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MATRIX REPRESENTATIONS FOR LINEAR TRANSFORMATIONS ON ANALYTIC SEQUENCES

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Let $\mathscr{A}$ be the space of all *analytic* sequences, those complex sequences α for which there is a positive number r such that $\sum \alpha_n r^n$ converges. Those linear transformations from $\mathscr{A}$ to $\mathscr{A}$ which have matrix representations are characterized in terms of various spaces and topologies associated with $\mathscr{A}$. An example is given of a linear transformation from $\mathscr{A}$ to $\mathscr{A}$ which has no matrix representation.

Louise Raphael [8] characterizes the matrix transformations from $\mathscr{A}$ to $\mathscr{A}$. She makes use of the following: if $q > 0$, A_q is the subspace of $\mathscr{A}$ to which α belongs only in case $\{|\alpha_n| q^n\}_{n=0}^{\infty}$ is a bounded sequence, and $\|\alpha\|_q$ denotes the least number less than no term of that bounded sequence. If $q > 0$, $\{A_q, \|\ \|_q\}$ is a complete normed linear space. (See also, I. Heller [5], I. M. Sheffer [10, Th. 6, p. 177], and the more fundamental work of Karl Zeller [12].)

Following M. G. Haplanov [4] and V. Ganapathy Iyer [3], S_q denotes the subset of $\mathscr{A}$ to which α belongs provided that $\sum \alpha_n z^n$ converges for $|z| < q$, and, if $0 < p < q$, $N_p(\alpha)$ denotes $\sum_{k=0}^{\infty} |\alpha_k| p^k$ for each α in S_q. If $q > p > 0$, $\{S_q, N_p\}$ is a normed linear space (not complete).

In [11] the author characterizes those linear transformations from S_1 to S_1 which have matrix representations. We continue here in much the same spirit. If $q > 0$ and $\alpha = \{\alpha_n\}_{n=0}^{\infty}$ is a sequence of sequences in $\mathscr{A}$ and f is a sequence of analytic functions such that if n is a nonnegative integer and $|z| < q$ then

$$f_n(z) = \sum_{k=0}^{\infty} \alpha_{nk} z^k$$

and f converges uniformly with limit 0 on each closed subset of the (open) disc with center 0 and radius q, then α is said to *have limit* 0 *analytically relative to* q. A sequence *has limit* 0 *analytically* if it has limit 0 analytically relative to some positive number.

We recall some fundamental notions from G. Köthe and O. Toeplitz [7] about sequence spaces:

Suppose that λ is a linear sequence space. λ^* (sometimes called the dual or α-dual of λ) is the collection of all complex sequences y such that $\sum |y_n x_n|$ converges for each x in λ. If x is in λ and y is in λ^*,

$$Q(x, y) = \sum_{n=0}^{\infty} x_n y_n .$$

A sequence $x = \{x_p\}_{p=0}^{\infty}$ of sequences in λ is said to *converge in* λ provided that, for each y in λ^*, the complex sequence $\{Q(x_p, y)\}_{p=0}^{\infty}$ converges. The transformation F is *sequentially continuous* from λ *to* λ provided that $\{F(x_p)\}_{p=0}^{\infty}$ converges in λ if $\{x_p\}_0^{\infty}$ converges in λ.

Theorems A and B are due to Köthe and Toeplitz.

THEOREM A. *If* $\lambda = \lambda^{**}$ *and the matrix* M *transforms* λ *to* λ (*if* x *is in* λ *and* $y_n = \sum_{k=0}^{\infty} M_{nk} x_k$, $n = 0, 1, \cdots$, *then* y *is in* λ), *then the transformation is sequentially continuous from* λ *to* λ [7, Satz 6, p. 206].

THEOREM B. *Each linear sequentially-continuous transformation from* λ *to* λ *has a matrix representation.* [7, Satz 7, p. 207].

