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PEANO MODELS WITH MANY GENERIC CLASSES

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The famous theorem of MacDowell and Specker asserts that every model of Peano arithmetic has a proper elementary end-extension. A consequence of their theorem (and its proof) is that every model of Peano arithmetic of cardinality less than κ has a κ-like elementary end-extension, and, in addition, if κ is regular, then there is such a κ-like model in which all classes are definable. However, under the assumption of the existence of a κ-Kurepa tree, each model of Peano arithmetic of cardinality less than κ does have a κ-like elementary end-extension in which there are more than κ generic classes.

Introduction. We assume that we have some fixed countable similarity type which includes symbols denoting the usual arithmetic operations and relations: $+$, $\cdot$, $<$, 0, 1. Then Peano arithmetic, denoted by $\boldsymbol{P}$, is the theory which, besides the sentences describing the trivial arithmetic properties, includes all the instances of the induction scheme. Thus, if $\varphi(x_0, \cdots, x_{n-1}, y)$ is an $(n+1)$-ary formula, then the universal closure of

$$[\varphi(\bar{x}, 0) \wedge \forall y(\varphi(\bar{x}, \mathrm{y}) \longrightarrow \varphi(x, y+1))] \longrightarrow \forall y \varphi(\bar{x}, y)$$

is in $\boldsymbol{P}$.

Henceforth, all models considered are models of Peano arithmetic. We denote these models by $\mathscr{N}$ and $\mathscr{M}$ with universes of N and M respectively.

For any set X we denote by X^n the set of n-tuples of elements of X. If $\bar{a} \in X^n$ (the bar being only for emphasis), we let a_i be the ith coordinate of $\bar{a}$. Thus if $\bar{a} \in X^n$, then $\bar{a} = \langle a_0, \cdots, a_{n-1}\rangle$. If $\bar{X} = \langle X_0, \cdots, X_{n-1}\rangle$ we occasionally write $\bar{x} \in \bar{X}$ for $\bar{x} \in X_0 \times \cdots \times X_{n-1}$.

For any $\mathscr{N}$ a set $R \subset N^n$ is *definable* (in $\mathscr{N}$) if it is defined in $\mathscr{N}$ by a formula in which parameters from N are allowed. A set $X \subset N$ is *$\mathscr{N}$-finite* iff X is bounded and definable. If X is $\mathscr{N}$-finite and $R \subset N^{n+1}$ is definable, then $\{Q \subset N^n: \{x\} \times Q \subset R$ for some $x \in X\}$ is an $\mathscr{N}$-finite collection of sets. We choose some binary formula, say $\varphi(x, y)$, which indexes finite sets. That is, for any model $\mathscr{M}$ let $F^{\mathscr{M}}$ be a function such that if $a \in M$, then $F^{\mathscr{M}}(a) = \{b \in M: \mathscr{M} \models \varphi(a, b)\}$. Then $F^{\mathscr{M}}$ is a bijection from M to the set of $\mathscr{M}$-finite sets. We define the binary relation $\lhd$ by $x \lhd y$ iff $F(x)$ is a proper initial segment of $F(y)$. Notice that $(M, \lhd)$ is treelike in the sense that $\lhd$ is a partial ordering of M such that the set of predecessors of any element is linearly ordered. In general, $(M, \lhd)$ is not well-founded although every nonempty definable set has a $\lhd$-minimal element. If $\bar{x} = \langle x_0, \cdots, x_{n-1}\rangle$ and $\bar{y} = \langle y_0, \cdots, y_{n-1}\rangle$, we set $\bar{x} \lhd \bar{y}$ iff $x_i \lhd y_i$ for each $i < n$.

We say that $\mathscr{M}$ is an *end-extension* of $\mathscr{N}$ iff $\mathscr{N} \subset \mathscr{M}$ and N is a $<$-initial segment of M. The model $\mathscr{N}$ is *κ-like* iff card $(N) = \kappa$ but every $\mathscr{N}$-finite set has cardinality $< \kappa$.

In proving the MacDowell-Specker Theorem one shows that an ultrafilter can be defined in any model and that the ultrapower is then a proper elementary end-extension. Of course, the ultrafilter is not over the set of all subsets of the model but only over the Boolean algebra of the definable sets. Then in constructing the ultrapower one only considers the definable functions. This type of construction originated with Skolem [6], who used it to get nonstandard models of arithmetic.

We use a slightly more general type of construction. Suppose that $\mathscr{N}$ is a model and U an ultrafilter over the Boolean algebra of all definable n-ary relations of $\mathscr{N}$. Then let $\mathscr{M}$ be the ultrapower of $\mathscr{N}$ restricted to definable n-ary functions. It is routine to verify in such a setting that Łoś' Theorem (see, for example, [1, Theorem 2.2]) still remains true; in particular, $\mathscr{N}$ is elementarily embeddable in $\mathscr{M}$ in a canonical way. We call such an $\mathscr{M}$ a *Skolem ultrapower* of $\mathscr{N}$, and denote it by $\mathscr{N}^U$.

In § 1 we give a proof of the MacDowell-Specker Theorem. It is then shown that the construction results in models with few classes. In particular, if κ is regular, then in any κ-like model constructed by their method all the classes are definable.

We discuss generic classes in § 2. Whenever $cf(\kappa) > \omega$, then in any κ-like model constructed by the MacDowell-Specker method there are no generic classes. This is in contrast with Theorem 2.7 which asserts that in any model with cofinality ω there are a host of generic classes.

In § 3 we show how to construct elementary end-extensions which have many classes. Theorem 3.1 extends the MacDowell-Specker Theorem. If $\mathscr{X}$ is an $\mathscr{N}$-generic collection of classes, then $(\mathscr{N}, \mathscr{X})$ has an elementary end-extension in which each $X \in \mathscr{X}$ is properly extended. From this we can deduce one of the main results of this paper: If there exists a κ-Kurepa tree and if $\mathscr{N}$ is a model such that card $(N) < \kappa$, then $\mathscr{N}$ has a κ-like elementary end-extension with at least κ^+ classes. The methods used for constructing such models owe much to similar methods developed by Keisler [3]. In fact, he constructed such models in the case $\kappa = \omega_1$.

