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A COMPARISON OF c-DENSITY AND k-DENSITY

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In this paper a comparison is made between c-density and k-density in the general setting of Freedman density spaces in additive number theory. The comparison is motivated by the following question of Freedman: Does there exist a density space and a set such that the c-density of that set is positive and the k-density is zero? The answer is yes. More generally, there exists a density space such that for any two real numbers ρ_1 and ρ_2 with $0 \leqq \rho_1 \leqq \rho_2 < 1$, a set can be constructed such that the k-density of the set is ρ_1 while the c-density is ρ_2.

Let S be any nonempty subset of an abelian group G with binary operation $+$ and identity element 0. We define a relation $<$ on S by saying $y < x$ whenever $x - y \in S\backslash\{0\}$. The set S is called an *s-set* whenever the following conditions hold:

(s.1) $0 \in S$

(s.2) $S\backslash\{0\} \neq \phi$

(s.3) $S\backslash\{0\}$ is closed with respect to $+$.

(s.4) $L(x) = \{y \mid y \in S, y < x \text{ or } y = x\}$ is finite for each $x \in S\backslash\{0\}$.

Corresponding to each $x \in S\backslash\{0\}$, let $H(x)$ be a subset of S satisfying the following three conditions:

(c.1) $\{0, x\} \subseteqq H(x)$

(c.2) $H(x) \subseteqq L(x)$

(c.3) if $y \in H(x)\backslash\{0\}$, then $H(y) \subseteqq H(x)$.

Let $\mathscr{F}(H) = \{F \mid F \subseteqq S, F \text{ finite}, \ 0 \in F, F\backslash\{0\} \neq \phi, x \in F\backslash\{0\} \text{ implies } H(x) \subseteqq F\}$.

Then $\mathscr{F}(H)$ is said to be a *fundamental family on S*.

Freedman [1] calls the ordered pair $(S, \mathscr{F}(H))$ a *density space* whenever S is an *s-set* and $\mathscr{F}(H)$ is a fundamental family on S.

For any two sets $X, D \subseteqq S$ with D finite, let $X(D)$ denote the number of nonzero elements in $X \cap D$.

DEFINITION. The *k-density* of a set $A \subseteqq S$ with respect to $\mathscr{F}(H)$, written α_k, is

$$\alpha_k = \operatorname{glb}\left\{\frac{A(F)}{S(F)} \,\middle|\, F \in \mathscr{F}(H)\right\} .$$

DEFINITION. The *c-density* of a set $A \subseteqq S$ with respect to $\mathscr{F}(H)$, written α_c, is

$$\alpha_c = \text{glb}\left\{\frac{A(H(x))}{S(H(x))}\,\middle|\, x \in S\backslash\{0\}\right\}.$$

We begin our comparison of k-density and c-density by stating without proof the following two results of Freedman.

THEOREM 1. *Let* $(S, \mathscr{F}(H))$ *be a density space. For any set* $A \subseteqq S$ *we have* $0 \leqq \alpha_k \leqq \alpha_c \leqq 1$.

THEOREM 2. *Let* $(S, \mathscr{F}(H))$ *be a density space and* A *be a subset of* S *with* $0 \in A$. *The following three conditions are equivalent*: (i) $\alpha_k = 1$, (ii) $\alpha_c = 1$, *and* (iii) $A = S$.

For the remainder of this paper we suppose that $\alpha_c < 1$. Freedman posed the following question [1]: Does there exist a density space $(S, \mathscr{F}(H))$ and a subset A of S such that $\alpha_c > 0$ and $\alpha_k = 0$? The answer is yes.

EXAMPLE 1. Let I be the set of nonnegative integers with the usual addition and let d be any positive integer. Let $H(x)$ be defined by

$$H(x) = \begin{cases}\{0, 1, 2, \cdots, d\} \cup \{x\} & \text{if} \quad x \geqq d + 1\,, \\ \{0, x\} & \text{otherwise}\,,\end{cases}$$

where $x \in I\backslash\{0\}$. Then $(I, \mathscr{F}(H))$ is a density space. Let $A = \{0, 1, 2, \cdots, d\}$. Then $\alpha_k = 0$, but $\alpha_c = d/(d + 1) > 0$.

Example 1 shows that there are density spaces for which $\alpha_k = 0$ and α_c is arbitrarily close to (but not equal) 1. Example 1 also answers a second question of Freedman [1]: Does $0 \in A$ and $\alpha_c > 0$ imply that A is a basis for I? The answer is, of course, no. The set A has finite cardinality and hence cannot be a basis for I.

