T) = c(L)$. Then $\ell(G) = k = \ell(T) \leqslant \ell(L) \leqslant \ell(G)$ shows that $\ell(L) = \pi(L) = k$. This proves the theorem in the case where G contains a proper subgroup with nilpotent length k. Now suppose that every proper subgroup of G has nilpotent length strictly less than k.

Since G is a $GSTG$, either G possesses nontrivial normal subgroups A and B with $A \cap B = 1$ or G contains a nontrivial normal Sylow subgroup P with $C_G(P) \subseteq P$. We consider these possibilities separately. First suppose that A and B are distinct minimal normal subgroups of G. If the Frattini subgroup ϕ of G is trivial then G contains a maximal subgroup M_1 not containing A and a maximal subgroup M_2 not containing B. Then M_1 complements A in G and M_2 complements B in G. Since we have assumed that all proper subgroups of G have nilpotent length less than k, the isomorphism of G/A and M_1 gives $\ell(G/A) < k$. Similarly one sees that $\ell(G/B) < k$. Since G is isomorphic to a subgroup of the direct product of G/A and G/B, it follows that $\ell(G) = \max\{\ell(G/A), \ell(G/B)\}$ is less than k, a contradiction. Therefore G has nontrivial Frattini subgroup. Since $\ell(G/\phi) = \ell(G) = k$, the induction hypothesis shows that G contains a Hall subgroup L satisfying $\ell(L\phi/\phi) = \pi(L\phi/\phi) = k$. Now

$$k = \ell(L\phi/\phi) \leqslant \ell(L) \leqslant \ell(G) = k$$

shows $\ell(L) = k$. Hence $L = G$. Since the Frattini subgroup of G contains no Sylow subgroup of G, $\pi(G) = \pi(L) = \pi(L\phi/\phi) = k$. Therefore $\ell(G) = \pi(G) = k$, which completes the argument in this case.

Now suppose G contains a nontrivial normal Sylow subgroup P with $C_G(P) \subseteq P$. It follows that P must be the Fitting subgroup of G. Therefore $\ell(G) = \ell(G/P) + 1$ or $G = P$. In the latter case the theorem is trivially true. If $\ell(G) = \ell(G/P) + 1$, the induction hypothesis shows that G/P contains a nontrivial Hall subgroup L/P satisfying $\ell(L/P) = \pi(L/P) = k - 1 = \ell(G/P)$. Clearly L is then a Hall subgroup of G with $\pi(L) = k$. Since $C_G(P) \subseteqq P$, P must be the Fitting subgroup of L. Hence $\ell(L) = \ell(L/P) + 1$. Therefore

$$\ell(L) = k = \pi(L) .$$

This completes the proof of the theorem.

Let G be a given $GSTG$ and suppose $\mathscr{S}$ is a Sylow basis for G. Define the relation R on the set of all primes as follows: for any primes p and q (possibly equal), pRq holds if and only if $p \notin c(G)$ or $q \notin c(G)$ or both p and q belong to $c(G)$ and the Sylow p-subgroup of G in $\mathscr{S}$ normalizes the Sylow q-subgroup of G in $\mathscr{S}$. Clearly G is a $GSTG$ of type R. If H is a Hall subgroup of G which is a Sylow tower group and H satisfies $\ell(H) = \pi(H) = \ell(G)$, then the restriction of R to $c(H)$ is a transitive relation (see the proof of Corollary 3). This leads to the following bound for the nilpotent length of G in terms of the relation R defined above. The nilpotent length of the $GSTG$ G cannot exceed n, where n is the largest integer such that the restriction of R to some subset of $c(G)$ having n elements is a transitive relation.

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