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IVERSEN'S THEOREM AND FIBER ALGEBRAS

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The main theorem asserts that $S_\zeta(D) = S(D) \cap \mathscr{M}_\zeta(D)$, where D is a polydomain in C^n, $S(D)$ is the Shilov boundary of $H^\infty(D)$, ζ is an essential distinguished boundary point of D, $\mathscr{M}_\zeta(D)$ is the fiber over ζ of the maximal ideal space of $H^\infty(D)$, and $S_\zeta(D)$ is the Shilov boundary of the fiber algebra. The main theorem includes Iversen's theorem for polydomains.

Introduction. This paper represents a continuation of the work in [5] and [7], dealing with the study of bounded analytic functions and cluster value theory from the "uniform algebra" point of view. The central theme will be Iversen's theorem, stating roughly that "the boundary cluster set contains the boundary of the cluster set". There is an elementary Banach algebra theorem asserting that "the image of the Shilov boundary contains the boundary of the image". When applied to certain fiber algebras, the latter assertion is closely related to Iversen's theorem, as had already been noted by Kakutani [9]. It is our purpose here to study the connection between Iversen's theorem and its abstract analogues. To describe more accurately what this amounts to, we introduce some notation and definitions.

Let D be a bounded open set in $\boldsymbol{C}^n$, let $H^\infty(D)$ denote the algebra of bounded analytic functions on D, and let $\mathscr{M}(D)$ denote the maximal ideal space of $H^\infty(D)$. The natural inclusion $D \hookrightarrow \mathscr{M}(D)$ embeds D homeomorphically as a subspace of $\mathscr{M}(D)$. [Problem: Is D open in $\mathscr{M}(D)$?.] The functions in $H^\infty(D)$ will be regarded as continuous functions on $\mathscr{M}(D)$. In particular, the coordinate functions $z_1, \cdots, z_n$ extend to $\mathscr{M}(D)$ and determine a projection

$$Z: \mathscr{M}(D) \longrightarrow \boldsymbol{C}^n,\ Z = (z_1, \cdots, z_n)\ .$$

If $\zeta \in \boldsymbol{C}^n$, then $Z^{-1}(\{\zeta\}) = \mathscr{M}_\zeta(D)$ is called the *fiber over* ζ, and the restriction of $H^\infty(D)$ to $\mathscr{M}_\zeta(D)$ is called the *fiber algebra*. The Shilov boundary of the fiber algebra is denoted by $S_\zeta(D)$, and the Shilov boundary of $H^\infty(D)$ is denoted by $S(D)$. In §1, we show that always

$$S_\zeta(D) \supseteq S(D) \cap \mathscr{M}_\zeta(D)\ .$$

Although trivial examples show that equality does not generally hold, it turns out that in some cases equality does hold, and that the reverse inclusion is in some sense an abstract form of Iversen's theorem.

From the work in [7] it follows that in the case $D \subset \boldsymbol{C}$ (i.e., $n = 1$), we have $S_\zeta(D) = S(D) \cap \mathscr{M}_\zeta(D)$ whenever ζ is an essential boundary point for D. In §§2 and 3 this result is used to give a proof of a strong form of the one-variable version of Iversen's theorem. This theorem is interpreted in §§4 and 5 as a theorem about representing measures. In §6, the representing measures are patched together to obtain a continuous family of representing measures for points of D satisfying a certain subsidiary condition which reflects Iversen's theorem.

The remaining sections are directed towards polydomain algebras. Section 7 contains background material on algebras on products $D = U_1 \times \cdots \times U_n$ of open planar sets. It includes a cluster value theorem and other results which can be obtained by a straight-forward iterated application of the techniques employed to obtain the corresponding results in the one-variable case. In §8, continuous families of representing measures are obtained for polydomain algebras, by simply taking the products of the representing measures of §6. The subsidiary condition on these families of representing measures is reinterpreted in §9, to obtain a version of Iversen's theorem for polydomains. In §10, the results of §7 are combined with the multivariable form of Iversen's theorem to prove that $S_\zeta(D) = S(D) \cap \mathscr{M}_\zeta(D)$ whenever $\zeta = (\zeta_1, \cdots, \zeta_n)$, where each ζ_j is an essential boundary point of U_j. This latter theorem includes both Iversen's theorem and the most important special case of the cluster value theorem.

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Notation and Conventions. All norms will be supremum norms, unless otherwise indicated. The supremum norm over a set E is denoted by

$$\|f\|_E = \sup\{|f(x)|: x \in E\}\ .$$

All measures are finite regular Borel measures. The closed support of a measure μ is denoted by $\operatorname{supp}\mu$. The restriction of a measure ν to a set E is denoted by ν_E. If ν is a measure on X, and π is a continuous map from X to Y, then $\pi^*\nu$ is the measure defined on Borel subsets E of Y by $(\pi^*\nu)(E) = \nu(\pi^{-1}(E))$. Then

$$\int f d(\pi^*\nu) = \int (f \circ \pi) d\nu$$

for bounded Borel functions f on Y.

If ν is a measure, then $\Sigma(\nu)$ will denote the maximal ideal space of $L^\infty(\nu)$. The Gelfand transform is an isometric isomorphism of $L^\infty(\nu)$ and $C(\Sigma(\nu))$. If E is a Borel set, the characteristic function of E in $L^\infty(\nu)$ corresponds in $C(\Sigma(\nu))$ to the characteristic function of a clopen subset of $\Sigma(\nu)$, which is denoted by $\tilde{E}$. The measure ν has a canonical lift to $\Sigma(\nu)$, which will be denoted by $\hat{\nu}$, or simply by ν when no confusion can arise. Elementary topological properties of $\Sigma(\nu)$, on the level of I. 9 of [4], will be used freely.

By $\Delta(\zeta; r)$ will denoted the open disc in the complex plane with center ζ and radius r, or the open polydisc in C^n with center ζ and multiradius r, depending on the context. If E is a subset of the complex plane, then

$$|E| = \sup\{|\zeta|: \zeta \in E\} .$$

A reference for background material on uniform algebra theory is [4].

1. The Inclusion $S_\zeta(D) \supseteq S(D) \cap \mathscr{M}_\zeta(D)$.

The main result of this section serves to clarify the state of affairs, but it will not be used in an essential way in anything that follows.

If $\{A_k\}_{k=1}^\infty$ is a sequence of uniform algebras, then $l^\infty(\{A_k\})$ will denote the algebra of sequences $\{f_k\}_{k=1}^*$, where $f_k \in A_k$, and $\sup \|f_k\| < \infty$. The following theorem is a variation of the "independence of fibers" theorem of J.-P. Rosay [11]. Generalized peak sets, or intersections of peak sets, are defined and discussed in [4].

THEOREM 1.1. *Regard $H^\infty(D)$ as a uniform algebra on the closure $\bar{D}$ of D in $\mathscr{M}(D)$. Let $\{E_k\}_{k=1}^\infty$ be a sequence of closed subsets of $\bar{D}$, and let E be the closure of $\cup E_k$ in $\bar{D}$. Suppose that for each k, E_k is a generalized peak set which does not meet the closure of $\bigcup_{j\neq k} E^j$. Then E is a generalized peak set. Moreover, the restriction algebra $H^\infty(D)|_E$ is isometrically isomorphic to $l^\infty(\{H^\infty(D)|_{E_k}\})$.*

The proof depends on the following lemma.

LEMMA 1.2. *Let N be a neighborhood of E in $\bar{D}$, let $\varepsilon > 0$, and let $g_k \in H^\infty(D)|_{E_k}$ satisfy $|g_k| \leqq 1$. Then there is $G \in H^\infty(D)$ such that $\|G\| \leqq 1 + \varepsilon$, $|G| \leqq \varepsilon$ off N, and $|G - g_k| \leqq \varepsilon$ on E_k.*

Proof. Shrinking N, if necessary, we can assume $\bar{D}\backslash N$ is adherent to $D\backslash N$. Take $\varepsilon_k > 0$ such that $\Sigma\varepsilon_k = \varepsilon$, and choose open neighborhoods M_k of E_k such that $M_k \subset N$ and $M_k \cap M_j = \emptyset$ for $j \neq k$. Since E_k is a generalized peak set, there are $f_k \in H^\infty(D)$ such that $f_k = g_k$ on E_k, $\|f_k\| \leqq 1$, and $|f_k| < \varepsilon_k$ off M_k. The series $G = \Sigma f_k$ then converges uniformly on compact subsets of D, uniformly on $D\backslash(\cup M_k) \supset D\backslash N$, and uniformly on each $M_k \cap D$. Evidently $G \in H^\infty(D)$ satisfies $\|G\| \leqq$

$1+\varepsilon$, $|G| \leqq \varepsilon$ on $D\backslash N$, and $|G-g_k| \leqq \varepsilon$ on $D \cap M_k$. Since $\bar{D}\backslash N$ is adherent to $D\backslash N$, and E_k is adherent to $D \cap M_k$, G has the desired properties.

Proof of Theorem 1.1. Suppose η is a measure on $\bar{D}$ orthogonal to $H^\infty(D)$. Let $\{N_j\}_{j=1}^\infty$ be a shrinking sequence of open neighborhoods of E in $\bar{D}$ such that $|\eta|(N_j\backslash E) \to 0$. By the lemma, there is $G_j \in H^\infty(D)$ such that $\|G_j\| \leqq 2$, $|G_j - 1| \leqq 1/j$ off E, and $|G_j| \leqq 1/j$ off N_j. Then $G_j\eta \perp H^\infty(D)$, and $G_j\eta$ converges weak-star to the restriction η_E of η to E. Consequently $\eta_E \perp H^\infty(D)$, this for all $\eta \perp H^\infty(D)$. By Glicksberg's theorem, E is a generalized peak set. In particular, $H^\infty(D)|_E$ is a closed subalgebra of $C(E)$. The lemma shows that the restriction algebra is dense in $l^\infty(\{H^\infty(D)|_{E_k}\})$, so that it must coincide with the latter algebra.

LEMMA 1.3. *For each $\zeta \in \boldsymbol{C}^n$, the generalized peak points in $S(D) \cap \mathscr{M}_\zeta(D)$ are dense in $S(D) \cap \mathscr{M}_\zeta(D)$.*

Proof. Suppose $\varphi \in S(D) \cap \mathscr{M}_\zeta(D)$. Let N be an open neighborhood of φ. It suffices to produce a generalized peak point in $\bar{N} \cap \mathscr{M}_\zeta(D)$. Since φ cannot be isolated in $S(D)$, and since the generalized peak points are dense in $S(D)$ we can find a generalized peak point φ_1 and an open neighborhood N_1 of φ_1 such that

$$N_1 \subset N,\ \varphi \notin \bar{N}_1\ , \quad \text{and} \quad |Z(\varphi_1) - \zeta| < 1\ .$$

Continuing in this fashion, we can construct a sequence $\{\varphi_j\}_{j=1}^\infty$ of generalized peak points, and open neighborhoods N_j of φ_j, such that the N_j are disjoint, $N_j \subset N$, and $|Z(\varphi_j) - \zeta| < 1/j$. Then Theorem 1.1 applies to the singletons $E_j = \{\varphi_j\}$. That theorem shows that the closure of $\{\varphi_j\}$ is a generalized peak interpolation set. In particular, any cluster point of the sequence $\{\varphi_j\}$ is a generalized peak point, which lies in $\bar{N} \cap \mathscr{M}_\zeta(D)$.

