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INTEGRAL OPERATORS ON $\mathscr{L}_p$-SPACES

D. R. LEWIS

It is shown that the complemented subspaces of $L^p(\mu)$-spaces are isomorphically and isometrically characterized by the behavior of the integral operators defined on such spaces. If the integral operators from E to any F are exactly those operators naturally inducing continuous maps from $l^q \check{\otimes} E$ to $l^q \hat{\otimes} F$ (where $p^{-1} + q^{-1} = 1$), then E is a $\mathscr{L}_p$-space or a $\mathscr{L}_2$-space. Further, if the integral norm always coincides with the operator norm of the induced mapping, then E is isometric to an $L^p(\mu)$-space.

Several recent papers ([9], [10], [5], [6]) have been concerned with the isomorphic and isometric characterization of the familiar Banach spaces by means of the behavior of the integral and absolutely summing operators defined on the spaces. Attention has mainly been focused on the $\mathscr{L}_\infty$-spaces, although most results have dual statements for $\mathscr{L}_1$-spaces. Here we consider a class of operators (first introduced by J. S. Cohen in [1], and called here the *cp*-operators) which can be used to provide both an isometric and isomorphic characterization of the complemented subspaces of $L^p(\mu)$, $1 < p < \infty$.

Since this result was proven the author has become aware of the elegant paper of Kwapien [4], in which it is shown that the *cp*-operators (called there the $\gamma_p{}^*$-operators) are exactly those maps of form $\beta\alpha$, where α is p-summing and β' is q-summing. Thus the isomorphic version of the theorem given here can be proven from [4]. However, the proofs have little in common, and we feel that the technique used below may be of use in other factorization problems.

1. All Banach spaces E, F, G, and H are over the real field. Operator (or map) means continuous linear operator, and by subspace we mean a closed linear subset. A map u from E onto F is called *quotient map* if the induced map from $E/u^{-1}(0)$ onto F is an isometry. The identity map on a space is written 1, and the restriction of a map u to a subspace H is written $u \mid H$.

For $1 \leqq p \leqq \infty$, l_p^n denotes the product of n copies of the scalar field under the norm $\|(a_i)\| = (\sum_{i \leqq n} |a_i|^p)^{1/p}$ for $1 \leqq p < \infty$, and under $\|(a_i)\| = \max_{i \leqq n} |a_i|$ for $p = \infty$. The Banach-Mazur distance between isomorphic Banach spaces E and F is $d(E, F) = \inf \|u\| \, \|u^{-1}\|$, where the infimum is taken over all isomorphisms from E onto F. For $1 \leqq p \leqq \infty$ and $1 \leqq \lambda$, a space E is a $\mathscr{L}_{p,\lambda}$-space if it has the following property: given $G \subset E$ finite dimensional, there is an n-

dimensional H with $G \subset H \subset E$ and $d(H, l_p^n) \leqq \lambda$. The space E is a $\mathscr{L}_p$-space if it is a $\mathscr{L}_{p,\lambda}$-space for some $\lambda \geqq 1$.

The notation and terminology about topological tensor products is that of [3]. The least and greatest tensor norms are written $|\ |_\vee$ and $|\ |_\wedge$ respectively, and $E \otimes_\alpha F$ is the completion of the algebraic tensor product under the tensor norm α. For $u: E \to F$ and $v: G \to H$, $u \otimes v$ denotes the map from $E \otimes G$ to $F \otimes H$ which satisfies $(u \otimes v)(x \otimes y) = u(x) \otimes v(y)$. An operator $u: E \to F$ is *integral* if the bilinear form taking (x, y') to $\langle u(x), y' \rangle$ naturally induces an element A of $(E \check{\otimes} F')'$, and the *integral norm* of u (written $\|u\|_\wedge$) is $\|A\|$. The space of integral operators from E to F is written $L^\wedge(E, F)$.