He also did so in the case that $\kappa = \lambda^+$, card $(N) = \lambda$, and $\mathscr{N}$ is saturated.

1. **The MacDowell-Specker Theorem.** In this section we sketch a proof of the MacDowell-Specker Theorem. We do this in order to show that if κ is regular and $\mathscr{N}$ is a κ-like model constructed according to their prescription, then all classes are definable. This fact, which is Theorem 1.5, was pointed out to me by S. Simpson.

THEOREM 1.1. (*MacDowell-Specker*). *Every model of* $\boldsymbol{P}$ *has a proper elementary end-extension.*

Proof. Let $\langle \varphi_i(x, y) : i < \omega \rangle$ be a list of all formulas of two variables. Then, using the fact that induction holds, we can find formulas $\psi_i(x)$ such that for each $n < \omega$ the following is a theorem of $\boldsymbol{P}$:

$$(\exists y > z)(\forall x < w) \bigwedge_{i<n} (\varphi_i(x, y) \longleftrightarrow \psi_i(x)) \, .$$

Let $\mathscr{N}$ be a model of $\boldsymbol{P}$ and and let

$$U = \{\{b \in N : \mathscr{N} \models \varphi_i(a, b) \longleftrightarrow \psi_i(a)\} : a \in N \text{ and } i < \omega\} \, .$$

Then U is an ultrafilter over the Boolean algebra of definable subsets of $\mathscr{N}$. Clearly $\mathscr{N}^U$ is a proper elementary end-extension of $\mathscr{N}$.

In the proof of Theorem 1.1 the ultrafilter U and, consequently, also the extension $\mathscr{N}^U$ are completely determined by the sequences $\langle \varphi_i(x, y) : i < \omega \rangle$ and $\langle \psi_i(x) : i < \omega \rangle$. We call any such extension $\mathscr{N}^U$ of $\mathscr{N}$ a *MacDowell-Specker extension* of $\mathscr{N}$. *A MacDowell-Specker chain* is a sequence $\langle \mathscr{N}_\nu : \nu < \alpha \rangle$ such that each $\mathscr{N}_{\nu+1}$ is a MacDowell-Specker extension of $\mathscr{N}_\nu$, and for δ a limit ordinal $\mathscr{N}_\delta = \bigcup \{\mathscr{N}_\nu : \nu < \delta\}$.

DEFINITION 1.2. If $\mathscr{N}$ is a model of $\boldsymbol{P}$ and $X \subset N$, then X is a *class* (of $\mathscr{N}$) iff X is not $\mathscr{N}$-finite but the intersection of X with each $\mathscr{N}$-finite set is $\mathscr{N}$-finite.

Every class is unbounded, and every unbounded, definable set is a class. A class can be thought of as a branch of the treelike structure $(N, \lhd)$ which has the same length as $(N, \lhd)$. Thus B is such a branch iff $\bigcup \{F(b) : b \in B\}$ is a class.

LEMMA 1.3. *If* $\mathscr{M}$ *is an elementary end-extension of* $\mathscr{N}$ *and* X *is a class of* $\mathscr{M}$, *then* $X \cap N$ *is a class of* $\mathscr{N}$.

Proof. One need only notice that a subset of N is $\mathscr{N}$-finite iff it is $\mathscr{M}$-finite.

LEMMA 1.4. *If $\mathscr{M}$ is a MacDowell-Specker extension of $\mathscr{N}$ and X is a class of $\mathscr{M}$, then $X \cap N$ is definable in $\mathscr{N}$.*

Proof. Let $X \subset M$ be a class of $\mathscr{M}$ and let $b \in X - N$. Then there is $c \in M$ such that $F(c) = \{x \in X: x < b\}$. But in the construction of $\mathscr{M}$, the element c corresponds to a function definable in $\mathscr{N}$. Hence clearly $\{a \in N: a \in F(c)\} = X \cap N$ is definable in $\mathscr{N}$.

THEOREM 1.5. *If $\langle \mathscr{N}_\nu: \nu \leqq \kappa \rangle$ is a MacDowell-Specker chain, where κ is regular and card $(N_0) < \kappa$, then $\mathscr{N}_\kappa$ is a κ-like elementary end-extension of $\mathscr{N}_0$ in which every class is definable.*

Proof. It is clear that $\mathscr{N}_\kappa$ is a κ-like elementary end-extension of $\mathscr{N}$. Now let X be a class of $\mathscr{N}_\kappa$. Then using the regularity of κ there is $\nu < \kappa$ such that $(\mathscr{N}_\nu, X \cap N_\nu) \prec (\mathscr{N}_\kappa, X)$. By Lemma 1.3, $X \cap N_{\nu+1}$ is a class of $\mathscr{N}_{\nu+1}$; then by Lemma 1.4, $X \cap N_\nu$ is definable in $\mathscr{N}_\nu$. But then X is definable in $\mathscr{N}_\kappa$.

Let $\mathscr{X}$ be a collection of classes of $\mathscr{N}$. We denote the structure $(\mathscr{N}, X)_{X \in \mathscr{X}}$ by $(\mathscr{N}, \mathscr{X})$. We say $\mathscr{X}$ *satisfies replacement* iff for each $(n+2)$-ary formula $\varphi(\bar{x}, u, v)$ in the language of $(\mathscr{N}, \mathscr{X})$ the following is true in $(\mathscr{N}, \mathscr{X})$:

$$\forall y \exists z (\forall u < y)[\exists v \varphi(\bar{x}, u, v) \longrightarrow (\exists v < z)\varphi(\bar{x}, u, v)] \ .$$

Similarly, we say that $\mathscr{X}$ *satisfies induction* iff for each $(n+1)$-ary formula $\varphi(\bar{x}, y)$, the following is true in $(\mathscr{N}, \mathscr{X})$:

$$[\varphi(\bar{x}, 0) \land \forall y(\varphi(\bar{x}, y) \longrightarrow \varphi(\bar{x}, y+1))] \longrightarrow \forall y \varphi(\bar{x}, y) \ .$$

We say that X satisfies replacement or induction whenever $\{X\}$ does. It is well-known that $\mathscr{X}$ satisfies replacement whenever it satisfies induction. In light of the fact that for regular κ the collection of all classes of *a* κ-like model satisfies replacement, the next theorem extends the previous one.