THEOREM 1.4. *If D is a bounded open subset of $\boldsymbol{C}^n$, and $\zeta \in \boldsymbol{C}^n$, then*

$$S_\zeta(D) \supseteqq S(D) \cap \mathscr{M}_\zeta(D)\ .$$

Proof. Generalized peak points in $S(D) \cap \mathscr{M}_\zeta(D)$ are also generalized peak points for $H^\infty(D)|_{\mathscr{M}_\zeta(D)}$, and hence belong to $S_\zeta(D)$. Taking closures and applying 1.3, we obtain the desired inclusion.

Now we specialize to the case of a bounded open set U in the complex plane $\boldsymbol{C}$. A point $\zeta \in \partial U$ is an *essential boundary* point for U if not all the functions in $H^\infty(U)$ extend analytically across ζ.

The set of essential boundary points E of U is a closed subset of ∂U, and $H^\infty(U)$ is isometrically isomorphic to $H^\infty(\bar{U}\backslash E)$. The projection $Z(S(U))$ of the Shilov boundary $S(U)$ of $H^\infty(U)$ coincides with E. If $\zeta \in \bar{U}\backslash E$, then $\mathscr{M}_\zeta(U)$ consists of only one point, the homomorphism "evaluation at ζ", so that $\mathscr{M}_\zeta(U) = S_\zeta(U) = \{\zeta\}$. On the other hand, if $\zeta \in E$, then $\mathscr{M}_\zeta(U)$ is quite large.

By Theorem 2.7 of [5], the restriction algebra $H^\infty(U)|_{\mathscr{M}_\zeta(U)}$ is a closed subalgebra of $C(\mathscr{M}_\zeta(U))$, whose maximal ideal space is $\mathscr{M}_\zeta(U)$. Theorem 6.8 of [7] asserts that if ζ is an essential boundary point, then every generalized peak point in $S_\zeta(U)$ for the fiber algebra is a generalized peak point for $H^\infty(D)$, and hence lies in $S(U)$. Since such points are dense in $S_\zeta(U)$, we obtain $S_\zeta(U) \subseteqq S(U) \cap \mathscr{M}_\zeta(U)$ whenever ζ is an essential boundary point. Combining this with 1.4, we obtain the following theorem.

THEOREM 1.5. *If U is a bounded open subset of $\boldsymbol{C}$, and if ζ is an essential boundary point for U, then*

$$S_\zeta(U) = S(U) \cap \mathscr{M}_\zeta(U) .$$

2. Harmonic measure on $\mathscr{M}(U)$.

For §§2 through 6, we let U be a fixed bounded domain in the complex plane. In this section we derive some elementary properties of the "harmonic measure" λ_z on $\mathscr{M}(U)$ for $z \in U$.

Let Δ be the open unit disc, and let $d\theta$ be the arc length measure on $\partial\Delta$. The correspondence between a function in $H^\infty(\Delta)$ and its radial boundary values is an isometric isomorphism of $H^\infty(\Delta)$ and a weak-star closed subalgebra $H^\infty(d\theta)$ of $L^\infty(d\theta)$. The functions in $H^\infty(d\theta)$ can consequently be regarded as continuous functions on the maximal ideal space $\Sigma(d\theta)$ of the algebra $L^\infty(d\theta)$. Since $H^\infty(d\theta)$ separates the points of $\Sigma(d\theta)$, $\Sigma(d\theta)$ can be regarded as a subset of $\mathscr{M}(\Delta)$. It turns out that $\Sigma(d\theta)$ is the Shilov boundary of $H^\infty(\Delta)$. If $w \in \Delta$, then the Poisson measure m_w on $\partial\Delta$ has a canonical lift to $\Sigma(d\theta)$, which will also be denoted by m_w. Then $f(w) = \int f dm_w$ for all $f \in H^\infty(\Delta)$ and all $w \in \Delta$.

Next we mention some facts which are developed in more detail in §6 of [8]. Let π be the universal covering map of Δ over U. Then π extends to a continuous map

$$\pi\colon \mathscr{M}(\Delta) \longrightarrow \mathscr{M}(U) .$$

For each $z \in U$, choose $w \in \Delta$ such that $\pi(w) = z$, and define

$$\lambda_z = \pi^*(m_w) ,$$

so that λ_z is the projection onto $\mathscr{M}(U)$ of m_w. Then λ_z does not depend on the choice of w. We call λ_z the *harmonic measure on* $\mathscr{M}(U)$ for $z \in U$. It satisfies

$$f(z) = \int f d\lambda_z , \qquad f \in H^\infty(U), z \in U .$$

Since the measures m_w are mutually boundedly absolutely continuous, so are the λ_z. When we are concerned only with the absolute continuity class of λ_z, we will abbreviate λ_z to λ. The inclusion map $H^\infty(U) \rightarrow L^\infty(\lambda)$ maps $H^\infty(U)$ isometrically and isomorphically onto a weak-star closed subalgebra of $L^\infty(\lambda)$, consisting of precisely those $g \in L^\infty(\lambda)$ such that $\int g d\lambda_z$ depends analytically on $z \in U$.

For $0 \leqq \theta \leqq 2\pi$, the image under π of the interval $\{re^{i\theta}: 0 \leqq r < 1\}$ is called a *conformal ray* and denoted by γ_θ. A property is said to hold for *almost all conformal rays* if it holds except for those *conformal rays* corresponding to a set of θ's with zero length. For almost all θ, the limit $\lim_{r\rightarrow 1} \pi(re^{i\theta})$ exists and will be denoted by $\tilde{\pi}(e^{i\theta})$. For such θ, the *conformal rays* γ_θ is said to *terminate* at $\tilde{\pi}(e^{i\theta})$. Each terminal point belongs to ∂U.

For each $z \in U$, we define a measure μ_z on the terminal points of the *conformal rays* as follows: select $w \in \varDelta$ such that $\pi(w) = z$, and set

$$\mu_z(E) = m_w(\{e^{i\theta}: \tilde{\pi}(e^{i\theta}) \in E\}) .$$

In other words, $\mu_z = \tilde{\pi}^*(m_w)$. Then μ_z does not depend on the choice of w. For an indication of the proof of the following simple lemma, see [5].

LEMMA 2.1. *The measure μ_z is the harmonic measure on ∂U for $z \in U$.*

Let $\rho: \Sigma(d\theta) \rightarrow \partial\varDelta$ be the natural projection, and let $d\hat{\theta}$ be the lift of $d\theta$ to $\Sigma(d\theta)$. If f is a Borel function on $\partial\varDelta$, then $f \circ \rho$ is a Borel function on $\Sigma(d\theta)$, which coincides a.e. $d\hat{\theta}$ with the Gelfand transform of f in $C(\Sigma(d\theta))$. In the case $f = \tilde{\pi}$, the Gelfand transform is the restriction of $Z \circ \pi$ to $\Sigma(d\theta)$, so that $Z \circ \pi = \tilde{\pi} \circ \rho$ a.e. $d\hat{\theta}$. It follows that

$$Z^*(\pi^*(\hat{m}_w)) = \tilde{\pi}^*(\rho^*(\hat{m}_w)) , \qquad w \in \varDelta .$$

Since the left hand side is $Z^*(\lambda_z)$, $z = \pi(w)$, and the right hand side is $\tilde{\pi}^*(m_w) = \mu_z$, we obtain the following.

LEMMA 2.2. *For $z \in U$, the projection $Z^*(\lambda_z)$ of λ_z onto $\bar{D}$ coincides with the harmonic measure μ_z on ∂D for z.*

Now let $f \in H^\infty(U)$, and let Q be a subset of ∂U. We define $\mathrm{Cl}_\Gamma(f, Q)$ to be the essential cluster set of f along *conformal rays* terminating in Q. In other words, a complex number b belongs to $\mathrm{Cl}_\Gamma(f, Q)$ if and only if for each $\varepsilon > 0$, there is a set of *conformal rays* of positive measure, each of which terminates in Q, and along each of which f has a limit within ε of b. The λ-essential range of f on $Z^{-1}(Q)$ is the essential range of f with respect to the restriction measure of λ to $Z^{-1}(Q)$.

THEOREM 2.3. *Let Q be a Borel subset of ∂U, and let $f \in H^\infty(U)$. Then $\mathrm{Cl}_\Gamma(f, Q)$ coincides with the λ-essential range of f on $Z^{-1}(Q)$.*

Proof. Let $\tilde{g} \in L^\infty(d\theta)$ be the nontangential boundary value function of $g = f \circ \pi \in H^\infty(\Delta)$. By definition, $\mathrm{Cl}_\Gamma(f, Q)$ coincides with the $d\theta$-essential range of $\tilde{g}$ on $\tilde{\pi}^{-1}(Q)$, and this coincides with the $d\hat{\theta}$-essential range of $g = f \circ \pi$ on $\rho^{-1}(\tilde{\pi}^{-1}(Q)) \subseteq \Sigma(d\theta)$. Since $Z^*(\pi^*(\hat{m}_w)) = \tilde{\pi}^*(\rho^*(\hat{m}_w))$, $\rho^{-1}(\tilde{\pi}^{-1}(Q))$ differs from $\pi^{-1}(Z^{-1}(Q))$ by a $d\hat{\theta}$-null set, so that $\mathrm{Cl}_\Gamma(f, Q)$ coincides with the $d\hat{\theta}$-essential range of $f \circ \pi$ on $\pi^{-1}(Z^{-1}(Q))$. Since $\pi^*(\hat{m}_w) = \lambda_z$, this coincides with the λ-essential range of f on $Z^{-1}(Q)$.

3. Iversen's theorem. The cluster set of $f \in H^\infty(U)$ at $\zeta \in \partial U$ is denoted by $\mathrm{Cl}(f, \zeta)$. It consists of all complex numbers b for which there is a sequence $z_n \in U$ satisfying $z_n \to \zeta$ and $f(z_n) \to b$. Evidently $\mathrm{Cl}(f, \zeta)$ coincides with the range of f on the adherence of U in $\mathscr{M}_\zeta(U)$, so that in particular

$$\mathrm{Cl}(f, \zeta) \subseteq f(\mathscr{M}_\zeta(U)) , \qquad \zeta \in \partial U, f \in H^\infty(U) .$$

It is shown in [5] that equality always holds here, but we do not need this fact for now. The following version of Iversen's theorem is already a consequence of our discussion so far. We denote by μ the harmonic measure on ∂U for some suppressed point z of U.

