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A CLASS OF INFINITE DIMENSIONAL SUBGROUPS OF DIFF$^r(X)$ WHICH ARE BANACH LIE GROUPS

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It is known that if X is a compact C^∞-manifold then Diff$^r(X)$ with the usual manifold structure is a Banach manifold but not a Banach Lie group. In this paper we construct a class of infinite dimensional subgroups of Diff$^r(X)$ which are Banach Lie groups.

If X and Y are C^∞-manifolds and X is compact the construction of a Banach manifold structure on $C^r(X, Y)$, the space of mappings of class C^r, has been given in [1]. The elementary theory of abstract Banach Lie groups has been given in [3]. In this paper we show that if $Y = G$, a finite dimensional Lie group, then $C^r(X, G)$ is a Banach Lie group. Now suppose that $\pi: X \to Z$ is a principal G-bundle. We show that the group of C^r-self-equivalences of π is a closed subgroup of $C^r(X, G)$ which inherits a natural Banach Lie group structure. This gives a class of examples of effective infinite dimensional Banach Lie group actions on compact manifolds.

The Lie group structure. Let X be a compact connected C^∞-manifold and G be a finite dimensional Lie group. There is a canonical right invariant C^∞ spray, $s: TG \to TTG$, on G which is defined as follows. If $v \in T_xG$ let $\bar{v}$ be the unique right invariant vector field which satisfies $\bar{v}(x) = v$. Then $T\bar{v}: TG \to TTG$ and we have $s(v) = T\bar{v}(v)$. Now s determines an exponential mapping whose domain is all of TG, $\exp: TG \to G$, and which satisfies $\exp(TR_g(v)) = R_g(\exp(v))$ where R_g is right translation by g. If we define $\operatorname{Exp}: TG \to G \times G$ by $\operatorname{Exp}(v) = (\pi(v), \exp(v))$, where $\pi: TG \to G$ is the natural projection, then it is well known that Exp maps some open neighborhood of the 0-section in TG diffeomorphically onto an open neighborhood of $D = \{(g, g) \mid g \in G\}$ in $G \times G$ [2].

LEMMA 1. *There is an open neighborhood S of the* 0*-section in TG and an open neighborhood U of D such that*

(a) Exp *maps S diffeomorphically onto U*

(b) *for all g in G we have that $TR_g(S) = S$ and $\{(hg, kg) \mid (h, k) \in U\} = U$.*

Proof. Note that $\exp \mid T_eG = \exp_e$ is the classical exponential mapping for the Lie group G. Choose an open set V in T_eG which contains 0_e and which satisfies

(1) there is a set M which is open in TG, which is mapped by Exp diffeomorphically onto an open subset of $G \times G$, and which satisfies $M \cap T_eG = V$.

(2) V is mapped diffeomorphically by $\exp_e$ onto an open set W in G which contains e.

Let $S = \bigcup\{TR_g(V) \mid g \in G\}$ and $U = \bigcup\{g \times Wg \mid g \in G\}$. It is easily checked that S and U have the desired properties.

The differential structure on $C^r(X, G)$ is now constructed in the usual way [1]. Let $f \in C^r(X, G)$. A manifold chart about f is constructed as follows. f^*TG, the pull-back of TG under f, is a bundle over X, and $f^*S = \{(x, v) \in f^*TG \mid v \in S\}$ is an open subset of this bundle. Let $N_f = \{g \in C^r(X, G) \mid (f(x), g(x)) \in U \text{ for all } x \in X\}$ and define $a_f\colon N_f \to \Gamma^r(f^*TG)$ by $a_f(g)(x) = (x, \operatorname{Exp}^{-1}(f(x), g(x)))$. a_f maps N_f bijectively onto an open subset of $\Gamma^r(f^*TG)$ and (N_f, a_f) gives a chart at f.

The coordinate chart at e is particularly nice. We use e to denote the identity in G and also to denote the constant map $e\colon X \to G$, $e(x) = e$ for all x. $N_e = \{g \in C^r(X, G) \mid (e, g(x)) \in U \text{ for all } x \in X\} = \{g \in C^r(X, G) \mid g(x) \in W \text{ for all } x \in X\}$. Here W is the set used in the proof of Lemma 1 to construct S and U. Now $e^*TG = X \times T_eG$ so we may identify $\Gamma^r(e^*TG)$ with $C^r(X, T_eG)$. With this identification we see that $a_e\colon N_e \to C^r(X, T_eG)$ is given by $a_e(g)(x) = \exp_e^{-1}(g(x))$.