THEOREM 1.6. *If $\langle \mathscr{N}_\nu: \nu \leqq \alpha \rangle$ is a MacDowell-Specker chain, where $cf(\alpha) > \omega$, then $\mathscr{N}_\alpha$ is an elementary end-extension of $\mathscr{N}_0$ in which every class satisfying replacement is definable.*

Proof. The proof is like that of the previous theorem; however, we use replacement as a substitute for regularity. Let X be a class of $\mathscr{N}_\alpha$ which satisfies replacement. Then we need only note that, due to the fact that $cf(\alpha) > \omega$, there is $\nu < \alpha$ such that $(\mathscr{N}_\nu, X \cap N_\nu) \prec (\mathscr{N}_\alpha, X)$.

2. **Generic classes.** A property that a class can have is that of being generic. In this section we consider generic classes. More generally, we consider generic collections of classes.

DEFINITION 2.1. If $R \subset N^n$, then R is *dense* (in $\mathscr{N}$) iff for each $\bar{a} \in N^n$ there is $\bar{b} \in R$ such that $\bar{a} \lhd \bar{b}$.

DEFINITION 2.2. Let $\mathscr{N}$ be a model of $\boldsymbol{P}$. A collection $\mathscr{X}$ of subsets of N is *$\mathscr{N}$-generic* iff whenever $\bar{X} \in \mathscr{X}^n$ and $R \subset N^n$ is a definable dense relation, then there is $\bar{a} \in R$ such that each $F(a_i)$ is an initial segment of X_i. Also, $X \subset N$ is $\mathscr{N}$-generic iff $\{X\}$ is $\mathscr{N}$-generic.

LEMMA 2.3. *If $\mathscr{X}$ is $\mathscr{N}$-generic, then each $X \in \mathscr{X}$ is a non-definable class.*

Proof. Let $X \in \mathscr{X}$. Clearly $\{a \in N\colon F(a)$ is not an initial segment of $X\}$ is dense, so that X is not definable. To show X is a class, for each $b \in N$ let $Y_b = \{a \in N\colon$ there is $c \in F(a)$ for some $c > b\}$. (Think of Y_b as the set of finite sets which have an element $> b$.) It is clear that Y_b is a definable dense set. Thus, for any $b \in N$ there is $a \in Y_b$ such that $F(a)$ is an initial segment of X. Thus, since $\{a \in X\colon a \leqq b\}$ is a proper subset of $F(a)$, it follows that X is a class.

LEMMA 2.4. *Let $\langle \mathscr{N}_\nu\colon \nu < \delta\rangle$ be such that if $\nu < \mu < \delta$ then $\mathscr{N}_\mu$ is an elementary end-extension of $\mathscr{N}_\nu$. Let $\mathscr{N} = \bigcup \{\mathscr{N}_\nu\colon \nu < \delta\}$ and let $\mathscr{X}$ be a collection of subsets of N such that whenever $\nu < \delta$, then $\{X \cap N_\nu\colon X \in \mathscr{X}\}$ is $\mathscr{N}_\nu$-generic. Then $\mathscr{X}$ is $\mathscr{N}$-generic.*

Proof. Let $R \subset N^n$ be a definable dense relation, and let $\bar{X} \in \mathscr{X}^n$. There is some $\nu < \delta$ such that $R \cap N_\nu^n$ is a definable dense relation in $\mathscr{N}_\nu$. Then there is $\bar{a} \in R \cap N_\nu^n$ such that each $F(a_i)$ is an initial segment of $X_i \cap N_i$. But then $\bar{a} \in R$ and each $F(a_i)$ is an initial segment of X_i.

The next two lemmas can be proved by forcing. We use as conditions the sets which are finite in the sense of the model, and p extends q when $q \lhd p$. Then standard forcing techniques are used. See, for example, [5].

LEMMA 2.5. *If $\mathscr{X}$ is $\mathscr{N}$-generic, then $\mathscr{X}$ satisfies induction.*

LEMMA 2.6. *If $\mathscr{M}$ is an elementary end-extension of $\mathscr{N}$ and $\mathscr{X}$ and $\mathscr{Y}$ are respectively $\mathscr{N}$-generic and $\mathscr{M}$-generic where $\mathscr{X} = \{Y \cap N\colon Y \in \mathscr{Y}\}$, then $(\mathscr{N}, \mathscr{X}) \prec (\mathscr{M}, \mathscr{Y})$.*

The existence of generic classes in certain models is shown in the next theorem. For any model $\mathscr{N}$ we let $cf(\mathscr{N})$ be the cofinality of the order type of $\mathscr{N}$.

THEOREM 2.7. *If $cf(\mathscr{N}) = \omega$, then there is an $\mathscr{N}$-generic $\mathscr{X}$ such that each $\mathscr{N}$-finite set is an initial segment of some $X \in \mathscr{X}$.*

Proof. The proof consists of showing that working inside of $\mathscr{N}$ we can find generic classes. For each $j < \omega$ let $\langle \varphi_i^j(x, \bar{y})\colon i < \omega\rangle$ be a list of all formulas with the $j + 1$ free variables $x, y_0, \cdots, y_{j-1}$. For each $x \in N$, let $R_{i,x}^j \subset N^j$ be such that $\mathscr{N} \models R_{i,x}^j(\bar{y}) \leftrightarrow \varphi_i^j(x, \bar{y})$.