THEOREM 3.1. *Let Q be a relatively open subset of* $\operatorname{supp} \mu$ *and let $f \in H^\infty(U)$. If $|\mathrm{Cl}_\Gamma(f, Q)| \leqq 1$, then $|\mathrm{Cl}(f, \zeta)| \leqq 1$ for all $\zeta \in Q$.*

Proof. The hypothesis and Theorem 2.3 show that the λ-essential range of f on $Z^{-1}(Q)$ is bounded in modulus by 1. Since f is continuous on $\mathscr{M}(U)$, we have $|f| \leqq 1$ on $Z^{-1}(Q) \cap \operatorname{supp} \lambda$, a relatively open subset of $\operatorname{supp} \lambda$. Now the closed support of λ contains the Shilov boundary for $H^\infty(U)$, so that $|f| \leqq 1$ on $S(U) \cap \mathscr{M}_\zeta(U)$ for all essential boundary points $\zeta \in Q$. By Theorem 1.5, $S(U) \cap \mathscr{M}_\zeta(U) = S_\zeta(U)$, so that we obtain in turn $|f(S_\zeta(U))| \leqq 1$, $|f(\mathscr{M}_\zeta(U))| \leqq 1$, and $|\mathrm{Cl}(f, \zeta)| \leqq 1$ for essential boundary points $\zeta \in Q$. Since each $f \in H^\infty(U)$ is analytic on the subset of $\bar{U}$ of nonessential boundary points of U, and since the terminal points of *conformal rays* are appropriately dense in supp μ, it is clear that the desired result holds also for those $\zeta \in Q$ which are not essential boundary points of U.

We will need a stronger version of Iversen's theorem. Note that Theorem 3.1 emerges from the following theorem as $a \to 1$.

THEOREM 3.2. *Let Q be a relatively open subset of* supp μ, *and fix $1 < a < M$. Then there exists an open subset V of U such that Q does not meet the closure of $U \setminus V$, and such that V has the following property: If $f \in H^\infty(U)$ satisfies $\|f\| \leqq M$ and $|\mathrm{Cl}_\Gamma(f, Q)| \leqq 1$, then $|f| \leqq a$ on V.*

Proof. We will argue by contradiction. Suppose the theorem is not true. Then there exist $f_n \in H^\infty(U)$ and $z_n \in U$ such that z_n converges to a point $\zeta \in Q$, $|f_n(z_n)| > a$, $\|f_n\| \leqq M$, and $|f_n| \leqq 1$ a.e. $d\lambda$ on $Z^{-1}(Q)$. We can assume that f_n converges weak-star to f in $L^\infty(\lambda)$, so that $f \in H^\infty(U)$, $\|f\| \leqq M$, and $|f| \leqq 1$ a.e. $d\lambda$ on $Z^{-1}(Q)$.

Suppose first that ζ is a regular boundary point of U. Then μ_{z_n} converges weak-star to the point mass at ζ, so that

$$\lambda_{z_n}(Z^{-1}(Q)) = \mu_{z_n}(Q) \to 1 .$$

The estimate

$$|f_n(z_n)| = \left|\int f_n d\lambda_{z_n}\right| \leqq M\lambda_{z_n}((\partial U)\setminus Z^{-1}(Q)) + \lambda_{z_n}(Z^{-1}(Q))$$

then shows that $\limsup |f_n(z_n)| \leqq 1$, a contradiction.

Suppose next that ζ is an irregular boundary point of U. Then $\{\zeta\}$ is a component of ∂U. For each $\varepsilon > 0$, we can find a simple closed Jordan curve J_ε in U which surrounds ζ and which is contained in the ε-disc centered at ζ. Let U_ε be the part of U inside J_ε. We assume ε is so small that $U_\varepsilon \cap \operatorname{supp} \mu \subseteqq Q$. By Theorem 3.1, we have $|f_n| \leqq a$ near points of $Q \cap \partial U_\varepsilon$. If n is large, then $z_n \in U_\varepsilon$. Since $|f_n(z_n)| > a$, the maximum modulus principle implies that there is some point $x_n \in J_\varepsilon$ such that $|f_n(x_n)| > a$. Let $\zeta_\varepsilon \in J_\varepsilon$ be a cluster point of the sequence $\{x_n\}$. By Schwarz's lemma, we will have $|f(\zeta_\varepsilon)| \geqq a$. Now $\zeta_\varepsilon \to \zeta$ as $\varepsilon \to 0$, so that $|\mathrm{Cl}(f, \zeta)| \geqq a$. This contradicts Theorem 3.1, and the theorem is established.

4. Representing measures. In this section, we give two abstract lemmas, which will be used to convert Iversen's theorem to information on representing measures.

Let X be a compact space, let A be a closed subalgebra of $C(X)$ containing the constants, and let φ be a nonzero complex-valued homomorphism of A. A *representing measure* for φ is a positive measure τ on X satisfying $\varphi(f) = \int f d\tau$ for all $f \in A$.

LEMMA 4.1. *Let T be closed subset of X, and let $1 < a < M$. Suppose that $|\varphi(f)| \leqq a$ whenever $f \in A$ satisfies $\|f\| \leqq M$ and $|f| \leqq 1$ on T. Then there is a representing measure τ for φ such that $\tau(T) \geqq 1 - (\log a)/\log M$.*

Proof. According to II.2.1 of [4], it suffices to show that $\varphi(u) \geqq 1 - (\log a)/\log M$ whenever $u \in \operatorname{Re} A$ satisfies $u \geqq 0$, and $u \geqq 1$ on T. So suppose $g \in A$ satisfies $\operatorname{Re} g \geqq 0$, and $\operatorname{Re} g \geqq 1$ on T. Set

$$f = M \exp(-g \log M) .$$

Then $\|f\| \leqq M$, and $|f| \leqq 1$ on T, so that

$$a \geqq |\varphi(f)| = M \exp(-\operatorname{Re} \varphi(g) \log M) .$$

This yields $\operatorname{Re} \varphi(g) \geqq 1 - (\log a)/\log M$, as required.

LEMMA 4.2. *Let σ be a positive measure on X, and let B be a weak-star closed subalgebra of $L^\infty(\sigma)$ containing the constants. Let φ be a nonzero complex-valued homomorphism of B which is continuous in the weak-star topology of $L^\infty(\sigma)$. Let F be a Borel subset of X, and let $1 < a < M$. Suppose that $|\varphi(f)| \leqq a$ whenever $f \in B$ satisfies $\|f\| \leqq M$, and $|f| \leqq 1$ a.e. on F. Then there is $h \in L^1(\sigma)$ such that $h\sigma$ is a representing measure for φ, and*

$$\int_F h d\sigma \geqq 1 - 2(\log a)/\log M .$$

Proof. B can be regarded as a closed subalgebra of $C(\Sigma(\sigma))$ containing the constants. According to Lemma 4.1, there is a representing measure τ for φ on $\Sigma(\sigma)$ such that $\tau(\tilde{F}) \geqq 1 - (\log a)/\log M$. The proof of the Hoffman-Rossi theorem (cf. IV. 2 of [4]) shows that there is a net of representing measures for φ of the form $h_\alpha d\sigma$, $h_\alpha \in L^1(\sigma)$, which converges to τ in the weak-star topology of measures on $\Sigma(\sigma)$. In particular, $\int_F h_\alpha d\sigma = (h_\alpha \sigma)(\tilde{F}) \to \tau(\tilde{F})$, so we can take $h = h_\alpha$ for an appropriate index α.

5. **Existence of representing measures for $H^\infty(U)$.** Let C be a Borel subset of $\mathscr{M}(U)$. There is then a Borel subset E of $\partial\Delta$ such that $\tilde{E}$ differs from $\pi^{-1}(C)$ by a set of m_w-measure zero. Then C differs from $\pi(\tilde{E})$ by a set of λ-measure zero, so that

$$\lambda_C = \lambda_{\pi(\tilde{E})} = \pi^*(d\theta_E) .$$

Let G be the group of covering transformations of Δ over U, so that $H^\infty(U) \circ \pi$ consists of precisely the functions in $H^\infty(\Delta)$ which are invariant under G. Evidently $\tilde{E}$ is invariant under G, and E is "almost" invariant under G, that is, for each $T \in G$, $T(E)$ differs from E by a set of zero length.

LEMMA 5.1. *The following are equivalent.*

(i) *The evaluation functionals at points of U are continuous in the weak-star topology of $L^\infty(\lambda_C)$.*

(ii) *The closed support of λ_C includes the Shilov boundary of $H^\infty(U)$.*

(iii) *$\tilde{E}$ is a boundary for $H^\infty(U) \circ \pi$.*

(iv) *$\|f\|_\Delta = \|f\|_E$ for all $f \in H^\infty(\Delta)$ which are invariant under G.*

Moreover, if these conditions are satisfied, then the natural inclusion embeds $H^\infty(U)$ isometrically as a weak-star closed subalgebra of $L^\infty(\lambda_C)$, the natural inclusion embeds $H^\infty(U) \circ \pi$ as a weak-star closed subalgebra of $L^\infty(d\theta_E)$, and the evaluation functionals at points of Δ are continuous on $H^\infty(U) \circ \pi$ in the weak-star topology of $L^\infty(d\theta_E)$.

Proof. If (i) is true, then each $z \in U$ has a representing measure absolutely continuous with respect to λ_C. Each $z \in U$ then belongs to the $H^\infty(U)$ – convex hull of the closed support of λ_C, so that the closed support of λ_C is a boundary for $H^\infty(U)$, and (ii) is true.

That (ii) implies (iii) follows immediately from the relation $\lambda_C = \lambda_{\pi(\tilde{E})}$. Evidently (iii) and (iv) are equivalent.

Suppose next that (iii) and (iv) are valid. Then the inclusion $H^\infty(U) \circ \pi \subsetneq L^\infty(d\theta_E)$ is an isometry. We claim that $H^\infty(U) \circ \pi$ is weak-star closed in $L^\infty(d\theta_E)$. Indeed, suppose $\{f_n\}$ is a bounded sequence in $H^\infty(U) \circ \pi$ which converges a.e. $d\theta_E$ to some function $f \in L^\infty(d\theta_E)$. Let F be a weak-star adherent point of the sequence $\{f_n\}$ in $H^\infty(d\theta)$. If $T \in G$, then $f_n \circ T = f_n$, so that $F \circ T = F$, and $F \in H^\infty(U) \circ \pi$. Evidently $F = f$ a.e. $d\theta_E$, so that F lies in the image of $H^\infty(U) \circ \pi$ in $L^\infty(d\theta_E)$. By the Krein-Schmulian theorem (cf. IV. 2.1 of [4]), $H^\infty(U) \circ \pi$ is weak-star closed in $L^\infty(d\theta_E)$. The same proof shows that if $z_0 \in U$, then the ideal of functions $f \circ \pi \in H^\infty(U) \circ \pi$ satisfying $f(z_0) = 0$ is weak-star closed in $L^\infty(d\theta_E)$, so the point evaluation functionals are continuous on $H^\infty(U) \circ \pi$ in the weak-star topology of $L^\infty(d\theta_E)$. It follows that $H^\infty(U)$ is weak-star closed in $L^\infty(\lambda_C)$, and also that (i) is valid.

The assertion that $H^\infty(U) \subsetneq L^\infty(\lambda_C)$ is an isometry follows immediately from (ii). That completes the proof.

Now we fix a Borel subset E of $\partial\Delta$ such that $\tilde{E}$ is invariant under G, and such that the conditions of Lemma 5.1 are met. We say that

$h \in L^1(d\theta_E)$ *represents* $\pi^{-1}(z)$ if $hd\theta_E$ is a representing measure for some point (and hence for all points) of $\pi^{-1}(z)$ on the algebra $H^\infty(U) \circ \pi$. This occurs if and only if $h \geqq 0$, and $\pi^*(hd\theta_E)$ represents z on $H^\infty(U)$.