THEOREM 1. *$C^r(X, G)$ is a Banach Lie group with respect to pointwise multiplication and inversion. If $E\colon C^r(X, T_eG) \to C^r(X, G)$ is the exponential of this Lie group then we have $E(f) = \exp \circ f$ where* $\exp\colon T_eG \to G$ *is the exponential of G. The Lie bracket in $C^r(X, T_eG)$ is the pointwise bracket; $[f, g](x) = [f(x), g(x)]$.*

Proof. Let $g \in C^r(X, G)$. We show that $R_g\colon C^r(X, G) \to C^r(X, G)$ is smooth. Fix $f \in C^r(X, G)$ and consider the chart (N_f, a_f). We will show that $R_g\colon C^r(X, G) \to C^r(X, G)$, $R_g(f) = fg$, is smooth. At fg there is the chart (N_{fg}, a_{fg}) and we first note that $R_g(N_f) = N_{fg}$. This follows from the definition of the coordinate neighborhoods and the property of U which is given in (b) of Lemma 1. To show the smoothness of R_g we need only show smoothness of the composite $a_{fg}R_g a_f^{-1}\colon a_f(N_f) \to \Gamma^r((fg)^*TG)$. Let $\xi \in \Gamma^r(f^*TG)$. Then $\xi(x) = (x, \xi_1(x))$ where $\xi_1\colon x \to TG$ is C^r and $\pi\xi_1 = f$. If $\xi \in a_f(N_f)$ then for each x in X we have $(a_{fg}R_g a_f^{-1}(\xi))(x) = (x, \operatorname{Exp}^{-1}(f(x)g(x), (R_g a_f^{-1}(\xi))(x)))$ and $(R_g a_f^{-1}(\xi))(x) = a_f^{-1}(\xi)(x)g(x) = \exp(\xi_1(x))g(x) = \exp(TR_{g(x)}(\xi_1(x)))$. Now $TR_{g(x)}(\xi_1(x))$ is in $T_{f(x)g(x)}G$ so that $\operatorname{Exp}^{-1}(f(x)g(x), \exp(TR_{g(x)}(\xi_1(x)))) = TR_{g(x)}(\xi_1(x))$. We thus have $(a_{fg}R_g a_f^{-1}(\xi))(x) = (x, TR_{g(x)}(\xi_1(x))$ which shows that $a_{fg}R_g a_f^{-1}$ is a continuous linear map, hence smooth.

In a similar manner we can use the canonical left invariant spray on G to show that left translation in $C^r(X, G)$ is smooth. Here we make use of the fact the Banach manifold structure on $C^r(X, G)$ does not depend upon the choice of spray [1]. Now to prove that multiplication $m: C^r(X, G) \times C^r(X, G) \to C^r(X, G)$ is smooth we need only prove smoothness in a neighborhood of (e, e) where $e: X \to G$ is the constant mapping which is the identity for $C^r(X, G)$. The coordinate neighborhood around e is given by $N_e = \{h \in C^r(X, G) \mid h(X) \subset W\}$. Let W_1 be a neighborhood of e in G such that $W_1^2 \subset W$, and $M = \{h \in C^r(X, G) \mid h(X) \subset W_1\}$, $V_1 = \exp^{-1}(W_1) \subset V$, $M_0 = \{g \in C^r(X, T_eG) \mid g(X) \subset V_1\}$. Then $\alpha_e(M) = M_0$, and $\alpha_e m(\alpha_e^{-1} \times \alpha_e^{-1}): M_0 \times M_0 \to \alpha_e(N_e)$, $M_0 \subset \alpha_e(N_e)$. We have $(\alpha_e m(\alpha_e^{-1} \times \alpha_e^{-1})(h, k))(x) = \mathrm{Exp}^{-1}(e, \alpha_e^{-1}(h)(x)\alpha_e^{-1}(k)(x)) = \exp^{-1}(\exp(h(x))\exp(k(x)))$. Now there is a C^∞-map $\nu: V_1 \times V_1 \to V$ given by $\nu = \exp^{-1}\bar{m}(\exp \times \exp)$ where $\bar{m}$ denotes the group multiplication in G. A basic result of [1] is that the mapping $\Omega_\nu: C^r(X, V_1 \times V_1) \to C^r(X, V)$ which is given by $\Omega_\nu(h) = \nu \cdot h$ is C^∞. Let $\psi: M_0 \times M_0 \to C^r(X, T_eG \times T_eG)$ be defined by $\psi(h, k)(x) = (h(x), k(x))$. Then ψ is continuous and linear so that $\Omega_\nu\psi: M_0 \times M_0 \to C^r(X, T_eG)$ is smooth. But we get that $(\Omega_\nu\psi(h, k))(x) = \exp^{-1}(\exp(h(x))\exp(k(x))) = (\alpha_e m(\alpha_e^{-1} \times \alpha_e^{-1})(h, k))(x)$ so that we have the joint smoothness of multiplication in $C^r(X, G)$.

