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MOMENT SEQUENCES IN HILBERT SPACE

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Suppose f is a real valued function of bounded variation on $[0, 1]$. Then for each nonnegative integer n, the Stieltjes integral $\int_0^1 j^n df$ exists, where for each number $x, j(x) = x$. A necessary and sufficient condition is given for f in order that the moment sequence for f, $\{C_n\}_{n=0}^{\infty}$, is square summable. A second result establishes that the set of all such square summable moment sequences is dense in l^2.

LEMMA 1. *If p is a number, $1/2 < p < 1$, and for each nonnegative integer n, $a_n = 1 - (n+1)^{-p}$ then*

$$1. \quad \lim_{n \to \infty} a_n^n = 0\ ,$$

$$2. \quad \sum_{n=0}^{\infty} a_n^{2n} \text{ exists}$$

and

$$3. \quad \sum_{n=0}^{\infty} (1 - a_n)^2 \text{ exists}\ .$$

Proof. To establish 1,

$$\lim_{n \to \infty} a_n^n = \lim_{n \to \infty} (1 - n^{-p})^n$$
$$= \exp [\lim_{n \to \infty} n \ \ln [1 - n^{-p}]]\ .$$

Since $1/2 < p < 1$,

$$\lim_{n \to \infty} n \ \ln [1 - n^{-p}] = -p \lim_{n \to \infty} n/[n^p - 1] = -\infty$$

and hence the result.

To establish 2, it will be sufficient to show that for sufficiently large n

$$a_n^n \leqq (1 + n)^{-p}$$

i.e., that $[1 - n^{-p}]^{n-1} \leqq n^{-p}$.

Let $n^p = k$ and $g = p^{-1} - 1$ (note that $g > 0$); we have then to show that

$$[[1 - k^{-1}]^k]^{k^g} \leqq k^{-1} - k^{-2}\ .$$

Recall that

$$[1 - k^{-1}]^k \leqq e^{-1}$$

and hence that

$$[[1 - k^{-1}]^k]^{kg} \leqq e^{-kg} .$$

Now if k is large we have

$$e^{-kg} \leqq k^{-1} - k^{-2}$$

and the result is established.

The third part follows immediately from the definition of a_n.

THEOREM 1. *If f is a real valued function of bounded variation on $[0, 1]$ and, for each nonnegative integer n, $\int_0^1 j^n df = C_n$ exists, then*

$$\sum_{n=0}^{\infty} C_n^2 < \infty$$

if and only if

$$\sum_{n=1}^{\infty} \left[f(1) - \int_{a_n}^1 f dj^n (1 - a_n^n) \right]^2 < \infty$$

where the sequence $\{a_n\}_{n=0}^{\infty}$ is as given in Lemma 1.

Proof. Let us first establish the necessity of the condition. Suppose $\sum_{n=0}^{\infty} C_n^2 < \infty$.

If n is a positive integer

$$\begin{aligned} C_n &= \int_0^1 j^n df \\ &= \int_0^{a_n} j^n df + \int_{a_n}^1 j^n df \\ &= a_n^n f(a_n) - \int_0^{a_n} f dj^n + \int_{a_n}^1 j^n df . \end{aligned}$$

Let $\gamma_n = \int_0^{a_n} f dj^n / a_n^n$, then

$$C_n = a_n^n [f(a_n) - \gamma_n] + f(1) - f(a_n) a_n^n - \int_{a_n}^1 f dj^n .$$

Let $\delta_n = \int_{a_n}^1 f dj^n / (1 - a_n^n)$, then

$$\begin{aligned} C_n &= a_n^n [f(a_n) - \gamma_n] + f(1) - f(a_n) a_n^n - (1 - a_n^n) \delta_n \\ C_n &= a_n^n [\delta_n - \gamma_n] + [f(1) - \delta_n] \end{aligned}$$

and

$$C_n^2 = (a_n^n[\delta_n - \gamma_n] + [f(1) - \delta_n])^2 .$$

Since the sequence $\{[\delta_n - \gamma_n]\}_{n=1}^{\infty}$ is bounded it follows from Lemma 1 that

$$\sum_{n=1}^{\infty} a_n^{2n} [\delta_n - \gamma_n]^2 < \infty .$$

Hence, since $\sum_{n=0}^{\infty} C_n^2 < \infty$, we have that

$$\sum_{n=1}^{\infty} [f(1) - \delta_n]^2 < \infty$$

i.e.,

$$\sum_{n=1}^{\infty} \left| f(1) - \int_{a_n}^{1} f dj^n/(1 - a_n^n) \right|^2 < \infty$$

and therefore the condition is necessary.

Now let us establish the sufficiency, i.e., suppose that

$$\sum_{n=1}^{\infty} \left| f(1) - \int_{a_n}^{1} f dj^n/(1 - a_n^n) \right|^2$$

exists.

Now $C_n = \left[f(1) - \int_{a_n}^{1} f dj^n\right] - \int_0^{a_n} f dj^n$ for $n = 0, 1, 2, \cdots$.