A subset X of the sequence space λ is *bounded in* λ if for each u in λ^* there is a number m such that if x is in X then $|Q(x, u)| \leqq m$. If F is a transformation from λ to λ, the *adjoint* F^* of F is the relation to which the ordered pair $\{x, y\}$ belongs only in the case that

$$Q(x, F(z)) = Q(y, z)$$

for each z in λ.

Let $\mathscr{E}$ be the space of all *entire* sequences, those complex sequences which are coefficient sequences for power-series expansions of entire functions. $\mathscr{E} = \mathscr{A}^*$ and $\mathscr{E}^* = \mathscr{A}$. The matrix transformations from $\mathscr{E}$ to $\mathscr{E}$ have been characterized by H. I. Brown [1] and, in another manner, by K. Chandrasekhara Rao [2].

THEOREM. *Let* L *be a linear transformation from* $\mathscr{A}$ *to* $\mathscr{A}$. *These statements are equivalent*:

(1) *L has a matrix representation.*

(2) *L is sequentially continuous from* $\mathscr{A}$ *to* $\mathscr{A}$.

(3) *If* $p > 0$ *there is a positive number* q *such that* L *maps* $\{A_p, \|\ \|_p\}$ *continuously into* $\{A_q, \|\ \|_q\}$ (*with respect to the norms*).

(3′) *If* $p > 0$ *there is a positive number* q *such that* L *maps* A_p *into* A_q.

(4) *If* X *is a set bounded in* $\mathscr{A}$ *then* $L(X)$ *is also.*

(5) *If* $0 < p < r$ *there is a positive number* R *such that, if* $0 < P < R$, *then* L *maps* $\{S_r, N_p\}$ *into* $\{S_R, N_P\}$ *continuously.*

(6) *L^* is a sequentially continuous transformations from* $\mathscr{E}$ *to* $\mathscr{E}$.

(7) *If* α *has limit* 0 *analytically, so does* $\{L(\alpha_n)\}_{n=0}^{\infty}$.

$\mathscr{A}^{**} = \mathscr{A}$ and $\mathscr{E}^{**} = \mathscr{E}$. This and the following lemmas are useful in the proof of our theorem.

LEMMA 0. *Suppose that λ is a sequence space and $\lambda^{**} = \lambda$ and T is a linear sequentially continuous transformation from λ to λ. Then T^* is a sequentially continuous transformation from λ^* into λ^*.*

Via [7, Satz 6, p. 200], a characterization of linear functionals, Lemma 0 is easy to prove. (See also [9, p. 158].)

LEMMA 1. *If B is a set bounded in $\mathscr{A}$, then there is a member α of $\mathscr{A}$ such that if β is in B then $|\beta_k| \leqq \alpha_k$, $k = 0, 1, \cdots$.*

Proof. Otherwise, there is a sequence β of sequences in B and an increasing sequence n of nonnegative integers such that, if k is a positive integer, $|\beta_{k,n_k}| > k^{1+n_k}$. Let us indicate how to define such a sequence. Let β_1 be a member of B and n_1 be a positive integer such that $|\beta_{1,n_1}| > 1^{1+n_1}$. Let t be a number such that if b is in B then $|b_k| \leqq t$, $k = 0, 1, \cdots, n_1$. Let β_2 be a member of B and n_2 be a positive integer such that $|\beta_{2,n_2}| > t \cdot 2^{1+n_2}$. $n_2 > n_1$. Please continue.

Let e be a sequence such that if k is a nonnegative integer then $e_{n_k} = k^{-n_k}$ and $e_k = 0$ if there is no positive integer j such that $n_j = k$. e is in $\mathscr{E}$.