Let us assume that in some definable way there is an enumeration of all $\mathscr{N}$-finite sequences of elements of N. We denote by (a) the sequence enumerated by a, and denote the length of (a) by $\ell(a)$. We say $(a) \lhd (b)$ iff $\ell(a) = \ell(b)$ and whenever $c < \ell(a)$, then $(a)_c \lhd (b)_c$.

Then for each $n < \omega$ it is easily seen that the following is a theorem of $\boldsymbol{P}$:

$(*_n)$ For every sequence (a) there is $(b) \rhd (a)$ such that for all $i, j < n$, whenever $x < \ell(a)$, $R_{i,x}^j$ is dense and $c_0, \cdots, c_{j-1} < \ell(a)$, then $\mathscr{N} \models R_{i,x}^j\ ((b)_{c_0}, \cdots, (b)_{c_{j-1}})$.

Now let $\langle c_n\colon n < \omega\rangle$ be an increasing cofinal sequence in $\mathscr{N}$. By induction we get, for each $n < \omega$, sequences (a_n) and (b_n) where $\ell(a_n) = \ell(b_n) = c_n$. Let (a_0) be the identity sequence of length c_0. Now suppose that we have (a_n). In $(*_n)$ take (a) to be (a_n) and set $(b_n) = (b)$. Now let (a_{n+1}) be the sequence of length c_{n+1} such that $(a_{n+1})\, c = (b_n)_c$ when $c < c_n$, and $(a_{n+1})_c = c$ when $c_n \leqq c < c_{n+1}$.

For each $c \in N$ there is $n < \omega$ such that $c_n \leqq c < c_{n+1}$. Let

$$X(c) = \bigcup \{F((a_i)_c)\colon n < i < \omega\}\ .$$

Clearly $c = (a_{n+1})_c \lhd (a_{n+2})_c \lhd \cdots$, so that $F(c)$ is an initial segment of $X(c)$. It is clear, because of sentences $(*_n)$, that $\mathscr{X} = \{X(c)\colon c \in N\}$ is generic.

In Theorem 2.7 suppose that card $(N) = \kappa$. Then it is possible to get such an $\mathscr{X}$ for which each $\mathscr{N}$-finite set is an initial segment of κ^ω different $X \in \mathscr{X}$. This is the best to expect since there are only κ^ω distinct classes.

In summary, let $\mathscr{N}$ be a model of cardinality $< \kappa$. We know that it has a κ-like elementary end-extension $\mathscr{M}$, and, if κ is regular then in any such κ-like $\mathscr{M}$ the collection of all classes satisfies replacement; however, according to Theorem 1.5 there is even such an $\mathscr{M}$ in which all the classes are definable. More generally, if $cf(\kappa) > \omega$ then by Theorem 1.6 we can choose $\mathscr{M}$ so that each class satisfying replacement is definable, and thus $\mathscr{M}$ has no generic classes. This is in contrast to the case when $cf(\kappa) = \omega$; then Theorem 2.7 tells us that $\mathscr{M}$ must have many generic classes. In the next section we show that, even if $cf(\kappa) > \omega$, we can get a κ-like $\mathscr{M}$ with many generic classes.

3. **Extending generic classes.** We show in this section how to construct elementary end-extensions which extend generic classes. Iterations of these extensions result in κ-like models which have many classes. The next theorem generalizes the MacDowell-Specker Theorem and Lemma 1.4.

THEOREM 3.1. *If $\mathscr{X}$ is $\mathscr{N}$-generic, then there is a proper elementary end-extension $(\mathscr{M}, \mathscr{Y})$ of $(\mathscr{N}, \mathscr{X})$ such that for any class Y of $\mathscr{M}$, $Y \cap N$ is definable in $(\mathscr{N}, \mathscr{X})$.*

Proof. We first describe the plan of attack. By Lemma 2.5 we know that the model $(\mathscr{N}, \mathscr{X})$ satisfies induction. Thus, for each finite subset $\mathscr{Y}$ of $\mathscr{X}$ there is a proper elementary end-extension of $(\mathscr{N}, \mathscr{Y})$. However, we want to construct these extensions in some uniform way so that they all cohere. That is, we want to be able to form their direct limit using Tarski's Union Theorem (Theorem 1.9 of [5]) so as to get a proper elementary end-extension of $(\mathscr{N}, \mathscr{X})$.

We use the following piece of notation. If $e: n \to j$ and $\bar{a} = \langle a_0, \cdots, a_{j-1} \rangle$, then we set $P_e(\bar{a}) = \bar{b}$, where $b_k = a_{e(k)}$ for each $k < n$. If R is a set of j-tuples, then we also set $P_e(R) = \{P_e(\bar{a}): \bar{a} \in R\}$. Our aim is to form a directed system $\langle \mathscr{M}(\bar{X}): \bar{X} \in D \rangle$, where D is the set of all finite sequences of distinct elements of $\mathscr{X}$, and $\bar{X} \prec \bar{Y}$ whenever there is some (necessarily unique) function $e: n \to j$ such that $P_e(\bar{Y}) = \bar{X}$. Each $\mathscr{M}(\bar{X})$ is to be an elementary end-extension of $\mathscr{N}$ such that each X_i is an initial segment of an $\mathscr{M}(\bar{Y})$-finite set. Furthermore, whenever $\bar{X} \prec \bar{Y}$, there is an elementary embedding $f_{\bar{X},\bar{Y}}: \mathscr{M}(\bar{X}) \to \mathscr{M}(\bar{Y})$ which fixes the elements of $\mathscr{N}$; and if $\bar{X} \prec \bar{Y} \prec \bar{Z}$, then $f_{\bar{X},\bar{Z}} = f_{\bar{Y},\bar{Z}} \circ f_{\bar{X},\bar{Y}}$. Then the direct limit of $\langle \mathscr{M}(\bar{X}): \bar{X} \in D \rangle$ is the desired model. As in the proof of Theorem 1.1, the models $\mathscr{M}(\bar{X})$ are constructed by getting a certain ultrafilter $V(\bar{X})$, and then letting $\mathscr{M}(\bar{X})$ be the Skolem ultrapower $\mathscr{N}^{V(\bar{X})}$. The problem of the coherence of all the $\mathscr{M}(\bar{X})$ is then transferred to that of the coherence of all the ultrafilters $V(\bar{X})$.