We could prove directly that inversion is smooth but since $C^r(X, G)$ is a Banach manifold this can be deduced from the implicit function theorem [4].

Now consider $E: C^r(X, T_eG) \to C^r(X, G)$ given by $E(f) = \exp \circ f$. Again, since exp is smooth we get that E is smooth. Let $h \in C^r(X, T_eG)$. Now the mapping $g: R \to C^r(X, G)$ given by $g(t) = E(th)$ satisfies $g(t_1 + t_2)(x) = \exp((t_1 + t_2)h(x)) = \exp(t_1h(x))\exp(t_2h(x)) = g(t_1)(x)g(t_2)(x) = g(t_1)g(t_2)(x)$ for all $x \in X$. Thus g is a one-parameter subgroup of $C^r(X, G)$. To show that E is the exponential map as asserted it is enough to show, $(d/dt)(\alpha_e E(th))|_{t=0} = h$. But for small t, $th \in C^r(X, V)$ so that $\alpha_e(th) = th$ and the result is immediate.

It remains to verify the Lie bracket formula as given in the theorem. We leave this as an exercise for the reader.

The group of self-equivalences of a principal bundle. Let X be as before, G be a compact Lie group, and suppose that $(x, g) \to xg$ is a free differentiable right action of G on X. Then the orbit projection has the structure of a principal G-bundle. Thus X/G has a differentiable structure making $\pi: X \to X/G$ a smooth map and for every $\pi(x) \in X/G$ there is an open set U in $X/G, \pi(x) \in U$, and an equivariant diffeomorphism $\varphi: \pi^{-1}(U) \to U \times G$ such that the following diagram commutes.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "\pi_1"] \\ & U . & \end{tikzcd}$$

Here G acts on the right of $U \times G$ by $(u, g)g' = (u, gg')$. A self-equivalence of this bundle is a G-equivariant diffeomorphism $f: X \to X$ so that $\pi f = \pi$.

LEMMA 2. *Let $f: X \to X$ be a C^r-self-equivalence of π. Then there is a unique C^r-map $\varphi: X \to G$ such that $f(x) = x\varphi(x)$ for each x in X. Conversely, suppose that $\varphi: X \to G$ is a C^r-map and define $f: X \to X$ by $f(x) = x\varphi(x)$. Then f is C^r and it is equivariant if and only if $\varphi(xg) = g^{-1}\varphi(x)g$ for all $x \in X$, $g \in G$.*

Proof. Given $f: X \to X$ a C^r-self-equivalence. The existence and uniqueness of φ are immediate from the assumptions that $\pi f = \pi$ and that the action is free. Smoothness is easily verified using the local triviality of π. Now suppose that we are given a C^r-map $\varphi: X \to G$ and we define $f: X \to X$ by $f(x) = x\varphi(x)$. Assume that f is equivariant. Then $f(xg) = (xg)\varphi(xg) = x(g\varphi(xg))$ and $f(xg) = f(x)g = (x\varphi(x))g = x(\varphi(x)g)$. Since the action is free we get the equation $g\varphi(xg) = \varphi(x)g$ which is the desired result. Conversely, if φ satisfies the stated condition then we have $f(xg) = (xg)\varphi(xg) = (xg)(g^{-1}\varphi(x)g) = (x\varphi(x))g = f(x)g$ so that f is equivariant.

We define $\mathscr{H} = \{f \in C^r(X, G) \mid f(xg) = g^{-1}f(x)g$ for all $x, g\}$, and $\mathscr{H}^* = \{h \in C^r(X, T_eG) \mid h(xg) = ad(g^{-1})(h(x))$ for all $x, g\}$. Here $ad: G \to \mathrm{Aut}\,(T_eG)$ is the adjoint representation of G. Now $\mathscr{H}^*$ is a Lie subalgebra of $C^r(X, T_eG)$ which is the Lie algebra of $C^r(X, G)$.