The set D to which d belongs only in case $|d_k| \leqq |e_k|$, $k = 0, 1, \cdots$, is bounded in $\mathscr{E}$. Since B is bounded, it is strongly bounded (see [7, Satz 1, p. 201] or [6, p. 413 (5)], so that there is a number c such that if b is in B and d is in D then $|Q(b, d)| \leqq c$. Let k be a positive integer. Let u be a complex sequence such that if j is a nonnegative integer then $|u_j| = 1$ and $\beta_{kj}u_j \geqq 0$. $u \cdot e$ is in D.

$$\begin{aligned} c \geqq |Q(\beta_k, u \cdot e)| &= \left|\sum_{j=0}^{\infty} \beta_{kj}u_je_j\right| = \sum_{j=0}^{\infty} \beta_{kj}u_je_j \\ &\geqq \beta_{k,n_k}u_{n_k}e_{n_k} = |\beta_{k,n_k}|\, e_{n_k} > k^{1+n_k}k^{-n_k} = k \,. \end{aligned}$$

So there is a member α of $\mathscr{A}$ such that if b is in B then $|b_k| \leqq \alpha_k$, $k = 0, 1, \cdots$.

LEMMA 2. *If α is a sequence of sequences in $\mathscr{A}$, then these are equivalent:*

(1) *α has limit 0 analytically.*

(2) *α has limit 0 in $\mathscr{A}$.*

Proof. Suppose that α has limit 0 analytically (relative to q). Then α has limit 0 in S_q (see [11, Lemma 1]). α is a sequence bounded in S_q. $\mathscr{A}^*$ is a subset of S_q^*, so α is a sequence bounded in $\mathscr{A}$, and there is a member β of $\mathscr{A}$ such that if each of j and k is a nonnegative integer then $|\alpha_{jk}| \leqq \beta_k$. Let t be a positive number such that $\beta_k \leqq t^{k+1}$, $k = 0, 1, \cdots$. Let e be in $\mathscr{E}$. ($\mathscr{E} = \mathscr{A}^*$.) Let ε be a positive number. Let m be a positive integer such that $2\sum_{k=m}^{\infty} |e_k| t^{k+1} < \varepsilon$. Let J be a positive interger such that if j is an integer exceeding J then $2\sum_{k=0}^{m-1} |\alpha_{jk}||e_k| < \varepsilon$. Then, if $j > J$,

$$|Q(\alpha_j, e)| = \left|\sum_{k=0}^{\infty} \alpha_{jk} e_k\right| \leqq \sum_{k=0}^{\infty} |\alpha_{jk}||e_k|$$
$$\leqq \sum_{k=0}^{m-1} |\alpha_{jk}||e_k| + \sum_{k=m}^{\infty} |e_k| t^{k+1} < \varepsilon .$$

So α has limit 0 in $\mathscr{A}$.

Now, suppose that α has limit 0 in $\mathscr{A}$. α is a sequence bounded in $\mathscr{A}$. There is a positive number t such that $|\alpha_{jk}| \leqq t^{k+1}$, $j, k = 0, 1, \cdots$. Let q be a number between 0 and $1/t$. Let ε be a positive number. Let m be a positive integer such that $2\sum_{k=m}^{\infty} q^k t^{k+1} < \varepsilon$. Let J be a positive integer such that if j is an integer exceeding J then $2\sum_{k=0}^{m-1} |\alpha_{jk}| q^k < \varepsilon$. Now, if $j > J$ and $|z| \leqq q$,

$$\left|\sum_{k=0}^{\infty} \alpha_{jk} z^k\right| \leqq \sum_{k=0}^{\infty} |\alpha_{jk}| q^k \leqq \varepsilon .$$

So α has limit 0 analytically relative to $1/t$.

LEMMA 3. *Suppose that $r > p > 0$ and $R > P > 0$ and L is a continuous linear transformation from $\{S_r, N_p\}$ to $\{S_R, N_P\}$. Then L has a matrix representation.*

Proof. By [11, Theorem 1] this is true if $r = R = 1$.

Suppose that, for each positive number ρ, $t(\rho)$ is the function from $\mathscr{A}$ to $\mathscr{A}$ such that if α is in $\mathscr{A}$ and n is a nonnegative integer then $t(\rho)(\alpha)_n = \alpha_n \rho^n$, so that, if $0 < q < \rho$, $t(\rho)$ maps $\{S_\rho, N_q\}$ continuously onto $\{S_1, N_{q/\rho}\}$.