To proceed with the proof, let $\langle \varphi_n(t, x, y_0, \cdots, y_{n-1}) \colon n < \omega \rangle$ be a list of all formulas with the free variables $t, x, y_0, y_1, \cdots$. Our first goal is to get formulas $\psi_n(z, s)$ with certain properties. Think of $\psi_n(z, s)$ in the following way: For each $n < \omega$ and each $a \in N$, let S_a^n be the set of $\mathscr{N}$-finite sequences p such that $p = (s)$ for some s for which $\psi_n(a, s)$ holds in $\mathscr{N}$. (Recall that (s) denotes the $\mathscr{N}$-finite sequence enumerated by s.) Then the sets S_a^n should have the following properties (all of which can be formulated as theorems of $\boldsymbol{P}$):

(1) S_a^n is a nonempty set of sequences each of length a.

(2) If $p \in S_a^n$ and $b < a$, then $b \lhd p_b$.

(3) If $p \in S_a^n$ and $p \lhd q$, then there exists $r \in S_a^n$ such that $q \lhd r$ (that is, S_a^n is *dense in itself*).

(4) Suppose that $p \in S_b^n$, and that f is a definable function such that $c \trianglelefteq f(c) < b$ for each $c < a$. Then $q \in S_a^n$ where $q_c = p_{f(c)}$ for each $c < a$.

(5) For each $n < \omega$ and each t, let f_t^n be the (definable) n-ary function such that $f_t^n(y_0, \cdots, y_{n-1})$ is the least x for which $\varphi_n(t, x, \bar{y}) \vee x = t$ holds in $\mathscr{N}$. Whenever $t < a$ and $a_0 < \cdots a_{n-1} < a$, then there is b such that for any $p \in S_a^n$, $f_t^n(p_{a_0}, \cdots, p_{a_{n-1}}) = b$.

(6) $S_a^m \subset S_a^n$ whenever $n < m < \omega$.

To show the existence of the formulas $\psi_n(z, s)$ satisfying properties (1)–(6), one uses induction on n, and then uses induction on the variable z inside of the model. The essence of the formal induction is the following lemma.

LEMMA. *For each n let T_n be a set of finite sequences all of the same length such that T_n is dense in itself. Furthermore, suppose that $T_m \supset T_n$ whenever $m < n$, and that P is a finite partition of T_0. Then there is $T \subset T_0$ such that $T \subset E$ for some $E \in P$, and for each n, $T \cap T_n$ is dense in itself.*

This lemma is easily proved. In fact it can easily be formulated and proved as a theorem scheme of $\boldsymbol{P}$. Using this lemma we prove the existence of sets S_a^m. This proof can of course be given in $\boldsymbol{P}$ so that we actually show the existence of formulas $\psi_m(z, s)$.

Let us suppose that $\varphi_0(t, x)$ is a universally valid formula so that we can set

$$S_a^0 = \{p \colon p \text{ is a sequence of length } a \text{ such that } b \lhd p_b \text{ for each } b < a\} .$$

Also set $S_0^m = \{\varnothing\}$ for each $m < \omega$. As an inductive hypothesis, suppose that S_x^m has been defined for each $m \leqq n$ and each x, and that $S_0^{n+1}, \cdots, S_a^{n+1}$ have also been defined. We define S_{a+1}^{n+1}. For every $c > a$, let T_c be the set of sequences p such that $p = q \mid (a+1)$ for some $q \in S_c^n$ and $p \mid a \in S_a^{n+1}$. Each T_c is dense in itself, and $c < d$ implies $T_c \supset T_d$. Partition T_{a+1} as follows: If $p, q \in T_{a+1}$, then p is equivalent to q iff whenever $m \leqq n+1$, $t < a+1$ and $a_0 < \cdots < a_{m-1} < a+1$, then $f_t^m(p_{a_0}, \cdots, p_{a_{m-1}}) = f_t^m(q_{a_0}, \cdots, q_{a_{m-1}})$. This partition is clearly $\mathscr{N}$-finite. The hypotheses of the lemma are now met. Hence, there is $T \subset T_{a+1}$ such that $T \subset E$ for some $E \in P$ and, for each $c > a$, $T \cap T_c$ is dense in itself. Let $S_{a+1}^{n+1} = T$. One easily checks that the S_a^m so defined satisfy the properties (1)-(6).

Next, we give the construction of the model $\mathscr{M}$. Let $\bar{x} = \langle x_0, \cdots, x_n \rangle$ be an $(n+1)$-tuple of elements of N. For each $a \in N$ and $m < \omega$, define

$R_a^m(\bar{x}) = \{\langle p_{x_0}, \cdots, p_{x_n} \rangle : p \in S_a^m\}$,

$U^m(\bar{x}) = \{R \subset N^{n+1} : R$ is definable and $R \supset R_a^m(\bar{x})$ for some $a \in N\}$,

$U(\bar{x}) = \bigcup \{U^m(\bar{x}) : m < \omega\}$.

The following properties hold:

(1*) $U(\bar{x}) \neq \varnothing$.

(2*) If $S \supset R \in U(\bar{x})$ and $S \in N^{n+1}$ is definable, then $S \in U(\bar{x})$.

(3*) If $R, S \in U(\bar{x})$, then $R \cap S \in U(\bar{x})$.

(4*) If $\bar{x} \lhd \bar{y}$, then $U(\bar{x}) \subset U(\bar{y})$.