THEOREM 5.2. *Assume every boundary point of U is essential. Let Q be a relatively open subset of ∂U and let $\varepsilon > 0$. Then there exists an open subset V of U such that Q does not meet the closure of $U \backslash V$, and such that for each $z \in V$, there is a function $h_z \in L^1(d\theta_E)$ representing $\pi^{-1}(z)$ which satisfies*

$$\int_{\pi^{-1}(Z^{-1}(Q))} h_z d\theta_E > 1 - \varepsilon .$$

Proof. Choose $1 < a < M$ so that $2(\log a)/\log M < \varepsilon$, and let V be the open subset of U given in Theorem 3.2. Fix $z_0 \in V$. We apply Lemma 4.2 to $\sigma = d\theta_E$, $B = H^\infty(U) \circ \pi$, and the homomorphism

$$\varphi(f \circ \pi) = f(z_0), f \in H^\infty(U) .$$

The Borel set $\pi^{-1}(Z^{-1}(Q)) \cap \Sigma(d\theta_E))$ differs from a clopen set $\tilde{F}$ by a set of $d\theta$-measure zero. Here F is a Borel subset of E, and we take this to be the F of Lemma 4.2.

Suppose $g = f \circ \pi \in H^\infty(U) \circ \pi$ satisfies $\|g\| \leqq M$, and $|g| \leqq 1$ a.e. on F. Then $|f| \leqq 1$ a.e. $\pi^* d\theta_E$ on $Z^{-1}(Q)$. Since $\pi^* d\theta_E = \lambda_C$, and since the closed support of λ_C includes $S(U)$, we have $|f| \leqq 1$ on $S(U) \cap Z^{-1}(Q)$. By Theorem 1.5, $|f| \leqq 1$ on $\mathscr{M}_\zeta(U)$, for all $\zeta \in Q$. Hence $|\mathrm{Cl}\,(f, Q)| \leqq 1$. By Iversen's Theorem 3.2, we have $|\varphi(g)| = |f(z_0)| \leqq a$. Consequently the hypotheses of Lemma 4.2 are satisfied. The existence of the h's then follows from 4.2.

6. Continuous selection of representing measures.

In this section we show how Theorem 5.2 can be used to obtain integral representation formulae for fuctions in $H^\infty(U)$, where the kernel functions are to satisfy an accessory condition [Theorem 6.6 (iii)] which embodies Iversen's theorem. The definitions and notation from the preceding section will be preserved. In order to carry the results over later to polydomain algebras, we must continue working with measures $hd\theta_E$ on $\partial\Delta$ representing fibers $\pi^{-1}(z)$.

LEMMA 6.1. *Suppose $z_0 \in U$, and suppose $h \in L^1(d\theta_E)$ represents $\pi^{-1}(z_0)$. Then for each compact subset J of U, there exists a constant c such that*

$$\|f\|_J \leqq c\|f \circ \pi\|_{L^1(hd\theta_E)} , \quad all \quad f \in H^\infty(U) .$$

Proof. Define an analytic function u on U by

$$u(\zeta) = \int \frac{(Z \circ \pi) - z_0}{(Z \circ \pi) - \zeta} h d\theta_E , \qquad \zeta \in U .$$

Since $u(z_0) = 1$, the zeros of u are isolated. By enlarging J, we can assume that u does not vanish on ∂J. For fixed $\zeta \in \partial J$, the function $[f - f(\zeta)](z - z_0)/(z - \zeta)$ belongs to $H^\infty(U)$ and vanishes at z_0, so

$$\int \frac{(f \circ \pi) - f(\zeta)}{(Z \circ \pi) - \zeta}(Z \circ \pi - z_0) h d\theta_E = 0 , \qquad f \in H^\infty(U) .$$

Solving for $f(\zeta)$, we obtain

$$f(\zeta) = \frac{1}{u(\zeta)} \int (f \circ \pi) \frac{(Z \circ \pi) - z_0}{(Z \circ \pi) - \zeta} h d\theta_E, f \in H^\infty(U), \zeta \in \partial J .$$

This representation yields an estimate of the desired form for $\zeta \in \partial J$, and by the maximum modulus principle the estimate persists on all of J.

LEMMA 6.2. *There are functions $h_z \in L^1(d\theta_E)$, $z \in U$, such that h_z represents $\pi^{-1}(z)$, and h_z moves continuously in $L^1(d\theta_E)$ with z.*

Proof. Suppose $h \in L^1(d\theta_E)$ represents $\pi^{-1}(z_0)$ for some fixed $z_0 \in U$. By 6.1, evaluation functionals on $H^\infty(U) \circ \pi$ at points of Δ are continuous in the norm of $L^1(hd\theta_E)$, and in particular they extend continuously to the closure $H^2(hd\theta_E)$ of $H^\infty(U) \circ \pi$ in $L^2(hd\theta_E)$. For $z \in U$, let $H^2_z(hd\theta_E)$ denote the closed subspace of functions in $H^2(hd\theta_E)$ which vanish on $\pi^{-1}(z)$. Using 6.1 and Schwarz's lemma, one sees that the evaluation functionals on $H^2(hd\theta_E)$ move continuously in the dual space of $H^2(hd\theta_E)$. Consequently the orthogonal projection F_z of 1 onto $H^2(hd\theta_E) \theta H^2_z(hd\theta_E)$ moves continuously in $H^2(hd\theta_E)$. Now $F_z H^2_z(hd\theta_E)$ is orthogonal to F_z in $L^2(hd\theta_E)$, so that

$$\int f |F_z|^2 h d\theta_E = 0 , \qquad f \in H^2_z(hd\theta_E) .$$

Then $h_z = |F_z|^2 h / \int |F_z|^2 h d\theta_E$ represents $\pi^{-1}(z)$. Using 6.1 and an elementary estimate, one sees that $|F_z|^2$ moves continuously in $L^1(hd\theta_E)$, so that h_z moves continuously in $L^1(d\theta_E)$.

LEMMA 6.3. *For each $z_0 \in U$, there is a neighborhood W of z_0 and functions $k_z \in L^1(d\theta_E)$, $z \in W$, such that k_z represents $\pi^{-1}(z)$, and the functions k_z/k_{z_0} are bounded and move continuously in $L^\infty(d\theta_E)$.*

Proof. Let J be an open disc centered at z_0 whose closure is contained in U, and let ν_z be the harmonic measure on ∂J for $z \in J$. Let h_z be as in 6.2, and define $k_z \in L^1(d\theta_E)$ by

$$k_z = \int_{\partial J} h_\zeta d\nu_z(\zeta) .$$

Fix $z_1 \in J$ and $\varepsilon > 0$. Then there exists $\delta > 0$ such that if $|z - z_1| < \delta$, we have $1/(1 + \varepsilon) \leqq d\nu_z/d\nu_{z_1} \leqq 1 + \varepsilon$. For such z we obtain

$$(1 + \varepsilon)k_{z_1} - k_z = \int_{\partial J} h_\zeta d[(1 + \varepsilon)\nu_{z_1} - \nu_z] \geqq 0 ,$$

so that $k_z \leqq (1 + \varepsilon)k_{z_1}$. Since the situation is symmetric, we obtain $1/(1 + \varepsilon) \leqq k_z/k_{z_1} \leqq 1 + \varepsilon$ whenever $|z - z_1| < \delta$, and this leads easily to the continuity assertions of the lemma.

LEMMA 6.4. *Let $z_0 \in U$, and suppose $p_0 \in L^1(d\theta_E)$ represents $\pi^{-1}(z_0)$. Then there is a neighborhood W of z_0, a positive function $p \in L^1(d\theta_E)$, and functions $q_z \in L^\infty(d\theta_E)$, $z \in W$, such that $q_z p$ represents $\pi^{-1}(z)$, the q_z move continuously in $L^\infty(d\theta_E)$ with z, and $q_{z_0}p = p_0$.*

Proof. Let W and k_z be as in 6.3. Shrinking W, if necessary, we can find a continuous function $c(z)$ on W such that $0 < c(z) \leqq k_z/k_{z_0}$, and $c(z_0) = 1$. Then $k_z + c(z)[p_0 - k_{z_0}]$ represents $\pi^{-1}(z)$. If $p = p_0 + k_{z_0}$, and q_z is defined so that $k_z + c(z)[p_0 - k_{z_0}] = q_z p$, then p and the q_z have the desired properties.

The result we have been aiming at is the following.

LEMMA 6.5. *Assume every boundary point of U is essential. There exists a positive function $P \in L^1(d\theta_E)$, and functions $K_z \in L^\infty(d\theta_E)$ for $z \in U$, such that*

(i) *K_zP represents $\pi^{-1}(z)$, that is, $K_z \geqq 0$, and*

$$f(z) = \int (f \circ \pi)K_zPd\theta_E , \qquad z \in U, f \in H^\infty(U) ;$$

(ii) *the K_z move continuously in $L^\infty(d\theta_E)$ with $z \in U$; and*

(iii) *if $z \in U$ approaches $\zeta \in \partial U$, then $Z^*\pi^*(K_zPd\theta_E)$ converges weak-star to the point mass at ζ, that is, the mass of $\pi^*(K_zPd\theta_E)$ accumulates at the fiber $\mathscr{M}_\zeta(U)$.*

Proof. For $n \geqq 1$, let $\Delta_{n,1}, \Delta_{n,2} \cdots$ be a finite cover of ∂U by open discs of radius $1/n$, and let Δ'_{nk} be the open disc with the same center as Δ_{nk} and radius $2/n$. By 5.2, we can find open subsets V_{nk} of U such that $\bar{\Delta}_{nk} \cap \partial U$ does not meet the closure of $U \backslash V_{nk}$, and such that each $\pi^{-1}(z)$, $z \in V_{nk}$, has a representing function in $L^1(d\theta_E)$

with mass greater than $1 - 1/n$ on $\pi^{-1}(Z^{-1}(\Delta'_{nk}))$. We can arrange that $V_{nk} \subseteqq \Delta'_{nk}$.

For $n \geqq 1$, let $V_n = \bigcup_k V_{nk}$, and set $V_0 = U$. Then V_n lies inside a $2/n$-neighborhood of ∂U, and $U \backslash V_n$ is at a positive distance from ∂U. By shrinking the V_{nk}, if necessary, we can arrange that $\bar{V}_{n+1} \cap U \subseteqq V_n$, so the V_n shrink towards ∂U.

For fixed $\xi \in U$, let n be the largest integer such that $\xi \in V_n$, say $\xi \in V_{nk}$. Then $\xi \notin \bar{V}_{n+2}$. Suppose $p_\xi \in L^1(d\theta_E)$ represents $\pi^{-1}(\xi)$ and has mass greater than $1 - 1/n$ on $\pi^{-1}(Z^{-1}(\Delta'_{nk}))$. Let $W(\xi)$ be an open disc centered at ξ of radius less than $1/n$, such that $W(\xi) \subseteqq V_n \backslash \bar{V}_{n+2}$, and such that the representing functions of 6.4 exist for $z \in W(\xi)$ and all have mass larger then $1 - 1/n$ on $\pi^{-1}(Z^{-1}(\Delta'_{nk}))$.