Let L' be the continuous linear transformation from $\{S_1, N_{p/r}\}$ into $\{S_1, N_{P/R}\}$ such that if x is in S_1 then

$$L'(x) = t(R)Lt(1/r)(x) .$$

L' has a matrix representation, so L has a matrix representation.

LEMMA 4. *Suppose that $0 < p < r$. If α is in A_r, then α is in S_r and*

$$N_p(\alpha) \leqq \|\alpha\|_r/(1 - p/r) .$$

If α is in S_r, then α is in A_p and

$$\|\alpha\|_p \leqq N_p(\alpha) .$$

The proof is straight-forward and omitted.

Proof of Theorem. $1 \leftrightarrow 2$. That statements (1) and (2) are equivalent is seen from Theorems A and B.

$1 \rightarrow 3$. Mrs. Raphael has shown that statement (3) follows from (1) [8, Theorem 4, p. 124].

$2 \rightarrow 4$. That statement (4) follows from (2) is a consequence of [7, Satz 5, p. 207].

$4 \rightarrow 3'$. Suppose that if X is a set bounded in $\mathscr{A}$ then $L(X)$ is too. Let p be a positive number. Let X be the set of all points x of A_p such that $\|x\|_p \leqq 1$. Let e be in E. Let x be in X.

$$| Q(x, e) | = \left| \sum_{k=0}^{\infty} e_k x_k \right| \leqq \sum_{k=0}^{\infty} | e_k | \, | x_k | \leqq \sum_{k=0}^{\infty} | e_k | p^{-k} ,$$

so X is bounded in A.

$L(X)$ is bounded in A. By Lemma 1 there is a positive number q such that if y is in $L(X)$ then $| y_n | \leqq q^{n+1}$, n $= 0, 1, \cdots$. So, if x is in A_p, $L(x)$ is in A_q. Therefore statement (3′) follows from statement (4).

$3' \rightarrow 1$. That statement 3′ implies that statement (1) is true is evident from part 4 of Karl Zeller's theorem in [12].

$2 \leftrightarrow 6$. That statements (2) and (6) are equivalent is a consequence of Lemma 0. One might also use Theorems A and B (of [7]) and [7, Satz 4, p. 206].

$2 \leftrightarrow 7$. That statements (2) and (7) are equivalent is evident from Lemma 2.

$3 \rightarrow 5$. Suppose that $0 < p < r$. Let q be a positive number such that L maps $\{A_p, \| \ \|_p\}$ continuously into $\{A_q, \| \ \|_q\}$. Let K be a positive number such that if x is in A_p then $\| L(x) \|_q \leqq K \| x \|_p$. Let P be a number between 0 and q. Then, by Lemma 4, if x is in S_r, x is in A_p, $L(x)$ in A_q, $L(x)$ is in S_q, and

$$N_p(L(x)) \leqq \frac{\| L(x) \|_q}{1 - P/q} \leqq \frac{K}{1 - P/q} \| x \|_p \leqq \frac{K}{1 - P/q} N_p(x) .$$

So statement (5) follows from statement (3).

$5 \rightarrow 1$. Since each point of A belongs to S_r for some positive number r, it follows from Lemma 3 that L has a matrix representation (statement (1)) if statement (5) is true.

One can add to the seven statements in the theorem by taking other combinations of these spaces and notions. I have presented those which seem most interesting.

EXAMPLE. Let S be a maximal linearly independent subset of A which contains the unit vectors $(1, 0, 0, \cdots)$, etc., and the constant sequence $k = (1, 1, \cdots)$. We define a function l from S to the plane such that if s is in S and $s \neq k$ then $l(s) = 0$ and $l(k) = 1$. Let l' be the linear extension of l to A. Let L be the linear transformation from A to A such that if x is in A and n is a nonnegative integer then

$$L(x)_n = l'(x) .$$

L is a linear transformation from A to A (indeed to the constant sequences) and, since l' cannot be represented by a sequence, L has no matrix representation.

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