As befor

$$\sum_{n=1}^{\infty} \left(\int_0^{a_n} f dj^n\right)^2$$

exists and hence we have only to consider

$$\begin{aligned} &\sum_{n=1}^{\infty} \left(f(1) - \int_{a_n}^{1} f dj^n\right)^2 \\ &= \sum_{n=1}^{\infty} \left(\left[f(1) - \int_{a_n}^{1} f dj^n\right]\Big/[1 - a_n^n]\right)^2 (1 - a_n^n)^2 \\ &\leqq \sum_{n=1}^{\infty} \left(\left[f(1) - \int_{a_n}^{1} f dj^n\right]\Big/[1 - a_n^n]\right)^2 \\ &= \sum_{n=1}^{\infty} \left(f(1) - \int_{a_n}^{1} f dj^n/[1 - a_n^n] + f(1) a_n^n/[1 - a_n^n]\right)^2 . \end{aligned}$$

Recall the assumption that

$$\sum_{n=1}^{\infty} \left(f(1) - \int_{a_n}^{1} f dj^n/[1 - a_n^n]\right)^2$$

exists and hence we need only consider

$$\sum_{n=1}^{\infty} (f(1)a_n^n/[1 - a_n^n])^2$$
$$= \sum_{n=1}^{\infty} (f(1))^2 a_n^{2n}/[1 - a_n^n]^2$$

which also exists. Hence it follows that $\sum_{n=0}^{\infty} C_n^2$ exists.

As an immediate consequence of this result we have the following results, which are stated here without proof.

PROPOSITION 1. *If there is a* δ, $0 < \delta < 1$, *such that* $f(1) - f(x) \leqq 1 - x$ *if* $\delta \leqq x \leqq 1$ *then* $\sum_{n=0}^{\infty} C_n^2 < \infty$.

PROPOSITION 2. *If there is a* $\delta, 0 < \delta < 1$, *such that* f *has a continuous derivative on* $[\delta, 1]$ *then* $\sum_{n=0}^{\infty} C_n^2 < \infty$.

PROPOSITION 3. *If there is a number* $\delta, 0 < \delta < 1$, *a number* $\alpha > 1/2$ *and a number* $B > 0$ *such that*

$$|f(1) - f(x)| \leqq B|1 - x|^{\alpha} \text{ for } x \text{ in } [\delta, 1]$$

then $\sum_{n=0}^{\infty} C_n^2 < \infty$.

Consider the following example. Let $f = 1 - (1 - j)^{1/2}$ on $[0, 1]$, then $C_n = \int_0^1 j^n df = 2n\int_0^1 j^2(1 - j^2)^{n-1}$ if $n \geqq 1$, and hence $C_{n+1} = 2(n + 1)\int_0^1 j^2(1 - j^2)^n$. It then follows that $(2n + 3)C_{n+1} = (2n + 2)C_n$ and this yields the following for $n = 1, 2, \cdots$, $C_{n+1} = C_1\prod_{t=0}^{n-1}[(2t + 4)((2t + 5)]$. By the use of Stirlings formula we have that $C_{n+1}^2 \geqq 6\sqrt{\pi}(n + 3/2)^{-1/2}$ and hence $\sum_{n=0}^{\infty} C_n^2$ does not exist.

The following lemma is stated without proof.

LEMMA 2. *If* t *is a positive integer and* n *is a nonnegative integer less than* t, *then*

$$\sum_{m=0}^{t} \binom{t}{m} m^t(-1)^m = (-1)^t t!$$

and

$$\sum_{m=0}^{t} \binom{t}{m} m^n(-1)^m = 0 .$$

DEFINITION 1. Suppose $\{C_m\}_{m=0}^{\infty}$ is a real number sequence and n is a positive integer. Let $\varphi_n(0) = 0$, $\varphi_n(1) = C_0$ and if x is in $(0, 1) \cap [k/n, (k + 1)/n)$ where $k = 0, 1, 2, \cdots, n - 1$ let

$$\varphi_n(x) = \sum_{t=0}^{k} \binom{n}{t} \sum_{i=0}^{n-t} \binom{n-t}{i} (-1)^i C_{i+t} .$$

THEOREM 2. *The set of all square summable moment sequences is dense in l^2.*

Proof. Let, for each nonnegative integer t, $\varepsilon_t = \{\delta_{it}\}_{i=0}^{\infty}$ where δ_{ij} is the Kronecker δ. Associated with each such sequence ε_t, there is a function sequence $\{\varphi_{k,t}\}_{k=1}^{\infty}$ as given in Definition 1. For each nonnegative integer t and each positive integer k there is a number sequence $C_{k,t} = \{C_{n,k,t}\}_{n=0}^{\infty}$ associated, where $C_{n,k,t} = \int_0^1 j^n d\varphi_{k,t}$.