For $1 < p < \infty$, a map $u: E \to F$ is a *cp-operator* if $1 \otimes u$ extends to a continuous linear operator from $l^p \check{\otimes} E$ into $l^p \hat{\otimes} F$, and the *cp-norm* of u (written $\|u\|_{cp}$) is the operator norm of $1 \otimes u$. As mentioned above, this class of operators was introduced by J. S. Cohen in [1]. We need the results of [1] that $C_p(E, F)$, the space of *cp*-operators from E to F, is a Banach space under $\|\ \|_{cp}$ and is a normed two sided ideal in the generalized sense. Cohen has also shown that $\|u\| \leqq \|u\|_{cp}$ for each *cp*-operator u, and that $\|u\|_\wedge \leqq \lambda \|\lambda\|_{cp}$ if the domain of u is $\mathscr{L}_{q,\lambda}$.

2. Throughout the remainder of this paper $1 < p < \infty$ and $q = p/(p-1)$.

THEOREM. *For $u \in L(E, F)$ and $b \geqq 1$, the following are equivalent:*

(1) *For every Banach space G and $v \in C_q(F, G)$, vu is integral and $\|vu\|_\wedge \leqq b\|v\|_{cq}$.*

(2) *There is a measure μ and operators $\alpha \in L(E, L^p(\mu))$, $\beta \in L(L^p(\mu), F'')$ such that $\|\alpha\|\ \|\beta\| \leqq b$ and $\beta\alpha = ju$, where j is the canonical embedding of F into F''.*

Proof. The implication (2) $\Rightarrow$ (1) follows directly from the result of Cohen cited above.

In proving that (1) implies (2), it is possible to reduce to the case in which E is finite dimensional by making the following two observations.

(a) If u satisfies (1) and $H \subset E$ is a finite dimensional subspace, then $u \mid H$ satisfies (1).

(b) Conclusion (2) holds if, whenever $H \subset E$ is finite dimensional and $\varepsilon > 0$, $u \mid H = \beta\alpha$, for some $\alpha \in L(H, l^p)$, $\beta \in L(l^p, F)$ with $\|\alpha\|\ \|\beta\| \leqq (1 + \varepsilon)b$.

In fact, (a) follows easily from the ideal structure of the *cq*-operators, and (b) from an inspection of the proofs of Proposition 7.1

and Theorem 7.1 of [7].

It is therefore possible to assume in the remainder of the proof that E is finite dimensional, and it is necessary to establish only that the statement in (b) holds.

For $\alpha \in E' \check{\otimes} l^p$ and $\beta \in l^q \check{\otimes} F$, the contraction of (α, β) is the element of $E' \check{\otimes} F$ defined by $\mathrm{Ctr}(\alpha, \beta) = (1 \otimes \beta)(\alpha)$, where β is considered as an operator from l^p to F. It is easily seen that contraction is a bilinear mapping from $(E' \check{\otimes} l^p) \times (l^q \check{\otimes} F)$ onto $E' \check{\otimes} F$ of norm at most one, and that the operator from E to F defined by $\mathrm{Ctr}(\alpha, \beta)$ is the composition of the operators defined by α and β. By the universal mapping property for the projective tensor product, contraction extends to a norm one linear operator from $(E' \check{\otimes} l^p) \hat{\otimes} (l^q \check{\otimes} F)$ onto $E' \check{\otimes} F$. Define a norm $| \ |$ on $E' \otimes F$ by setting

$$|w| = \inf \{|\varphi|_\wedge : w = \mathrm{Ctr}(\varphi)\} .$$

Then $| \ |$ is a crossnorm on $E' \otimes F$ under which this space is complete, and further contraction is now a quotient map onto $E' \tilde{\otimes} F$ (we will write $E' \tilde{\otimes} F$ for $E' \otimes F$ under $| \ |$).

For $A \in (E' \tilde{\otimes} F)'$ let v be the map from F to E defined by $\langle v(y), x' \rangle = \langle x' \otimes y, A \rangle$. It follows that $v \in C_q(F, E)$ and that $\|v\|_{cq} = \|A\|$; in fact, the adjoint of the contraction map is an isometric embedding of $(E' \tilde{\otimes} F)'$ into the dual of $(E' \check{\otimes} l^p) \hat{\otimes} (l^q \check{\otimes} F)$, which by [2] may be naturally identified as

$$\begin{aligned} ((E' \check{\otimes} l^p) \hat{\otimes} (l^q \check{\otimes} F))' &= B(E' \check{\otimes} l^p, l^q \check{\otimes} F) \\ &= L(l^q \check{\otimes} F, (E' \check{\otimes} l^p)') \\ &= L(l^q \check{\otimes} F, l^q \hat{\otimes} E) . \end{aligned}$$

Tracing through all the identifications involved shows that $(\mathrm{Ctr})'(A) = 1 \otimes v$, and so the claim is established.