THEOREM 2. *$\mathscr{H}$ is a closed subgroup of $C^r(X, G)$. In fact $\mathscr{H}$ is an imbedded submanifold, hence a Banach Lie group, with Lie algebra $\mathscr{H}^*$.*

Proof. Clearly $\mathscr{H}$ is a subgroup of $C^r(X, G)$ and since the manifold topology on $C^r(X, G)$ is finer than pointwise convergence it follows that $\mathscr{H}$ is closed. Similarly $\mathscr{H}^*$ is a closed subalgebra of $C^r(X, T_eG)$. Consider the exponential $E: C^r(X, T_eG) \to C^r(X, G)$. We will show that there is a neighborhood M_0^* of 0 in $C^r(X, T_eG)$, and a neighborhood M^* of e in $C^r(X, G)$ such that E maps M_0^* diffeomorphically onto M^* and $E(M_0^* \cap \mathscr{H}^*) = M^* \cap \mathscr{H}$. Using the sets constructed in the proof of Theorem 1, we know that E maps M_0 diffeomorphically onto M. If h is in $M_0 \cap \mathscr{H}^*$ then $E(h)$ is in $M \cap \mathscr{H}$ since $E(h)(xg) = \exp\,(h(xg)) = \exp\,(ad(g^{-1})(h(x)) = g^{-1}\exp\,(h(x))g = g^{-1}E(h)(x)g$.

Now using the continuity of the adjoint representation and the compactness of G there is a neighborhood V^* of 0 in T_eG such that $V^* \subset V_1$ and if $v \in V^*$, $g \in G$ then $ad(g)(v) \in V_1$. Let $M_0^* = \{h \in C^r(X, T_eG) \mid h(X) \subset V^*\}$ and $M^* = E(M_0^*)$. Then E maps M_0^* diffeomorphically onto M^* and $E(M_0^* \cap \mathscr{H}^*) \subset M^* \cap \mathscr{H}$. If, conversely, $E(h) \in M^* \cap \mathscr{H}$, $h \in M_0^*$, then for g in G, x in X we have $E(h)(xg) = \exp(h(xg)) = g^{-1}\exp(h(x))g = \exp(ad(g^{-1})(h(x)))$. Now $h(xg) \in V^* \subset V$, $ad(g^{-1})(h(x)) \subset V$ and since exp is injective on V we can conclude that $h(xg) = ad(g^{-1})(h(x))$. Thus $h \in \mathscr{H}^*$ which completes the proof.

Now $\mathscr{H}$ acts on X by $f * x = xf(x)$. Since the evaluation map $C^r(X, G) \times X \to G$ is smooth and the group action is smooth it follows that $*: \mathscr{H} \times X \to X$ is smooth.

LEMMA 3. *$*$ is an effective action of the Banach Lie group $\mathscr{H}$ on X.*

Proof. $f*(h*x) = (h*x)f(h*x) = (xh(x))f(xh(x)) = x(h(x)f(xh(x))) = x(f(x)h(x))$ where the last equality follows from the assumption that $f \in \mathscr{H}$. We have shown that $f*(h*x) = x((fh)(x)) = (fh)*x$ so that $*$ is a group action. Suppose that $f*x = x$ for all x in X. Then $xf(x) = x$ for all x and since the original action of G on X is free we get that $f(x) = e$ for all x. Thus f is the identity in $C^r(X, G)$ and we have shown that the action of $*$ is effective.

This effective action of $\mathscr{H}$ on X allows us to identify $\mathscr{H}$ with a subgroup of $\mathrm{Diff}^r(X)$. As noted before, this subgroup is precisely the group of self-equivalences of the bundle $\pi: X \to X/G$. We have shown:

THEOREM 3. *Let the compact Lie group G act freely and smoothly on a compact C^∞-manifold X. Let $E^r(X, G)$ be the group of C^r-self-equivalences as defined above. Then $E^r(X, G)$ has the structure of a Banach Lie group.*