A straight forward computation yields

$$\begin{aligned} C_{n,k,t} &= \sum_{m=0}^{t} (-1)^{t-m} (m/k)^n \binom{k}{m} \binom{k-m}{t-m} \\ &= (-1)^t \binom{k}{t} \sum_{m=0}^{t} \binom{t}{m} (-1)^m (m/k)^n \end{aligned}$$

and therefore

$$\sum_{n=0}^{\infty} C_{n,k,t}^2 = \sum_{n=0}^{\infty} \binom{k}{t}^2 \Big[\sum_{m=0}^{t} \binom{t}{m} (-1)^m (m/k)^n \Big]^2 .$$

This, using Lemma 2, becomes

$$\begin{aligned} &\sum_{n=t}^{\infty} \binom{k}{t}^2 \Big[\sum_{m=0}^{t} \binom{t}{m} (-1)^m (m/k)^n \Big]^2 \\ &= \sum_{n=0}^{\infty} \binom{k}{t}^2 \Big[\sum_{m=0}^{t} \binom{t}{m} (-1)^m (m/k)^n (m/k)^t \Big]^2 \\ &= \sum_{n=0}^{\infty} \binom{k}{t}^2 k^{-2t} \Big[\sum_{m=0}^{t} \binom{t}{m}^2 (m^2/k^2)^n m^{2t} \\ &\quad + 2 \sum_{m=0}^{t-1} \binom{t}{m} m^t (m/k)^n (-1)^m \sum_{i=m+1}^{t} \binom{t}{i} i^t (i/k)^n (-1)^i \Big] \\ &= \binom{k}{t}^2 k^{-2t} \Big[\sum_{m=0}^{t} \binom{t}{m}^2 m^{2t} k^2/(k^2 - m^2) \\ &\quad + 2 \sum_{m=0}^{t-1} \binom{t}{m} m^t (-1)^m \sum_{i=m+1}^{t} \binom{t}{i} i^t (-1)^i k^2/(k^2 - mi) \Big] \\ &= \sum_{m=0}^{t} \binom{t}{m}^2 m^{2t} \binom{k}{t}^2 k^{-2t} k^2/(k^2 - m^2) \\ &\quad + 2 \sum_{m=0}^{t-1} \binom{t}{m} m^t (-1)^m \sum_{i=m+1}^{t} \binom{t}{i} i^t (-1)^i \binom{k}{t}^2 k^{-2t} k^2/(k^2 - mi) \end{aligned}$$

if $k > t$.

Note that

$$\lim_{k\to\infty} k^{-2t}k^2\binom{k}{t}^2\Big/(k^2 - m^2) = (t!)^{-2}$$

and that

$$\lim_{k\to\infty} k^{-2t}k^2\binom{k}{t}^2\Big/(k^2 - mi) = (t!)^{-2} .$$

Then it follows that

$$\begin{aligned}\lim_{k\to\infty}\sum_{n=0}^{\infty} C_{n,k,t}^2 &= \sum_{m=0}^{t}\binom{t}{m}^2 m^{2t}(t!)^{-2} \\ &\quad + 2\sum_{m=0}^{t-1}\binom{t}{m} m^t(-1)^m \sum_{i=m+1}^{t}\binom{t}{i} i^t(-1)^i(t!)^{-2} \\ &= (t!)^{-2}\left[\sum_{m=0}^{t}\binom{t}{m} m^t(-1)^m\right]^2 \\ &= 1 .\end{aligned}$$

Hence, if t is a nonnegative integer

$$\lim_{k\to\infty} \|C_{k,t}\| = 1. \text{ } (\|\cdot\| \text{ is } l^2 \text{ norm})$$

Let us now show that

$$\lim_{k\to\infty} \|\varepsilon_t - \varepsilon_{k,t}\| = 0 .$$

Suppose t is a nonnegative integer and k is a positive integer greater than t.

$$\begin{aligned}&\sum_{n=0}^{\infty}(\delta_{n,t} - C_{n,k,t})^2 \\ &\quad = \sum_{n=t}^{\infty}(\delta_{n,t} - C_{n,k,t})^2 \\ &\quad = (\delta_{t,t} - C_{t,k,t})^2 + \sum_{n=t+1}^{\infty} C_{n\cdot k,t}^2 \\ &\quad = (1 - C_{t,k,t})^2 + \sum_{n=t+1}^{\infty} C_{n,k,t}^2 .\end{aligned}$$

Now

$$(1 - C_{t,k,t})^2 = \left(1 - \binom{k}{t}k^{-t}\sum_{m=0}^{t}\binom{t}{m} m^t(-1)^{m+t}\right)^2$$

and

$$\lim_{k\to\infty}(1 - C_{t,k,t})^2 = \left[1 - (t!)^{-1}\sum_{m=0}^{t}\binom{t}{m}m^t(-1)^{m+t}\right]^2$$

since

$$\lim_{k\to\infty}\binom{k}{t}k^{-t} = (t!)^{-1}$$

and hence by Lemma 2

$$\lim_{k\to\infty}(1 - C_{t,k,t})^2 = 0$$

Combining this with the fact that

$$\sum_{n=0}^{\infty} C_{n,k,t}^2 = 1$$

yields, $\lim_{k\to\infty}||\varepsilon_t - \varepsilon_{k,t}|| = 0$ for each nonnegative integer t.

Since $\{\varepsilon_t: t = 0, 1, 2, \cdots\}$ is a complete orthonormal set for l^2 and each point can be approximated by a square summable moment sequence, it follows that the set of all square summable moment sequences is dense in l^2 and hence the theorem is established.

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