We can select a sequence $\{z_j\}$ in U such that the open discs $\{W(z_j)\}$ form a locally finite cover of U. Let $p_j \in L^1(d\theta_E)$ and $q_{zj} \in L^\infty(d\theta_E)$, $z \in W(z_j)$, be the functions whose existence is asserted by 6.4, so that $q_{zj}p_j$ represents $\pi^{-1}(z)$ for $z \in W(z_j)$, and the q_{zj} move continuously in $L^\infty(d\theta_E)$ with $z \in W(z_j)$. Let $\{g_j\}$ be a continuous partition of unity subordinate to the cover $\{W(z_j)\}$, satisfying $0 \leqq g_j \leqq 1$, and define

$$h_z = \sum_{j=1}^{\infty} g_j(z) q_{zj} p_j \, .$$

Evidently $h_z \in L^1(d\theta_E)$ represents $\pi^{-1}(z)$. If we define $P = \sum p_j/2^j$, and if we define K_z so that $h_z = K_z P$, then evidently (i) and (ii) are valid.

Let $z \in U$, and let $m = m(z)$ satisfy $z \in V_{m+1} \backslash V_{m+2}$. Fix z_j such that $z \in W(z_j)$. Suppose $z_j \in V_n \backslash V_{n+1}$, and let V_{nk} be the set entering in the choice of the representing function $q_{zj}p_j$, so that the mass of $q_{zj}p_j$ on $\pi^{-1}(Z^{-1}(\Delta'_{nk}))$ is larger that $1 - 1/n$. Since $|z - z_j| < 1/n$, and since $z_j \in \Delta'_{nk}$, we have $\Delta(z; 5/n) \supseteqq \Delta'_{nk}$. Hence

$$(q_{zj} p_j d\theta_E)(\pi^{-1}(Z^{-1}(\Delta(z; 5/n)))) > 1 - 1/n \, .$$

Now by construction, $W(z_j)$ is disjoint from V_{n+2}, so that $z \notin V_{n+2}$, and hence $m \geqq n$. Consequently the above estimate remains valid if we replace n by m. Multiplying by g_j and summing over j, we obtain

$$(h_z d\theta_E)(\pi^{-1}(Z^{-1}(\Delta(z; 5/m)))) > 1 - 1/m, \; z \in V_{m+1} \backslash V_{m+2} \, .$$

If now z tends to $\zeta \in \partial U$, the index $m = m(z)$ approaches $+\infty$, so that the mass of $\pi^*(h_z d\theta_E)$ concentrates at the fiber over ζ. That concludes the proof of the lemma.

If we define $K_z^* = \pi^*(K_z)$ and $P^* = \pi^*(P)$, we obtain immediately the following theorem.

THEOREM 6.6. *Let U be a bounded domain in $\mathbf{C}$ such that every point of ∂U is essential, and let λ be the harmonic measure on $\mathscr{M}(U)$.*

Let C *be a Borel subset of* $\mathscr{M}(U)$ *such that the closed support of* λ_C *includes* $S(U)$. *Then there are real-valued nonnegative functions* $P^* \in L^1(\lambda_C)$ *and* $K_z^* \in L^\infty(\lambda_C)$, $z \in U$, *which satisfy*

(i) $f(z) = \int f K_z^* P^* d\lambda_C$, $z \in U$, $f \in H^\infty(U)$;

(ii) K_z^* *moves continuously in* $L^\infty(\lambda_C)$ *with* $z \in U$; *and*

(iii) *as* $z \in U$ *approaches* $\zeta \in \partial U$, *the mass of* $K_z^* P^* \lambda_C$ *accumulates at the fiber* $\mathscr{M}_\zeta(U)$.

In many cases it is possible to find a subset C of $\mathscr{M}(U)$ such that $S(U)$ is homeomorphic to $\Sigma(\lambda_C)$. This occurs for instance when U is finitely connected, or when U is one of the domains "of type L" treated by Zalcman [14]. In this case the functions K_z^* can be regarded as continuous functions on $S(U)$, and the K_z^* will move continuously in $C(S(U))$ with $z \in U$.

7. **Fiber algebras for polydomains.** In this section we treat bounded open sets D in C^n of the form

$$D = U_1 \times \cdots \times U_n ,$$

where each U_j is a bounded open subset of C. We begin with some elementary observations concerning the fibers $\mathscr{M}_\zeta(D)$.

In the case at hand, the coordinate projection Z maps $\mathscr{M}(D)$ onto $\bar{D} = \bar{U}_1 \times \cdots \times \bar{U}_n$. If $\zeta \in D$, then $\mathscr{M}_\zeta(D)$ consists of only the homomorphism "evaluation at ζ". Indeed, if $f \in H^\infty(D)$, there are $f_1, \cdots, f_n \in H^\infty(D)$ such that

$$f(z) = f(\zeta) + \sum_{j=1}^{n} (z_j - \zeta_j) f_j(z) .$$

Applying $\varphi \in \mathscr{M}_\zeta(D)$ to this identity, we obtain $\varphi(f) = f(\zeta)$ for all $f \in H^\infty(D)$, so that φ is the evaluation homomorphism at ζ.

Suppose next that $\zeta \in \partial D$, and suppose that the indices are arranged so that $\zeta_j \in \partial U_j$ for $1 \leqq j \leqq k$, while $\zeta_j \in U_j$ for $k+1 \leqq j \leqq n$. Let $D' = U_1 \times \cdots \times U_k$. There are natural maps

$$H^\infty(D') \xrightarrow{\iota} H^\infty(D) \xrightarrow{\rho} H^\infty(D') ,$$

defined by

$$\iota g(z_1, \cdots, z_n) = g(z_1, \cdots, z_k) , \qquad g \in H^\infty(D') ,$$
$$\rho f(z_1, \cdots, z_k) = f(z_1, \cdots, z_k, \zeta_{k+1}, \cdots, \zeta_n) , \qquad f \in H^\infty(D) .$$

These maps induce adjoint maps

$$\mathscr{M}(D') \xleftarrow{\iota^*} \mathscr{M}(D) \xleftarrow{\rho^*} \mathscr{M}(D') .$$

Since $\rho \circ \iota$ is the identity, $\iota^* \circ \rho^*$ is also the identity, and ρ^* must be a homeomorphism of $\mathscr{M}(D')$ and a compact subset of $\mathscr{M}(D)$. We claim that

$$\rho^* \mathscr{M}(D') = \{\varphi \in \mathscr{M}(D): Z_j(\varphi) = \zeta_j,\ k+1 \leqq j \leqq n\} .$$

Indeed, if $\varphi \in \mathscr{M}(D')$ and $j > k$, then $Z_j(\rho^*\varphi) = \varphi(\rho z_j) = \zeta_j$, so we have the inclusion "$\subseteqq$". On the other hand, suppose $\varphi \in \mathscr{M}(D)$ satisfies $Z_j(\varphi) = \zeta_j$, $k+1 \leqq j \leqq n$. If $f \in H^\infty(D)$, we can find

$$f_n, \cdots, f_{k+1} \in H^\infty(D)$$

such that

$$\begin{aligned} f(z) &= f(z_1, \cdots, z_k, \zeta_{k+1}, \cdots, \zeta_n) + \sum_{j=k+1}^{n} (z_j - \zeta_j) f_j(z) \\ &= (\iota\rho f)(z) + \sum_{j=k+1}^{n} (z_j - \zeta_j) f_j(z) . \end{aligned}$$

If $\psi = \iota^*\varphi$, then $f(\rho^*\psi) = (\iota\rho f)(\varphi) = f(\varphi)$, so that $\varphi = \rho^*\psi$, and the reverse inclusion "$\supseteqq$" also obtains.

Set $\zeta' = (\zeta_1, \cdots, \zeta_k) \in \partial U_1 \times \cdots \times \partial U_k$. We have shown that $\mathscr{M}_{\zeta'}(D')$ is homeomorphic in a natural way to $\mathscr{M}_\zeta(D)$. Note that the expression for f above also shows that the cluster set of f from D at $\zeta \in \partial D$ coincides with the cluster set of f from D' at $\zeta' \in \partial D'$. Consequently both the study of fibers and cluster sets at $\zeta \in \partial D$ can be reduced to the case in which ζ belongs to the "distinguished boundary" of D, that is, in which $\zeta \in \partial U_1 \times \cdots \times \partial U_n$.

Now we turn to the analogues for polydomains for some results contained in [5] and [7]. The proofs will often amount to writing down the integral formulae in [5] and [7] in one variable at a time, regarding the other variables as analytic parameters, and checking that the same estimates obtain as in the one variable case. For this reason, only sketches are given at certain points in the proofs. The iteration procedure was first employed by Bekken [1], who dealt with rational approximation on product sets. The fact that the iterative methods yield 7.1-7.4 and the cluster value Theorem 7.5 has been observed by several people.

LEMMA 7.1. *Let $\zeta_1 \in \bar{U}_1$, let $\delta > 0$, and let $D_0 = [U_1 \cap \Delta(\zeta_1; \delta)] \times U_2 \times \cdots \times U_n$. If $f \in H^\infty(D_0)$, then there exists $F \in H^\infty(D)$ and $f_1 \in H^\infty(D_0)$ such that*

$$F = f + (z_1 - \zeta_1) f_1 \quad on \quad D_0$$
$$\| F \|_D \leqq 33 \| f \|_{D_0} .$$

Proof. Let g be a smooth function of one complex variable such that g is supported on $\Delta(\zeta_1; \delta)$, $g = 1$ on $\Delta(\zeta_1; \delta/2)$, and $|\partial g/\partial \bar{w}| \leqq 4/\delta$. Define

$$F_1(z) = g(z_1)f(z) + \frac{1}{\pi}\iint \frac{f(w, z_2, \cdots, z_n)}{z_1 - w}\frac{\partial g}{\partial \bar{w}} dudv ,$$

where $w = u + iv$. Then $F_1 - f = G$ is analytic and bounded on $\Delta(\zeta_1; \delta/2) \times U_2 \times \cdots \times U_n$, so that the function

$$f_1(z_1, \cdots, z_n) = [G(z_1, z_2, \cdots, z_n) - G(\zeta_1, z_2, \cdots, z_n)]/(z_1 - \zeta_1)$$

is bounded, and

$$f(z) + (z_1 - \zeta_1)f_1(z) = F_1(z) + G(\zeta_1, z_2, \cdots, z_n) .$$

This is the desired decomposition. For the analyticity and estimates, see [7].