EXAMPLE 2. Let I^∞ denote the set of all zero terminating sequences of nonnegative integers. Then $(I^\infty, \mathscr{F}(L))$ is a density space. For any positive integer $N \geqq 2$, let

$$I^\infty\backslash A = \{(x_1, x_2, \cdots) \mid x_i \leqq N \text{ for all } i \text{ and } x_i = N \text{ for exactly one } i\}\,.$$

Then $\alpha_k = 0$ and $\alpha_c = (N - 1)/N$ which again answers Freedman's first question. This density space is less artificial than the space in Example 1. However, it does not serve as an answer to Freedman's second question.

In the final theorem of this paper we show that it is possible to create a density space for which there exist sets having any k-density and c-density we want as long as Theorems 1 and 2 are not violated.

THEOREM 3. *There exists a density space* $(I, \mathscr{F}(H))$ *such that if* $0 \leqq \rho_1 \leqq \rho_2 < 1$, *then there is a set* $A \subseteqq I$ *such that* $\alpha_k = \rho_1$ *and* $\alpha_c = \rho_2$.

Proof. Let $\{(d_i, b_i)\}$ be a sequence of ordered pairs of positive integers satisfying $1 < d_i \leqq b_i$ where all possible such pairs occur and occur infinitely often.

For all $x \in I \backslash \{0\}$, define

$$(1)\quad H(x) = \begin{cases} \{0, 1, 2, \cdots, x\} & \text{if } 1 \leqq x \leqq d_1 - 1\,, \\ \{0, 1, 2, \cdots, d_1 - 1, x\} & \text{if } d_1 \leqq x \leqq b_1\,, \\ \{0, b_1 + 1, b_1 + 2, \cdots, x\} & \text{if } b_1 + 1 \leqq x \leqq b_1 + d_2 - 1\,, \\ \{0, b_1 + 1, b_1 + 2, \cdots, b_1 + d_2 - 1, x\} & \text{if } b_1 + d_2 \leqq x \leqq b_1 + b_2\,, \\ \vdots & \\ \{0, \sum_{i=1}^{j} b_i + 1, \sum_{i=1}^{j} b_i + 2, \cdots, x\} & \text{if } \sum_{i=1}^{j} b_i + 1 \leqq x \leqq \sum_{i=1}^{j} b_i + d_{j+1} - 1\,, \\ \{0, \sum_{i=1}^{j} b_i + 1, \sum_{i=1}^{j} b_i + 2, \cdots, \sum_{i=1}^{j} b_i + d_{j+1} - 1, x\} & \text{if } \sum_{i=1}^{j} b_i + d_{j+1} \leqq x \leqq \sum_{i=1}^{j+1} b_i\,, \\ \vdots & \end{cases} .$$

The space $(I, \mathscr{F}(H))$ is a density space.

Let ρ_1 and ρ_2 satisfy the hypothesis of the theorem. Let $\{u_i\}$ and $\{v_i\}$ be strictly decreasing sequences of positive rational numbers less than 1 such that

$$\rho_1 = \text{glb } \{u_i \mid i = 1, 2, \cdots\},$$
$$\rho_2 = \text{glb } \{v_i \mid i = 1, 2, \cdots\}, \text{ and } u_i \leqq v_i \text{ for each } i\,.$$

Since u_i and v_i are positive rationals, there exist positive integers a_i, b'_i, and d'_i such that $u_i = a_i/b'_i$ and $v_i = a_i/d'_i$. Since $0 < u_i \leqq v_i < 1$, we have $1 \leqq a_i < d'_i \leqq b'_i$. Furthermore, there is a strictly increasing function $r(s)$ such that $b'_s = b_{r(s)}$ and $d'_s = d_{r(s)}$ for $s = 1, 2, \cdots$. Let

$$A = \{0\} \cup \left\{x \,\middle|\, x \in I, \sum_{i=1}^{j} b_i + 1 \leqq x \leqq \sum_{i=1}^{j+1} b_i, \text{ where } j \geqq 0 \text{ and } j + 1 \neq r(s) \text{ for all } s\right\}$$
$$\cup \bigcup_{s=1}^{\infty} \left\{x \,\middle|\, x \in I, \sum_{i=1}^{r(s)-1} b_i + 1 \leqq x \leqq \sum_{i=1}^{r(s)-1} b_i + a_s\right\}.$$

We now show that $\alpha_c = \rho_2$ and $\alpha_k = \rho_1$.