(5*) $\{\bar{y} \in N^{n+1} : \bar{x} \lhd \bar{y}\} \in U(\bar{x})$.

(6*) If $R \subset N^{n+1}$ is definable, then for some $m < \omega$ the set

$$\{\bar{x} \in N^{n+1} : R \in U^m(\bar{x}) \text{ or } (N^{n+1} - R) \in U^m(\bar{x})\}$$

is dense. More generally, if $f: N^{n+1} \to N$ is definable with bounded range, then for some m the set $\{\bar{x} \in N^{n+1} : f^{-1}(b) \in U^m(\bar{x})$ for some $b \in N\}$ is dense.

Property (1*) follows immediately from (1), property (2*) is trivial, and (3*) can easily be proved from (4) and (6). Property (5*) is immediate from (2), and (6*) is a consequence of (5). Property (4*) follows from (4). For, suppose that $\bar{x} \lhd \bar{y}$ and $R \in U(\bar{x})$. Then $R = R_a^m(\bar{x})$ for some m and a. Let b be sufficiently large: choose $b > \max(y_0, \cdots, y_n, a)$. Now let f be the definable function such that

$$f(x_i) = y_i \text{ for } i \leqq n ,$$

$$f(x) = x \text{ otherwise} .$$

Then using (4) we see that $R_b^m(\bar{y}) \subset R_a^m(\bar{x})$. Hence $R \in U(\bar{y})$, so that (4*) holds.

Now let $\bar{X} = \langle X_0, \cdots, X_{n-1} \rangle$ be a sequence of distinct classes of $\mathscr{N}$. Define

$$V(\bar{X}) = \{R \subset N^n : R \in U(\bar{x}) \text{ and } \bar{x} \in \bar{X}\} .$$

Properties (1*)-(4*) imply that $V(\bar{X})$ is a filter in the Boolean algebra of definable n-ary relations. (5*) implies that this filter is non-principal. Suppose, furthermore, that each X_i is a member of the generic collection $\mathscr{X}$. Hence it follows from (6*) (more accurately, from the formalization of (6*) as a theorem scheme of $\boldsymbol{P}$) that $V(\bar{X})$ is an ultrafilter. More generally, (6*) implies that some member of each $\mathscr{N}$-finite collection whose union is N^n is a member of $V(\bar{X})$.

We can now form the Skolem ultrapower $\mathscr{M}(\bar{X}) = \mathscr{N}^{V(\bar{X})}$ of $\mathscr{N}$. Since $V(\bar{X})$ is closed under $\mathscr{N}$-finite intersections, the model $\mathscr{M}(\bar{X})$ is an elementary end-extension of $\mathscr{N}$. We wish to show that each X_i is an initial segment of an $\mathscr{M}(\bar{X})$-finite set. It suffices to show that there is $f \in \mathscr{M}(\bar{X})$ such that for all $a \in X_i$, $\mathscr{M}(\bar{X}) \models a \lhd f$. By (5*) the function f defined by $f(\bar{a}) = a_i$ clearly has this property.

The ultrafilters $V(\bar{X})$ have a certain property which guarantees that the models $\mathscr{M}(\bar{X})$ cohere. Suppose that $e: n \to j$, $\bar{Y} \in \mathscr{X}^j$ and $\bar{X} = P_e(\bar{Y}) \in \mathscr{X}^n$. We easily get from (4) that

$$V(\bar{X}) = \{P_e(R): R \in V(\bar{Y})\} .$$

Thus there is an elementary embedding $f_{\bar{X},\bar{Y}}: \mathscr{M}(\bar{X}) \to \mathscr{M}(\bar{Y})$ which fixes the elements of $\mathscr{N}$. If $g: N^n \to N$ is definable in $\mathscr{N}$, then let

$$f_{X,\bar{Y}}(g) = g \circ P_e .$$

By the use of Łoś' Theorem for Skolem ultrapowers, it is easily verified that $f_{\bar{X},\bar{Y}}$ is an elementary embedding.

We form a directed system of structures. Recall that D is the set of finite sequences of distinct elements of $\mathscr{X}$. Thus $(D, \prec)$ is a directed set such that whenever $\bar{X} \prec \bar{Y}$, then $f_{\bar{X},\bar{Y}}: \mathscr{M}(\bar{X}) \to \mathscr{M}(\bar{Y})$ is an elementary embedding. Also it is clear that if $\bar{X} \prec \bar{Y} \prec \bar{Z}$, then $f_{\bar{X},\bar{Z}} = f_{\bar{Y},\bar{Z}} \circ f_{X,Y}$. Thus, $\langle \mathscr{M}(\bar{X}): \bar{X} \in D\rangle$ is a directed system of structures, so that now applying Tarski's Union Theorem we get the direct limit $\mathscr{M}_0$, which is an elementary end-extension of $\mathscr{N}$, and each $X \in \mathscr{X}$ is an initial segment of some $\mathscr{M}_0$-finite set.

We want, however, a proper elementary end-extension of $(\mathscr{N}, \mathscr{X})$. To this end, let $\langle \mathscr{M}_i: i \leqq \omega\rangle$ be a MacDowell-Specker chain, and set $\mathscr{M} = \mathscr{M}_\omega$. Since each $X \in \mathscr{X}$ is an initial segment of an $\mathscr{M}$-finite set and $cf(\mathscr{M}) = \omega$, then according to Theorem 2.7 there is an $\mathscr{M}$-generic $\mathscr{Y}$ such that $(\mathscr{N}, \mathscr{X}) \subset (\mathscr{M}, \mathscr{Y})$. But then $(\mathscr{N}, \mathscr{X}) \prec (\mathscr{M}, \mathscr{Y})$ by Lemma 2.6.