LEMMA 7.2. *Let $\zeta \in \bar{D}$, let $\delta > 0$, and let D_0 be the intersection of D with the polydisc $\Delta(\zeta; \delta)$. If $f \in H^\infty(D_0)$, then there exists $F \in H^\infty(D)$ and $f_1, \cdots, f_n \in H^\infty(D_0)$ such that*

$$F = f + \sum_{j=1}^{n} (z_j - \zeta_j)f_j \quad on \quad D_0 ,$$
$$||F||_D \leqq (33)^n||f||_{D_0} .$$

Proof. For the first step, we apply Lemma 7.1, with U_j replaced by $U_j \cap \Delta(\zeta_j; \delta)$ for $2 \leqq j \leqq n$. This yields

$$F_1 \in H^\infty(U_1 \times \textstyle\prod_{j=2}^{n} [U_j \cap \Delta(\zeta_j; \delta)])$$

and $f_1 \in H^\infty(D_0)$ such that $F_1 = f + (z_1 - \zeta_1)f_1$ on D_0, and $||F_1|| \leqq 33\,||f||$. Now we apply 7.1 again, interchanging the variables appropriately, and replacing the U_j's of 7.1 by the appropriate sets. After n applications of 7.1, we arrive at 7.2.

LEMMA 7.3. *Let $\zeta \in \bar{D}$. If $f \in H^\infty(D)$ satisfies*

$$\lim_{D \ni z \to \zeta} f(z) = 0 ,$$

then $f = 0$ on $\mathscr{M}_\zeta(D)$.

Proof. Let $\varepsilon > 0$. Choose $\delta > 0$ so small that $|f| \leqq \varepsilon$ on $D \cap \Delta(\zeta; \delta)$, and choose F as in 7.2. The functions $f_1, \cdots, f_n$ constructed there actually belong to $H^\infty(D)$, because f does. For $\varphi \in \mathscr{M}_\zeta(D)$, we then have $\varphi(f) = \varphi(F)$, so that $|\varphi(f)| \leqq ||F|| \leqq (33)^n\varepsilon$. Letting $\varepsilon \to 0$, we have the result.

THEOREM 7.4. *Let $\zeta \in \bar{D}$. Let $V_1, \cdots, V_n$ be open subsets of C such that $V_j \subset U_j$, $U_j \backslash V_j$ is at a positive distance from ζ_j. Then the restriction map $H^\infty(D) \to H^\infty(V_1 \times \cdots \times V_n)$ induces a homeomorphism*

$$\mathscr{M}_\zeta(V_1 \times \cdots \times V_n) \cong \mathscr{M}_\zeta(D) .$$

Proof. The proof is exactly the same as that of the corresponding result in [5].

Recall that $\mathrm{Cl}(f; \zeta)$ is the cluster set of $f \in H^\infty(D)$ at ζ. The following cluster value theorem, which includes Lemma 7.3, follows easily from 7.4.

THEOREM 7.5. *If $f \in H^\infty(D)$, and $\zeta \in \bar{D}$, then*

$$f(\mathscr{M}_\zeta(D)) = \mathrm{Cl}(f; \zeta) .$$

Proof. The proof is the same as that of the corresponding result of [5].

Recall that the fiber algebra associated with $\zeta \in \bar{D}$ is the restriction of $H^\infty(D)$ to $\mathscr{M}_\zeta(D)$. From 7.2 and 7.5, it follows that each function f in the fiber algebra has an extension $F \in H^\infty(D)$ satisfying $||F||_D \leqq [(33)^n + 1]||f||_{\mathscr{M}_\zeta(D)}$. It follows that the fiber algebra is a closed subalgebra of $C(\mathscr{M}_\zeta(D))$. In order to study the Shilov boundary $S_\zeta(D)$ of the fiber algebra, we need sharper information on the extension of functions in the fiber algebra. If $\mathscr{M}_\zeta(D)$ is a peak set, then abstract results yield sharp extension theorems, and the equality $S_\zeta(D) = S(D) \cap \mathscr{M}_\zeta(D)$. In order to treat the case in which $\mathscr{M}_\zeta(D)$ is not a peak set, we must introduce a certain ideal I_ζ which plays the role of the kernel of the distinguished homomorphism from the one-variable case, and we must prove the analogue of the extension Theorem 6.2 of [7] for I_ζ.

Suppose that $\zeta_1 \in \bar{U}_1$ is such that $\mathscr{M}_{\zeta_1}(U_1)$ is not a peak set for $H^\infty(U_1)$. Let φ be the distinguished homomorphism in $\mathscr{M}_{\zeta_1}(U_1)$. For fixed $\xi_j \in U_j$, $2 \leqq j \leqq n$, we have an identification

$$\mathscr{M}(U_1) \sim Z^{-1}(\{z_j = \xi_j, 2 \leqq j \leqq n\}) ,$$

and we denote by $(\varphi, \xi_2, \cdots, \xi_n)$ the image of the distinguished homomorphism under this identification. There is a sequence $\{z_1^{(j)}\}$ in U_1 which converges to ζ_1 in the topology of $\bar{U}_1$, and which converges to φ in the norm of the dual space of $H^\infty(U_1)$. If $f \in H^\infty(D)$, then $f(z_1^{(j)}, z_2, \cdots, z_n)$ converges uniformly in $(z_2, \cdots, z_n) \in U_2 \times \cdots \times U_n$ to $f(\varphi, z_2, \cdots, z_n)$. In particular, $f(\varphi, z_2, \cdots, z_n)$ depends analytically on the z_j's, $j \geqq 2$. [Actually, this latter assertion is true for any $\varphi \in \mathscr{M}(U_1)$, distinguished or not.]

Now suppose $\zeta \in \bar{D}$, and suppose for convenience that the variables are ordered so that $\mathscr{M}_{\zeta_j}(U_j)$ is a peak set for $H^\infty(U_j)$ if and only if $k < j \leqq n$. Let φ_j be the distinguished homeomorphism in $\mathscr{M}_{\zeta_j}(U_j)$ for $1 \leqq j \leqq k$. We define I_ζ to be the ideal of functions $f \in H^\infty(D)$ such that $f(z_1, \cdots, z_{j-1}, \varphi_j, z_{j+1}, \cdots, z_n) = 0$ for $1 \leqq j \leqq k$ and $z_i \in U_i$.

THEOREM 7.6. *Let $\zeta \in \bar{D}$, and let $f \in I_\zeta$. Suppose p is a strictly positive continuous function on $\mathscr{M}(D)$ such that $|f| \leqq p$ on $\mathscr{M}_\zeta(D)$. There exists $F \in I_\zeta$ such that $F = f$ on $\mathscr{M}_\zeta(D)$, and $|F| \leqq p$ on $\mathscr{M}(D)$.*

Proof. According to abstract principles (cf. Lemma 6.1 of [7]), it suffices to find a sequence $\{H_m\}$ in I_ζ such that the H_m are uniformly bounded, H_m coincides with f on $\mathscr{M}_\zeta(D)$, and H_m converges uniformly to zero on compact subsets of $\mathscr{M}(D)\backslash\mathscr{M}_\zeta(D)$. By 7.5, this latter requirement will be met whenever H_m converges uniformly to zero on subsets of D at a positive distance from ζ.

So fix $c > 0$ and $\varepsilon > 0$. It suffices to find $H \in H^\infty(D)$ such that $H = f$ on $\mathscr{M}_\zeta(D)$, $||H|| \leqq (33)^k||f||$, and $|H| \leqq (33)^k\varepsilon$ off $D \cap \Delta(\zeta; c)$. To produce the function H, we apply k times the procedure in Lemma 7.1, and then we handle the remaining $n - k$ variables with a peaking function. Each time we apply 7.1, we proceed as in the proof of Theorem 6.2 of [7], the crucial point being the uniform estimate there for $L_\delta(f)$, defined in Lemma 2.1 of [7] and estimated in Corollary 3.7 of [7].

To simplify matters, let us assume that $k = 2$. Consider the decomposition $F = f + (z_1 - \zeta_1)f_1 = F_1 + G(\zeta_1, z_2, \cdots, z_n)$ obtained in the proof of 7.1. For each fixed $(z_2, \cdots, z_n)$ in $U_2 \times \cdots \times U_n$, $z_1 \rightarrow F(z_1, \cdots, z_n)$ is analytic off $\Delta(\zeta_1; \delta)$ and vanishes at $z_1 = \infty$. By Schwarz's lemma, we can make $F_1(z)$ uniformly small whenever $|z_1 - \zeta_1| > c$, by taking $\delta > 0$ sufficiently small. Since $f(\varphi_1, z_2, \cdots, z_n) = 0$, the estimate of Corollary 3.7 of [7] shows that the term $G(\zeta_1, z_2, \cdots, z_n)$ is uniformly small if $\delta > 0$ is sufficiently small. Hence for $\delta > 0$ small we have $|F| < \varepsilon$ off $[U_1 \cap \Delta(\zeta_1; c)] \times U_2 \times \cdots \times U_n$. Always $||F|| \leqq 33||f||$, and $F = f$ on $\mathscr{M}_\zeta(D)$. Since $\varphi_1(z_1 - \zeta_1) = 0$, we have $F(\varphi_1, z_2, \cdots, z_n) = 0$. By inspecting the definition of F_1 and G, we see that $F(z_1, \varphi_2, z_3, \cdots, z_n) = 0$, so that $F \in I_\zeta$.

Now we apply the same procedure on the second variable to F, obtaining $F' \in I_\zeta$ such that $F' = f$ on $\mathscr{M}_\zeta(D)$, $||F'|| \leqq (33)^2||f||$, and $|F'| < \varepsilon$ off $U_1 \times [U_2 \cap \Delta(\zeta_2; c)] \times U_3 \times \cdots \times U_n$. The formula for F' shows that if $z_1 \in U_1\backslash\Delta(\zeta_1; c)$, then $|F'| \leqq 33\,\varepsilon$. Consequently $|F'| \leqq 33\,\varepsilon$ off $[U_1 \cap \Delta(\zeta_1; c)] \times [U_2 \cap \Delta(\zeta_2; c)] \times U_3 \times \cdots \times U_n$.

We are assuming that $k = 2$, so that for $3 \leqq j \leqq n$, there is a function $h_j \in H^\infty(U_j)$ which peaks on $\mathscr{M}_{\zeta_j}(U_j)$, that is, $h_j = 1$ on $\mathscr{M}_{\zeta_j}(U_j)$, and $|h_j| < 1$ on $\mathscr{M}(U_j)\backslash\mathscr{M}_{\zeta_j}(U_j)$. Then $h = h_3 \cdots h_n \in H^\infty(D)$ peaks on the set of $\varphi \in \mathscr{M}(U)$ satisfying $Z_j(\varphi) = \zeta_j$, $3 \leqq j \leqq n$. In particular, $h = 1$ on $\mathscr{M}_\zeta(D)$. If m is a large integer, then h^m is small off

$$U_1 \times U_2 \times [U_3 \cap \Delta(\zeta_3; c)] \times \cdots \times [U_n \cap \Delta(\zeta_n; c)] .$$

Consequently for m sufficiently large, the function $H = h^m F' \in I_\zeta$ has the desired properties.