Concluding remarks. In addition to the assumptions made above suppose that compact Lie group G is abelian. Then the condition $\varphi(xg) = g^{-1}\varphi(x)g$ simplifies to $\varphi(xg) = \varphi(x)$ so that we get $\mathscr{H} = \{f \mid f \in C^r(X, G)$ and f is constant on the G-orbits$\}$. Thus $\mathscr{H}$ is isomorphic to $C^r(X/G, G)$ and the Lie algebra of $\mathscr{H}$ can be identified with $C^r(X/G, T_eG)$. An example of this is given by the standard action of S^1 on S^{2n+1}. The action is obtained by representing S^{2n+1} as $\{(z_0, z_1, \cdots, z_n) \in C^{n+1} \mid |z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 1\}$ and defining $(z_0, z_1, \cdots, z_n)z = (z_0z, z_1z, \cdots, z_nz)$. This action satisfies all of our hypothesis and the orbit manifold is CP^n, complex projective n-space. Thus $C^r(CP^n, S^1)$ acts on S^{2n+1} and the resulting subgroup of $\mathrm{Diff}^r(S^{2n+1})$ consists of the equivariant diffeomorphisms of S^{2n+1} which cover the identity map on CP^n.

If the Lie group P acts differentiably on the manifold X and $p \in P$ is in the image of the exponential map then we know that the diffeomorphism $f_p: X \to X$ by $f_p(x) = xp$ is imbeddable in a smooth flow. Since we have the Banach Lie group $\mathscr{H}$ acting on X we can say that any self-equivalence which is of the form $f(x) = x\varphi(x)$, $\varphi \in \mathrm{image}(E)$, is imbeddable in a flow. More specifically we have

THEOREM 4. *Let G be a compact, abelian, connected Lie group (i.e., a torus). Let X be a compact, connected, simply connected, C^∞-manifold and suppose that G acts freely and differentiably on X. Then every C^r-self-equivalence of the action is imbeddable in a C^r-flow.*

Proof. $C^r(X/G, G)$ acts on X by $\varphi * x = x\varphi(\pi(x))$ where $\pi: X \to X/G$ is the projection. Let $f: X \to X$ be a C^r-self-equivalence of the action of G. Choose φ such that $f(x) = \varphi * x$ for all x. It is enough to show that there is an h in $C^r(X/G, T_eG)$ such that $E(h) = \varphi$. But this is just the lifting problem for φ. Since $\exp: T_eG \to G$ is a covering and X is 1-connected it follows that given $\varphi \in C^r(X/G, G)$ there is h in $C^r(X/G, T_eG)$ so that $\exp h = \varphi$; i.e., $E(h) = \varphi$.

We now comment on our assumption that G acts on the right of X. If G acts on the left we still get a subgroup of $C^r(X, G)$ acting as a group of diffeomorphisms of X. The appropriate subgroup is $H_1 = \{f \in C^r(X, G) \mid f(gx) = f(x) \text{ for all } g \in G, x \in X\}$. However, the diffeomorphism $x \to f*x$, $f \in H_1$, is G-equivariant if and only if $f(X)$ is contained in the center of G. Thus H_1 does not act as group of equivariant diffeomorphisms and, conversely, not every equivariant diffeomorphism which covers the identity of X/G is representable as $x \to f*x$ for some $f \in H_1$. Letting G act on the right and $C^r(X, G)$ act on the left obviates these difficulties.

Finally let us note a comparison between our results and those in N. Kopell's paper, "Commuting Diffeomorphisms". Assume G is a compact, connected Lie group acting smoothly and freely on the compact connected manifold X with $\dim(G) < \dim(X)$. Suppose a diffeomorphism f is imbedded in this action, that is, there exists $g \in G$ such that $f(x) = xg$. Then there is a whole Banach manifold of diffeomorphisms which commute with f. To see this choose a torus $T \subset G$ with $g \in T$. Then for $h \in C^r(X/T, T)$ the diffeomorphism $x \to h*x$ commutes with f. (Of course we need $\dim(T) > 0$ in order that $C^r(X/T, T)$ be infinite dimensional.) Since $\dim(X/T) > 0$ we can easily define $h \in C^r(X/T, T)$ so that $h(V) = \{e\}$ for a nonvoid open $V \subset X/T$ but h is not globally constant. Then $x \rightarrow h * x$ is the identity on an open set but not globally. In contrast if f is a special M. -S. diffeomorphism and $h: X \rightarrow X$ is a diffeomorphism which commutes with f then if $h | V = id.$ for some open set $V \neq \emptyset$ then Kopell shows that h is the identity diffeomorphism. It follows for example that no special M. -S. diffeomorphism is imbeddable in a group action of the type we have considered. The authors wish to thank the referee for directing our attention to Kopell's paper.

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