Finally we must show that for any class Y of $\mathscr{M}$, $Y \cap N$ is definable in $(\mathscr{N}, \mathscr{X})$. Let Y be a class of $\mathscr{M}$. From Lemmas 1.3

and 1.4 it is clear that $Y \cap M_0$ is definable in $\mathscr{M}_0$. We now proceed as in the proof of Lemma 1.4. Let $b \in (Y \cap M_0) - N$. Then there is $c \in M$ such that $F(c) = \{x \in Y\colon \mathrm{x} < b\}$. Let $\bar{X} \in D$ be such that $c \in \mathscr{M}(\bar{X})$. Then, in the construction of $\mathscr{M}(\bar{X})$, the element c corresponds to a function definable in $\mathscr{N}$. Now the ultrafilter $V(\bar{X})$ is definable in $(\mathscr{N}, \bar{X})$; hence, it is clear that $F(c) \cap N = Y \cap N$ is definable in $(\mathscr{N}, \bar{X})$.

We use Theorem 3.1 along with the existence of certain trees to build models with special properties. The method is based on [3].

We say $(T, <)$ is a *tree* iff $<$ is a partial ordering of T such that the set of predecessors of any element is well-ordered. We let T_α be the set of elements whose set of predecessors has order type α. A tree $(T, <)$ is a κ-tree iff both (i) $T_\alpha = \varnothing$ iff $\kappa \leqq \alpha$, and (ii) for $\alpha < \kappa$, card $(\bigcup\{T_\nu\colon \nu < \alpha\}) < \kappa$. A branch of a tree is a maximal, linearly ordered subset of the tree. We consider κ-trees which have exactly λ branches of length κ. Such a tree with $\lambda > \kappa$ is a *κ-Kurepa tree*. It is known that under the assumption of Gödel's Axiom of Constructibility $V = L$, there exist κ-Kurepa trees for each $\kappa \geqq \omega$. (See [2].) Notice that whenever $1 \leqq \lambda \leqq \kappa$, there are trivial κ-trees with exactly λ branches of length κ.

Suppose that $\lambda = 0, 1$ or λ is an infinite cardinal. We say that $\mathscr{N}$ has *generic dimension* λ iff there is an $\mathscr{N}$-generic $\mathscr{X}$ such that each class of $\mathscr{N}$ is definable in $(\mathscr{N}, \mathscr{X})$ and card $(\mathscr{X}) = \lambda$. The generic dimension, if it exists, is unique. For, suppose that $\mathscr{X}$ and $\mathscr{Y}$ are $\mathscr{N}$-generic such that each class of $\mathscr{N}$ is definable in $(\mathscr{N}, \mathscr{X})$ and $(\mathscr{N}, \mathscr{Y})$. If $\omega \leqq$ card $(\mathscr{X}) \leqq$ card $(\mathscr{Y})$, then there is $\mathscr{Y}_0 \subset \mathscr{Y}$ such that card $(\mathscr{X}) =$ card $(\mathscr{Y}_0)$ and each $X \in \mathscr{X}$ is definable in $(\mathscr{N}, \mathscr{Y}_0)$, so that each $Y \in \mathscr{Y}$ is definable in $(\mathscr{N}, \mathscr{Y}_0)$. Hence $\mathscr{Y}_0 = \mathscr{Y}$ and so card $(\mathscr{X}) =$ card $(\mathscr{Y})$. Argue similarly for finite generic dimension. Theorem 1.5 asserts the existence of κ-like models with generic dimension 0 for each regular $\kappa > \omega$; the next theorem generalizes this assertion. Notice that if $\lambda \geqq \kappa$ and $\mathscr{N}$ is κ-like with generic dimension λ, then $\mathscr{N}$ has exactly λ classes.

THEOREM 3.2. *Suppose that κ is regular, $\lambda \geqq \omega$ or $\lambda = 1$, and that there is a κ-tree with exactly λ branches of length κ. Then for each model $\mathscr{N}$ of cardinality $< \kappa$ there is a κ-like elementary end-extension $\mathscr{M}$ with generic dimension λ.*

Proof. Let $(T, <)$ be a κ-tree with exactly λ branches of length κ. Without loss of generality we can assume that each element of T is contained in some branch of length κ. By induction we get $\langle \mathscr{N}_\nu\colon \nu < \kappa\rangle$ and $\langle X_t\colon t \in T\rangle$ as follows.

Suppose $\nu = 0$. Let $\alpha = \mathrm{card}\,(T_0)\cdot\omega$ (ordinal multiplication). Let $\langle \mathscr{M}_\xi : \xi \leqq \alpha\rangle$ be a MacDowell-Specker chain such that $\mathscr{M}_0 = \mathscr{N}$. We set $\mathscr{N}_0 = \mathscr{M}_\alpha$. According to Theorem 2.7 there is an $\mathscr{N}_0$-generic $\mathscr{X}$ such that $\mathrm{card}\,(\mathscr{X}) = \mathrm{card}\,(T_0)$. So we let $\{X_t : t \in T_0\}$ be $\mathscr{N}_0$-generic, and X_s and X_t are distinct whenever s and t are.

Now suppose $\nu < \kappa$ and that we have $\mathscr{N}_\nu$ and $\{X_t : t \in T_\nu\}$ which is $\mathscr{N}_\nu$-generic, and X_s and that X_t are distinct whenever s and t are. We use Theorem 3.1 with $\mathscr{N} = \mathscr{N}_\nu$ and $\mathscr{X} = \{X_t : t \in T_\nu\}$ to get $\mathscr{M}$ as in the theorem. Furthermore, we can get $\mathscr{M}$ such that $cf(\mathscr{M}) = \omega$ and $\mathrm{card}\,(M) = \mathrm{card}\,(N_\nu) + \mathrm{card}\,(T_\nu)$. Set $\mathscr{N}_{\nu+1} = \mathscr{M}$. Now using Theorem 2.7 again we can find $\mathscr{M}_{\nu+1}$-generic $\mathscr{X}$ such that $\mathrm{card}\,(\mathscr{X}) = \mathrm{card}\,(T_{\nu+1})$. Actually we can get $\{X_t : t \in T_{\nu+1}\}$ which, besides having distinct elements and being $\mathscr{N}_{\nu+1}$-generic, has the property that whenever $s \in T_\nu$, $t \in T_{\nu+1}$, $s < t$, then X_s is a proper initial segment of X_t.