The next claim is that the operator $1 \otimes u$ from $E' \check{\otimes} E$ into $E' \tilde{\otimes} F$ has norm at most b, where b is the constant occurring in the statement (1) of the the theorem. To this end consider the adjoint of $1 \otimes u$. By preceding paragraph $(E' \tilde{\otimes} F)' \subset C_q(F, E)$ isometrically, and $(E' \check{\otimes} E)' = L^\wedge(E, E)$ isometrically by [2]. Further, after mrking these two identifications, $(1 \otimes u)'$ is the restriction of the map from $C_q(F, E)$ to $L^\wedge(E, E)$ taking v to vu, which has norm $\leqq b$ by (1).

By the preceding paragraph the tensor $w = (1 \otimes u)(1_E)$ has norm at most b in $E' \tilde{\otimes} F$, and clearly u is the operator from E to F defined by w. To complete the proof is sufficient to produce a pair (α, β) in $(E' \check{\otimes} l^p) \times (l^q \check{\otimes} F)$ so that $w = \mathrm{Ctr}(\alpha, \beta)$ and $|\alpha|_\vee |\beta|_\vee \leqq (1 + \varepsilon)b$. To do the former, we need only produce a pair (α, β) so that u is the operator defined by $\mathrm{Ctr}(\alpha, \beta)$ (since $| \ |_\vee$ and $| \ |$ are equivalent norms on $E' \otimes F = L(E, F)$, the inclusion of $E' \tilde{\otimes} F$ into $E' \check{\otimes} F$ is one-to-one).

Since $|w| \leqq b$ and Ctr is a quotient map, there is a φ in $(E' \check{\otimes} l^p) \hat{\otimes} (l^q \check{\otimes} F)$ such that $w = \mathrm{Ctr}(\varphi)$ and $|\varphi|_\wedge < (1 + \varepsilon)b$. By [2] φ has a representation

$$\varphi = \sum_{i \geqq 1} \lambda_i \alpha_i \otimes \beta_i$$

where (λ_i) is a positive sequence in l^1, $\alpha_i \in E' \check{\otimes} l^p$, $\beta_i \in l^q \check{\otimes} F$ $||\alpha_i|| \leqq 1$, $||\beta_i|| \leqq 1$ and $||(\lambda_i)|| < (1 + \varepsilon)b$.

Let $(\Gamma_i)_{i \geqq 1}$ be a partition of the natural numbers with each Γ_i countably infinite, U_i be the natural embedding of $l^p(\Gamma_i)$ into l^p, V_i the natural projection of l^p onto $l^p(\Gamma_i)$, $S_i: l^p \to l^p(\Gamma_i)$ any onto isometry and $T_i = (S_i')^{-1}$. We claim that the series

$$\sum_{i \geqq 1} \lambda_i^{1/p} (1 \otimes U_i S_i)(\alpha_i)$$

is unconditionally Cauchy in $E' \check{\otimes} l^p$, and that its sum, α, has norm at most $(\sum_{i \geqq 1} \lambda_i)^{1/p}$. To see this, let I be any finite set of indices, and consider the unordered sum over I as an operator from E to l^p. For any $x \in E$, the terms of the unordered sum evaluated at x are disjointly supported in l^p and so

$$\begin{aligned} \Big|\Big| \sum_{i \in I} \lambda_i^{1/p} (1 \otimes U_i S_i)(\alpha_i)(x) \Big|\Big|^p &= \sum_{i \in I} \lambda_i \, ||(1 \otimes U_i S_i)(\alpha_i)(x)||^p \\ &\leqq ||x||^p \sum_{i \in I} \lambda_i \, . \end{aligned}$$