For each positive integer s, we have

$$\frac{A\left(H\left(\sum_{i=1}^{r(s)-1} b_i + d_{r(s)}\right)\right)}{I\left(H\left(\sum_{i=1}^{r(s)-1} b_i + d_{r(s)}\right)\right)} = \frac{a_s}{d_{r(s)}} = v_s .$$

Since $\rho_2 = \text{glb}\,\{v_s \mid s = 1, 2, \cdots\}$, we have $\alpha_c \leqq \rho_2$. Also for any positive integer m there is an integer $j \geqq 0$ such that $\sum_{i=1}^{j} b_i + 1 \leqq m \leqq \sum_{i=1}^{j+1} b_i$. If $j + 1 = r(s)$ for some s, then

$$\frac{A(H(m))}{I(H(m))} \geqq \frac{A\left(H\left(\sum_{i=1}^{r(s)-1} b_i + d_{r(s)}\right)\right)}{I\left(H\left(\sum_{i=1}^{r(s)-1} b_i + d_{r(s)}\right)\right)} = \frac{a_s}{d_{r(s)}} = v_s .$$

Otherwise $A(H(m)) = I(H(m))$.
Therefore,

$$\alpha_c = \text{glb}\left\{\frac{A(H(m))}{I(H(m))}\,\middle|\, m = 1, 2, \cdots\right\} \geqq \text{glb}\,\{v_s \mid s = 1, 2, \cdots\} = \rho_2 .$$

Hence we have $\alpha_c = \rho_2$.

It is more difficult to show that $\alpha_k = \rho_1$. For each integer $j \geqq 0$, define that set F_j by

$$F_j = \left\{0, \sum_{i=1}^{j} b_i + 1, \sum_{i=1}^{j} b_i + 2, \cdots, \sum_{i=1}^{j+1} b_i\right\} .$$

By formula (1) we have

$$F_j = \bigcup_{m=m_1(j)}^{m_2(j)} H(m) ,$$

where $m_1(j) = \sum_{i=1}^{j} b_i + 1$ and $m_2(j) = \sum_{i=1}^{j+1} b_i$, and hence $F_j \in \mathscr{F}(H)$. If $j + 1 = r(s)$ for some s, then

$$\frac{A(F_j)}{I(F_j)} = \frac{a_s}{b_{j+1}} = \frac{a_s}{b_{r(s)}} = u_s . \tag{2}$$

Since $\rho_1 = \text{glb}\,\{u_s \mid s = 1, 2, \cdots\}$ we have $\alpha_k \leqq \rho_1$. Now consider any $F \in \mathscr{F}(H)$. For each integer $j \geqq 0$, let $G_j = F \cap F_j$. Now since $F \in \mathscr{F}(H)$ and $F_j \in \mathscr{F}(H)$, we have $G_j \in \mathscr{F}(H) \cup \{\{0\}\}$. Now $i \neq j$ implies $F_i \cap F_j = \{0\}$ and hence $G_i \cap G_j = \{0\}$. Also F is finite. Hence there is a finite integer

$$J = \max\,\{j \mid G_j \backslash \{0\} \neq \phi\} .$$

Now if $G_j\backslash\{0\} \neq \phi$, then $G_j \in \mathscr{F}(H)$. If $G_j\backslash A \neq \phi$, then $j + 1 = r(s)$ for some s and

$$\left\{\sum_{i=1}^{r(s)-1} b_i + 1, \sum_{i=1}^{r(s)-1} b_i + 2, \cdots, \sum_{i=1}^{r(s)-1} b_i + a_s\right\} \subseteqq G_j \subseteqq F_j ,$$

and so

$$\frac{A(G_j)}{I(G_j)} \geqq \frac{A(F_j)}{I(F_j)} . \tag{3}$$

If $G_j \backslash A = \phi$, then $A(G_j) = I(G_j)$ and inequality (3) still holds. Therefore, by statement (3) and since $G_i \cap G_j = \{0\}$ for $i \neq j$, we have

$$\frac{A(F)}{I(F)} = \frac{A(\bigcup_{j=1}^{J} G_j)}{I(\bigcup_{j=1}^{J} G_j)} = \frac{\sum_{j=1}^{J} A(G_j)}{\sum_{j=1}^{J} I(G_j)} \geqq \frac{A(G_i)}{I(G_i)} \geqq \frac{A(F_i)}{I(F_i)}$$

for some i $(1 \leqq i \leqq J)$. If $i + 1 = r(s)$ for some s, then by statement (2), we have

$$\frac{A(F_i)}{I(F_i)} = u_s .$$

If $i + 1 \neq r(s)$ for all s, then $A(F_i) = I(F_i)$. In either case, $A(F)/I(F) \geqq u_s$ for some s. Therefore,

$$\alpha_k = \text{glb}\left\{\frac{A(F)}{I(F)} \,\middle|\, F \in \mathscr{F}(H)\right\} \geqq \text{glb}\,\{u_s \mid s = 1, 2, \cdots\} = \rho_1 .$$

Hence we have $\alpha_k = \rho_1$.

REFERENCE

1. Allen R. Freedman, *Some generalizations in additive number theory*, Reine Angew. Math., **235**: (1969), 1-19.

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