8. Representation formulae for polydomains. In this section, we will extend Theorem 6.6 to polydomains.

Let $d\Theta = d\theta_1 \cdots d\theta_n$ denote the volume measure on the distinguished boundary $(\partial\Delta)^n$ of the open unit polydisc Δ^n in $\boldsymbol{C}^n$. If $f \in H^\infty(\Delta^n)$, then $\lim_{r\to 1} f(re^{i\theta_1}, \cdots, re^{i\theta_n})$ exists for almost all $\Theta = (\theta_1, \cdots, \theta_n)$, defining a boundary value function almost everywhere on $(\partial\Delta)^n$. Again the correspondence between a function and its radial boundary values is an isometric isomorphism of $H^\infty(\Delta^n)$ and a weak-star closed subalgebra $H^\infty(d\Theta)$ of $L^\infty(d\Theta)$, so that the functions in $H^\infty(\Delta^n)$ can be regarded as continuous functions on $\Sigma(d\Theta)$. It is known (cf. [10]) that $H^\infty(\Delta^n)$ does not separate the points of $\Sigma(d\Theta)$. However, once appropriate point identifications are made, one can regard $\Sigma(d\Theta)$ as a subset of $\mathscr{M}(\Delta^n)$ which contains the Shilov boundary $S(\Delta^n)$. The natural lift of the measure $d\Theta$ to $\Sigma(d\Theta)$ will also be denoted by $d\Theta$, and the projection of $d\Theta$ onto $\mathscr{M}(\Delta^n)$ will be denoted also by $d\Theta$.

Fix bounded domains $U_1, \cdots, U_n$ of $\boldsymbol{C}$ such that each boundary point of each U_j is essential, and let $D = U_1 \times \cdots \times U_n$. Let π_j be the universal covering map of Δ over U_j, and let $\pi = (\pi_1, \cdots, \pi_n)$ be the universal covering map of Δ^n over D. Then π extends to a continuous map

$$\pi \colon \mathscr{M}(\Delta^n) \longrightarrow \mathscr{M}(D) .$$

Indeed, the map $f \to f \circ \pi$ of $H^\infty(D)$ into $H^\infty(\Delta^n)$ has an adjoint whose restriction to $\mathscr{M}(\Delta^n)$, regarded as a subset of $H^\infty(\Delta^n)^*$, yields the desired extension of π.

Let G_j be the group of covering transformations of Δ over U_j, so that $\pi_j \circ T = \pi_j$ for all $T \in G_j$, and let $G = G_1 \times \cdots \times G_n$ be the group of covering transformations of Δ^n over D. Then $H^\infty(D)$ is isomorphic to the algebra of functions in $H^\infty(\Delta^n)$ which are invariant under G.

LEMMA 8.1. *Suppose $f \in H^\infty(\Delta^n)$. For almost all θ_n, the function*

$$(e^{i\theta_1}, \cdots, e^{i\theta_{n-1}}) \longrightarrow f(e^{i\theta_1}, \cdots, e^{i\theta_{n-1}}, e^{i\theta_n})$$

is defined almost everywhere $d\theta_1 \cdots d\theta_{n-1}$, and is the boundary value function of a function $F(w_1, \cdots, w_{n-1}, e^{i\theta_n})$ in $H^\infty(\Delta^{n-1})$. For each *fixed* $(w_1, \cdots, w_{n-1}) \in \Delta^{n-1}$, *the function* $e^{i\theta_n} \to F(w_1, \cdots, w_{n-1}, e^{i\theta_n})$ *is the boundary value function of the function* $w_n \to f(w_1, \cdots, w_{n-1}, w_n)$ *in* $H^\infty(\Delta)$. *If* f *is invariant under* G, *then for almost all* θ_n,

$$F(w_1, \cdots, w_{n-1}, e^{i\theta_n})$$

is invariant under $G_1 \times \cdots \times G_{n-1}$.

Proof. The first two assertions are true if f is an analytic polynomial. In the general case, take a sequence of analytic polynomials $\{f_k\}_{k=1}^\infty$ such that the f_k are uniformly bounded, and $f_k \to f$ a.e. $d\theta_1 \cdots d\theta_n$ on $(\partial\Delta)^n$. Fix θ_n so that $f_k(e^{i\theta_1}, \cdots, e^{i\theta_n})$ converges a.e. $d\theta_1 \cdots d\theta_{n-1}$. For that fixed θ_n, the functions $f_k(w_1, \cdots, w_{n-1}, e^{i\theta_n})$ converge for all $(w_1, \cdots, w_{n-1}) \in \Delta^{n-1}$, to some function $F(w_1, \cdots, w_{n-1}, e^{i\theta_n})$ in $H^\infty(\Delta^{n-1})$, whose boundary value function with respect to the first $n-1$ variables coincides with $f(e^{i\theta_1}, \cdots, e^{i\theta_{n-1}}, e^{i\theta_n})$. In particular, for each $(w_1, \cdots, w_{n-1}) \in \Delta^{n-1}$, the functions $f_k(w_1, \cdots, w_{n-1}, e^{i\theta_n})$ converge to $F(w_1, \cdots, w_{n-1}, e^{i\theta_n})$ a.e. $d\theta_n$. Since f_k converges pointwise to f on Δ^n, the function $F(w_1, \cdots, w_{n-1}, e^{i\theta_n})$ must be the boundary value function of $f(w_1, \cdots, w_{n-1}, w_n)$.

If f is invariant under G, then in particular f is invariant under $G_1 \times \cdots \times G_{n-1}$, so taking boundary values in the last variable, we see that F has the asserted invariance property.

Now for each U_j, we select a Borel subset E_j of $\partial\Delta$ as in §5, so that $\tilde{E}_j$ is invariant under G_j, and E_j enjoys the equivalent properties of Lemma 5.1. Choose $P_j \in L^1(d\theta_{E_j})$ and functions $K^j_{z_j} \in L^\infty(d\theta_{E_j})$, $z_j \in U_j$, which have the properties of Lemma 6.5, and define kernel functions

$$\begin{aligned} P(e^{i\theta_1}, \cdots, e^{i\theta_n}) &= P_1(e^{i\theta_1}) \cdots P_n(e^{i\theta_n}) \\ K_z(e^{i\theta_1}, \cdots, e^{i\theta_n}) &= K^1_{z_1}(e^{i\theta_1}) \cdots K^n_{z_n}(e^{i\theta_n}),\ z \in D\ , \end{aligned}$$

on the subset

$$E = E_1 \times \cdots \times E_n$$

of $(\partial\Delta)^n$.

LEMMA 8.2. *If* $f \in H^\infty(D)$, *then*

$$f(z) = \int_E (f \circ \pi)(e^{i\theta}) K_z(e^{i\theta}) P(e^{i\theta}) d\vartheta,\ z \in D\ .$$

Proof. Here $e^{i\theta} = (e^{i\theta_1}, \cdots, e^{i\theta_n})$, and $(f \circ \pi)(e^{i\theta})$ is interpreted as the boundary values of the function $f \circ \pi$ on Δ^n. The lemma is true for $n = 1$, and we can suppose that the lemma is true with n replaced by $n-1$. Let F be the function on $\Delta^{n-1} \times \partial\Delta$ related to $f \circ \pi \in H^\infty(\Delta^n)$

by Lemma 8.1. For almost all θ_n, F is automorphic under $G_1 \times \cdots \times G_{n-1}$, so by the induction hypothesis, $F(w_1, \cdots, w_{n-1}, e^{i\theta_n})$ is given by the integral

$$\int_{E_{n-1}} \cdots \int_{E_1} (f \circ \pi)(e^{i\theta}) K^1_{z_1}(e^{i\theta_1}) \cdots K^{n-1}_{z_{n-1}}(e^{i\theta_{n-1}}) P_1(e^{i\theta_1}) \cdots \times P_{n-1}(e^{i\theta_{n-1}}) d\theta_1 \cdots d\theta_{n-1} ,$$

where $\pi_j(w_j) = z_j \in U_j$, $1 \leqq j \leqq n-1$. Now the function

$$F(w_1, \cdots, w_{n-1}, e^{i\theta_n})$$

is automorphic under G_n, so when we multiply by $K^n_{z_n}(e^{i\theta_n}) P_n(e^{i\theta_n}) d\theta_n$ and integrate over E_n, we obtain the desired result.

The point identification mapping $\Sigma(d\Theta) \to \mathscr{M}(\Delta^n)$, carrying $d\Theta$ to a measure by the same name on $\mathscr{M}(\Delta^n)$, preserves absolute continuity and bounded absolute continuity. Hence the functions P and K_z, determine (a.e. $d\Theta_E$) functions on $\mathscr{M}(\Delta^n)$, which will be denoted ambiguously by the same symbols. For these, we have the following extension of Lemma 6.5, which leads also immediately to an extension of Theorem 6.6.

THEOREM 8.3. *Let $D = U_1 \times \cdots \times U_n$ be a bounded polydomain in C^n such that each boundary point of each U_j is essential.*

Let $P \in L^1(d\Theta_E)$ and $K_z \in L^\infty(d\Theta_E)$ be the positive functions constructed above, for $z \in D$. Then

(i) $f(z) = \int_{\mathscr{M}(\Delta^n)} (f \circ \pi) K_z P d\Theta_E$, $f \in H^\infty(D)$, $z \in D$.

(ii) *K_z moves continuously in $L^\infty(d\Theta_E)$ with $z \in D$; and*

(iii) *if $\zeta \in \partial U_1 \times \cdots \times \partial U_n$, then the mass of $\pi^*(K_z P d\Theta_E)$ accumulates at the fiber $\mathscr{M}_\zeta(D)$ as $z \in D$ approaches ζ, that is, $(Z \circ \pi)^*(K_z P d\Theta_E)$ converges weak-star to the point mass at ζ as $z \in D$ approaches ζ.*

Proof. Property (i) follows from Lemma 7.2, while property (ii) follows from the corresponding property of Lemma 6.5 for the factors of K_z.

Let Ψ_j be the natural map of $\mathscr{M}(D)$ onto $\mathscr{M}(U_j)$, obtained by restricting $\psi \in \mathscr{M}(D)$ to the functions in $H^\infty(D)$ which depend only on the jth variable. These yield a map

$$(\Psi_1, \cdots, \Psi_n) = \Psi: \mathscr{M}(D) \longrightarrow \mathscr{M}(U_1) \times \cdots \times \mathscr{M}(U_n) .$$

Similarly we have natural maps $\Phi: \mathscr{M}(\Delta^n) \to \mathscr{M}(\Delta)^n$ and

$$\Sigma(d\Theta) \to \Sigma(d\theta_1) \times \cdots \times \Sigma(d\theta_n) ,$$

and one verifies that the following diagram is commutative:

$$
\begin{tikzcd}
 & \Sigma(d\theta_1) \times \cdots \times \Sigma(d\theta_n) \arrow[rr] & & \mathscr{M}(\Delta)^n \arrow[rr, "\pi_1 \times \cdots \times \pi_n"] & & \mathscr{M}(U_1) \times \cdots \times \mathscr{M}(U_n) \arrow[d, "Z_1 \times \cdots \times Z_n"] \\
\Sigma(d\Theta) \arrow[ur] \arrow[rr] & & \mathscr{M}(\Delta_n) \arrow[ur, "\Phi"] \arrow[rr, "\pi"] & & \mathscr{M}(D) \arrow[ur, "\psi"] \arrow[r, "Z"] & \bar{U}_1 \times \cdots \times \bar{U}_n .
\end{tikzcd}
$$

If we start with the measure $K_z P d\Theta_E$ on $\Sigma(d\Theta)$ and take the low road, we obtain the measure $(Z \circ \pi)^*(K_z P d\Theta_E)$ of the theorem. On the other hand, if we take the high road around we obtain a measure on $\bar{U}_1 \times \cdots \times \bar{U}_n$ which is a product of the projections of the kernel measures dealt with in Lemma 6.5. Applying Lemma 6.5 (iii) to each factor, we obtain assertion (iii). That concludes the proof of Theorem 8.3.