For the case of limit ordinals, let $\delta < \kappa$ be a limit ordinal. Then let $\mathscr{N}_\delta = \bigcup \{\mathscr{N}_\alpha : \alpha < \delta\}$, and for each $t \in T_\delta$, let $X_t = \bigcup \{X_s : s < t\}$. By Lemma 2.4 $\{X_t : t \in T_\delta\}$ is $\mathscr{N}_\delta$-generic.

Now let $\mathscr{M} = \bigcup \{\mathscr{N}_\nu : \nu < \kappa\}$. We claim that $\mathscr{M}$ has the desired properties. Clearly $\mathscr{M}$ is a κ-like elementary end-extension of $\mathscr{N}$. Now let

$$\mathscr{Y} = \{\bigcup \{X_t : t \in B\} : B \text{ is a branch of length } \kappa\} .$$

By the construction, $\mathrm{card}\,(\mathscr{Y}) = \lambda$. We continue as in the proof of Theorem 1.5. Let X be a class of $\mathscr{M}$. By constructing an elementary tower of models, we easily see that there is a $\nu < \kappa$ and a sequence $\langle Y_t : t \in T_\nu\rangle$ of elements of $\mathscr{Y}$ such that

$$(\mathscr{N}_\nu, X \cap \mathscr{N}_\nu, X_t)_{t \in T_\nu} \prec (\mathscr{M}, X, Y_t)_{t \in T_\nu} .$$

By Lemma 1.3, $X \cap N_{\nu+1}$ is a class of $\mathscr{N}_{\nu+1}$ so that $X \cap N_\nu$ is definable in $(\mathscr{N}_\nu, X_t)_{t \in T_\nu}$. But then X is definable in $(\mathscr{M}, \mathscr{Y})$.

Theorem 3.2 is the best possible in the following sense: if κ is regular, $\lambda \geqq \omega$, and there is a κ-like model $\mathscr{N}$ with generic dimension λ, then there is a κ-tree with exactly λ branches. We need only consider the case that $\lambda > \kappa$. Let $\mathscr{N}$ be such a κ-like model and let $\mathscr{X}$ be the collection of all classes of $\mathscr{N}$. Now choose $C \subset N$ to have order type κ. Let $T = \{(c, A) : c \in C$ and $A = \{x \in X : x < c\}$ for some $X \in \mathscr{X}\}$, and let $(c, A) < (d, B)$ iff $c < d$ and $A = \{x \in B : x < c\}$. Then $(T, <)$ is the desired κ-tree, each branch of length κ corresponding to a unique element of $\mathscr{X}$.

The next theorems can be proved with a construction like that in the proof of Theorem 3.2.

THEOREM 3.3. *Suppose that there is a κ-tree with at least λ branches of length κ. Then for each model $\mathscr{N}$ of cardinality $< \kappa$ there is a κ-like elementary end-extension $\mathscr{M}$ and an $\mathscr{M}$-generic $\mathscr{Y}$ of cardinality λ.*

THEOREM 3.4. *Suppose that $cf(\kappa) > \omega$ and that there is a κ-tree with exactly λ branches of length κ. Then for each model $\mathscr{N}$ of cardinality $< \kappa$ there is a κ-like elementary end-entension $\mathscr{M}$ and an $\mathscr{M}$-generic $\mathscr{Y}$ of cardinality λ such that each class of $\mathscr{M}$ which satisfies replacement is definable in $(\mathscr{M}, \mathscr{Y})$.*

If we do not require that the models we get be κ-like but merely that they have cardinality κ, then we can do better. Let $D(\kappa)$ be the least cardinal λ such that every tree with at most κ nodes has fewer than λ branches having length the length of the tree. It is easily shown that $\kappa^+ < D(\kappa)$ for all κ. The following theorem can be proved by the same construction as in Theorem 3.2.

THEOREM 3.5. *For each model $\mathscr{N}$ such that $\mathrm{card}(N) \leqq \kappa$ and $\lambda < D(\kappa)$, there is an elementary end-extension $\mathscr{M}$ and an $\mathscr{M}$-generic $\mathscr{Y}$ of cardinality λ.*

Theorem 3.5 is the best possible for, in general, if $\mathscr{X}$ is $\mathscr{N}$-generic, then $\mathrm{card}(\mathscr{X}) < D(\mathrm{card}(N))$.

We can apply these results to models of ZF set theory. For models of ZF a generic collection of classes is simply a collection of mutually Cohen-generic reals. The theory ZF does not, however, fall into the general scheme of Peano arithmetic described at the beginning of the Introduction. But this is easily remedied by collapsing the universe onto ω. In a model $\mathscr{M}$ of ZF, consider the notion of forcing in which the forcing conditions are one-to-one functions with natural number domains and which are ordered by inclusion. If G is a generic filter, then $F = \bigcup G$ is a one-to-one function from ω onto M. Then the structure $(\mathscr{M}, F)$ still satisfies (finite) induction and can be considered (essentially) as a model of some theory $\boldsymbol{P}$. Since generic filters always exist in countable models, we get the following theorems:

THEOREM 3.6. *Suppose that there is a κ-tree with at least λ branches of length κ. Then any model of ZF of cardinality $< \kappa$ has an elementary extension $\mathscr{M}$ such that*

(i) *each $\mathscr{M}$-finite set has cardinality $< \kappa$;*
(ii) *there are exactly κ natural numbers in $\mathscr{M}$;*
(iii) *$\mathscr{M}$ has λ mutually Cohen-generic reals.*

THEOREM 3.7. *If $\lambda < D(\kappa)$, then any model of ZF of cardinality $\leqq \kappa$ has an elementary extension of cardinality κ which has λ mutually Cohen-generic reals.*

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