Taking the supremum over the closed unit ball of E shows that the norm of the unordered sum over I is at most $(\sum_{i \in I} \lambda_i)^{1/p}$, which establishes the claim. Similarly, the series

$$\sum_{i \geqq 1} \lambda_i^{1/q} (V_i' T_i \otimes 1)(\beta_i)$$

converges unconditionally in $l^q \check{\otimes} F$ to an element β of norm at most $(\sum_i \lambda_i)^{1/q}$. Clearly $|\alpha|_\vee \, |\beta|_\vee \leqq (1 + \varepsilon)b$. For τ a tensor, write $op(\tau)$ for the operator defined by τ. Then, for each i and k,

$$\begin{aligned} &op[\mathrm{Ctr}\,((1 \otimes U_i S_i)(\alpha_i), (V_k' T_k \otimes 1)(\beta_k))] \\ &= op[(V_k' T_k \otimes 1)(\beta_k)] \, op[(1 \otimes U_i S_i)(\alpha_i)] \\ &= op(\beta_k)(V_k' T_k)'(U_i S_i) op(\alpha_i) \\ &= op(\beta_k)(T_k' V_k U_i S_i) op(\alpha_i) \\ &= op(\beta_k)(\delta_{ik} 1_{l^p}) op(\alpha_i) \\ &= \delta_{ik} op(\beta_k) op(\alpha_i) \\ &= \delta_{ik} op[\mathrm{Ctr}\,(\alpha_i, \beta_k)] \end{aligned}$$

and so, by the continuity of the maps Ctr and op,

$$\begin{aligned} op[\mathrm{Ctr}\,(\alpha, \beta)] &= \textstyle\sum_{i,k} \lambda_i^{1/p}\lambda_k^{1/q}\delta_{ik} op[\mathrm{Ctr}\,(\alpha_i, \beta_k)] \\ &= \textstyle\sum_i \lambda_i op[\mathrm{Ctr}\,(\alpha_i, \beta_i)] \\ &= op[\mathrm{Ctr}\,(\textstyle\sum_i \lambda_i \alpha_i \otimes \beta_i)] \\ &= u. \end{aligned}$$

This completes the proof.

As corollaries to the proof we have the following.

COROLLARY 1. *If every cq-operator on E is integral, then E is a $\mathscr{L}_2$-space or $\mathscr{L}_p$-space.*

COROLLARY 2. *If every cq-operator on E is integral with equality of the integral norm and cq-norm, then E is isometric to some space $L^p(\mu)$.*

Proof. Both corollaries follow by applying the theorem to the identity operator on E. The existence of a constant b satisfying condition (1) of the theorem in the situation of Corollary 1 can easily be shown by contradiction. In either case, the injection of E into E'' factors

$$E \xrightarrow{\alpha} L^p(\mu) \xrightarrow{\beta} E''$$

with $\|\alpha\|\,\|\beta\| \leqq b$, so that E is reflexive and isomorphic to a complemented subspace of $L^p(\mu)$. In general the result of Lindenstrauss and Rosenthal ([8]) shows that E is a $\mathscr{L}_p$-space or a $\mathscr{L}_2$-space. If $b = 1$, the theorem of Tzafriti [11] shows that E is isometric to some space $L^p(\mu)$.

COROLLARY 3. *If every c2-operator on E is integral, then E is isomorphic to a Hilbert space.*

Proof. It is well-known that a complemented subspace of a Hilbert space is itself a Hilbert space.

COROLLARY 4. *If $1 < s, t < \infty$ and s, t and 2 are distinct, then there are cs-operators which are not ct-operators.*

Proof. By [8] it is impossible that $l^{t'}$ be isomorphic to a complemented subspace of $L^{s'}(\mu)$, where s' and t' are the conjugates of s and t, so by the theorem, $C_s(l^{t'}, F) \not\subset L^\wedge(l^{t'}, F)$ for some F. By Cohen's result quoted at the beginning of the paper, $L^\wedge(l^{t'}, F) = C_t(l^{t'}, F)$.

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VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY

Current address: University of Florida