As in §2, we could define a harmonic measure λ on $\mathscr{M}(D)$, by defining $\lambda_z = \pi^*(m_w)$, when m_w is the Poisson kernel for $w \in \Delta^n$, and $z = \pi(w)$. The commutative diagram above serves to compensate for the fact that m_w and λ_z are not product measures on $\mathscr{M}(\Delta^n)$ and $\mathscr{M}(D)$ respectively. The commutative diagram and results from §2 show for instance that $Z^*(\lambda_z)$ is the product of harmonic measures $\mu_{z_1} \times \cdots \times \mu_{z_n}$ on $\partial U_1 \times \cdots \times \partial U_n$.

9. Iversen's theorem in polydomains.

We retain the notation of the preceding section. Preceding in analogy with §2, the *conformal ray* γ_θ in D is defined to be the image under π of the interval $\{re^{i\Theta}: 0 \leqq r < 1\}$. Almost all $(d\Theta)$ *conformal ray* terminate at a point of $\partial U_1 \times \cdots \times \partial U_n$, and every $f \in H^\infty(D)$ has limits along almost all *conformal ray*. Let $\Gamma(E)$ denote the family of *conformal ray* γ_θ, where Θ belongs to the set $E = E_1 \times \cdots \times E_n$ of the preceding section. If Q is a subset of $\partial U_1 \times \cdots \times \partial U_n$, and $f \in H^\infty(D)$, we define $\mathrm{Cl}_{\Gamma(E)}(f, Q)$ to be the essential cluster set of f along *conformal ray* in $\Gamma(E)$ which terminate in Q. In other words, $b \in \mathrm{Cl}_{\Gamma(E)}(f, Q)$ if and only if for each $\varepsilon > 0$, there is a set of *conformal ray* in $\Gamma(E)$ corresponding to a subset of E of positive $d\Theta$-measure, each of which terminates in Q, and along each of which f has a limit within ε of b.

LEMMA 9.1. *Let Q be a Borel subset of $\partial U_1 \times \cdots \times \partial U_n$, and let $f \in H^\infty(D)$. The $\pi^*(d\Theta_E)$-essential range of f on $Z^{-1}(Q)$ coincides with* $\mathrm{Cl}_{\Gamma(E)}(f, Q)$.

Proof. The proof is essentially the same as that of Theorem 2.3. Let T be the set of $e^{i\Theta} \in E$ such that the *conformal ray* γ_θ terminates in Q. By definition, $\mathrm{Cl}_{\Gamma(E)}(f, Q)$ coincides with the $d\Theta$-essential range of the boundary value function of $f \circ \pi$ on V, and this is the range of $f \circ \pi$ on $\tilde{T}$. As in 2.3, $\tilde{T}$ differs from $\tilde{E} \cap \pi^{-1}(Z^{-1}(Q))$ by a set of $d\Theta$-measure zero. Consequently $\mathrm{Cl}_{\Gamma(E)}(f, Q)$ coincides with the $d\Theta_E$-essential range of $f \circ \pi$ on $\pi^{-1}(Z^{-1}(Q))$, and this is the $\pi^*(d\Theta_E)$-essential range of f on $Z^{-1}(Q)$.

As before, we denote by $\mathrm{Cl}\,(f, Q)$ the cluster set from D of a function $f \in H^\infty(D)$ at the set Q.

THEOREM 9.2. *Let U_j be the bounded domain in the complex plane such that each point of ∂U_j is essential, $1 \leqq j \leqq n$. Let $D = U_1 \times \cdots \times U_n$, and let Q be a relatively open subset of $\partial U_1 \times \cdots \times \partial U_n$. If $f \in H^\infty(D)$ satisfies $|\mathrm{Cl}_{\Gamma(E)}(f, Q)| \leqq 1$, then $|\mathrm{Cl}\,(f, Q)| \leqq 1$.*

Proof. Fix $\zeta \in Q$, and represent f by the integral formula of §8.

$$f(z) = \int (f \circ \pi) K_z P d\Theta_E = \int f \pi^*(K_z P d\Theta_E), \quad z \in D .$$

According to the hypothesis and Lemma 9.1, we have $|f| \leqq 1$ a.e. $\pi^*(d\Theta_E)$ on the relative neighborhood $Z^{-1}(Q)$ of $\mathscr{M}_\zeta(D)$ in the closed support of $\pi^*(d\Theta_E)$. By Theorem 8.3, the mass of $\pi^*(K_z P d\Theta_E)$ concentrates at ζ as $z \in D$ tends to ζ, so that

$$\limsup_{D \ni z \to \zeta} |f(z)| \leqq 1 .$$

That establishes the theorem.

10. The Shilov boundary of the fiber algebra.

Now we are in a position to prove a multivariable version of Theorem 1.5.

THEOREM 10.1. *Let $D = U_1 \times \cdots \times U_n$ be a bounded polydomain in $\boldsymbol{C}^n$. Let $\zeta = (\zeta_1, \cdots, \zeta_n)$, where each ζ_j is an essential boundary point of U_j. Then*

$$S_\zeta(D) = S(D) \cap \mathscr{M}_\zeta(D) .$$

We can assume that each point of ∂U_j is essential, $1 \leqq j \leqq n$. The proof will be broken into several lemmas. We fix ζ as above.

LEMMA 10.2. *Suppose $E = E_1 \times \cdots \times E_n$ is chosen as in §8. Then the closed support of $\pi^*(d\Theta_E)$ contains $S_\zeta(D)$.*

Proof. Suppose $f \in H^\infty(D)$ is such that $|f| < 1$ on the intersection of $\mathscr{M}_\zeta(D)$ and the closed support of $\pi^*(d\Theta_E)$. Then there is $\varepsilon > 0$ such that $|f| < 1$ a.e. $\pi^*(d\Theta_E)$ on $Z^{-1}(\varDelta(\zeta; \varepsilon))$. By Lemma 9.1 and Theorem 9.2, or by the proof of Theorem 9.2, we have $|\mathrm{Cl}\,(f, \zeta)| \leqq 1$. By Theorem 7.5, $|f| \leqq 1$ on $\mathscr{M}_\zeta(D)$. This shows that the intersection of $\mathscr{M}_\zeta(D)$ with the closed support of $\pi^*(d\Theta_E)$ is a boundary for the fiber algebra, so that it includes $S_\zeta(D)$.

Now we will work with the distinguished ideal I_ζ introduced in §7.

LEMMA 10.3. *There exists $F \in I_\zeta$ such that $||F|| = 1$, while $|F| = 1$ on $S_\zeta(D)$.*

Proof. Assume that the indices are arranged so that there is a distinguished homomorphism φ_j in $\mathscr{M}_{\zeta_j}(U_j)$ for $1 \leqq j \leqq k$, while $\mathscr{M}_{\zeta_j}(U_j)$ is a peak set for $H^\infty(U_j)$ when $k+1 \leqq j \leqq n$. Fix $1 \leqq j \leqq k$. According to a theorem of Fisher (cf. [2], [6]), there is a function $F_j \in H^\infty(U_j)$ such that $|F_j| < 1$ on U_j, while $|F_j| = 1$ on $S(U_j)$. Then $|F_j| < 1$ on the Gleason part of U_j, so that $|F_j(\varphi_j)| < 1$. By composing F_j with an analytic automorphism of the unit disc, we can assume that $F_j(\varphi_j) = 0$. Then $F = F_1 \cdots F_k$ belongs to I_ζ and satisfies $||F|| = 1$.

Let $\varepsilon > 0$ be small. The subset of $\mathscr{M}(U_j)$ on which $|F_j| > 1 - \varepsilon$ is a neighborhood of $S(U_j)$. By taking this to be the set C of Lemma 5.1, we see that we can choose the set E_j so that $|F_j \circ \pi_j| \geqq 1 - \varepsilon$ a.e. $d\theta_{E_j}$, $1 \leqq j \leqq k$. Then $|F \circ \pi| \geqq (1-\varepsilon)^k$ a.e. $d\Theta_E$, so that $|F| \geqq (1-\varepsilon)^k$ a.e. $\pi^*(d\Theta_E)$. According to Lemma 10.2, we must have $|F| \geqq (1-\varepsilon)^k$ on $S_\zeta(D)$. Letting $\varepsilon \to 0$, we obtain $|F| = 1$ on $S_\zeta(D)$.

Proof of Theorem 10.1. Let $\varphi \in S_\zeta(D)$ be a generalized peak point for the fiber algebra. Let p be a continuous function on $\mathscr{M}(D)$ such that $0 < p \leqq 1$, and $p(\varphi) = 1$. Then there is $g \in H^\infty(D)$ such that $g(\varphi) = 1$, and $|g| \leqq p$ on $\mathscr{M}_\zeta(D)$. Multiplying g by a unitary multiple of the function F of Lemma 10.2, we can assume furthermore that $g \in I_\zeta$. By Theorem 7.6, there is $G \in H^\infty(D)$ such that $G(\varphi) = 1$, and $|G| \leqq p$ on $\mathscr{M}(D)$. It follows that φ is a generalized peak point for $H^\infty(D)$, so that $\varphi \in S(D)$. Such φ's are dense in $S_\zeta(D)$, so that $S_\zeta(D) \subseteqq S(D) \cap \mathscr{M}_\zeta(D)$. The reverse inclusion follows from Theorem 1.4. That completes the proof.

Several remarks are in order, concerning 10.1. The first observation is that the multivariable version 9.2 of Iversen's theorem follows from Theorem 10.1, just as the single-variable version 3.1 followed from Theorem 1.5. However, Theorem 10.1 is somewhat stronger that 9.2. It implies immediately a weak form of the cluster value theorem, that $|f(\mathscr{M}_\zeta(D))| \leqq |\mathrm{Cl}\,(f, \zeta)|$.

Secondly, in order to prove Theorem 10.1 one can bypass the work in §6, basing the proof on Theorem 5.2.

Finally Theorem 10.1 remains valid whenever $D = U_1 \times \cdots \times U_n$ where each U_j is an open (not necessarily connected) subset of $\boldsymbol{C}$ on which there exists a nonconstant bounded analytic function. To see this in the case in which the U_j are bounded, one proceeds as follows. Let U be a bounded open subset of $\boldsymbol{C}$, and let $V_1, V_2, \cdots$ be the components of U. Then each $\mathscr{M}(V_j)$ can be regarded as a clopen subset of $\mathscr{M}(U)$. Let λ_j be harmonic measure on $\mathscr{M}(V_j)$, and define harmonic measure on $\mathscr{M}(U)$ to be $\lambda = \Sigma\lambda_j/2^j$. Working with the measure λ, and working with a covering space over U consisting of a disjoint union of discs, one for each V_j, one can carry through the analogues of the results in §§2 through 6. The results in §§8 through 10 then can also be modified to cover the disconnected case.

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