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ON QUASI-COMPLEMENTS

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Results of H. P. Rosenthal and the author on w^*-basic sequences are combined with known techniques and applied to quasi-complementation problems in Banach spaces.

1. Introduction. Recall that (closed, linear) subspaces Y, Z of the Banach space X are *quasi-complements* (respectively *complements*) provided $Y \cap Z = \{0\}$ and $Y + Z$ is dense in X (respectively, $Y + Z = X$).

Suppose that Y, Z are quasi-complements, but not complements, for the separable space X. We show that there exist closed subspaces Y_1 and Y_2 of X with $Y_1 \subset Y \subset Y_2$, $\dim Y/Y_1 = \infty = \dim Y_2/Y$, such that Y_1, Z are quasi-complements and Y_2, Z are quasi-complements. This generalizes a theorem of James [5], who proved the existence of Y_1 for the case of general separable X and the existence of Y_2 for separable, reflexive X. Our proof uses James' method (and w^*-basic sequences), but seems simpler than James' construction. Also, our argument provides information for some nonseparable spaces.

We show also the following.

THEOREM 2. *Suppose Y is a subspace of X and Y^* is weak*-separable. If X/Y has a separable, infinite dimensional quotient space, then Y is quasi-complemented in X.*

Theorem 2 was discovered by J. Lindenstrauss and H. P. Rosenthal [unpublished], both of whom apparently use an idea from [3]. Our argument uses w^*-basic sequences and Rosenthal's proof of Theorem 2 in the case where X/Y has a reflexive, infinite dimensional quotient (cf. [12]).

The final result of the paper is that every subspace of a separable conjugate space admits a weak*-closed quasi-complement which is spanned by a boundedly complete w^*-basic sequence.

The notation and terminology agree with [6]. In particular, subspaces and quotients are assumed to be infinite dimensional and complete. For $A \subset X$, $A^\perp$ is the annihilator of A in X^*, while for $B \subset X^*$, $B^\top$ is the annihilator of B in X and $\widetilde{B}$ is the weak*-closure of B in X^*.

II. THE THEOREMS. We recall the definition of w^*-basic sequence

[6]: A sequence $(y_n) \subset X^*$ is called w^*-basic provided that there exists $(x_n) \subset X$ biorthogonal to (y_n) and, for each y in the weak*-closure $\widetilde{[y_n]}$ of the closed linear span $[y_n]$ of (y_n), $y = w^*\text{-}\lim_n \sum_{i=1}^n y(x_i)y_i$.

In [6] it was proved that, when X is separable, if $(y_n) \subset X^*$, $y_n \xrightarrow{w^*} 0$, but $\liminf \|y_n\| > 0$, then (y_n) contains a w^*-basic subsequence. Let us note that the same result is true when X admits a weakly compact fundamental set. Indeed, in this case there exists by [1] a norm one projection P on X with PX separable and $(y_n) \subset P^*X^*$. P^*X^* is isometric to $(PX)^*$ and the relative weak* topology on P^*X^* from X^* agrees with the weak* topology on P^*X^* considered as the conjugate of PX. Therefore, the above mentioned result from [6] applies to show that (y_n) has a w^*-basic subsequence.

First we prove the extension of James' theorem:

THEOREM 1. *Suppose that Y, Z are quasi-complements, but not complements, for X.*

(a) *If Y has a weakly compact fundamental subset, then there exists a subspace Y_1 of Y with dim $Y/Y_1 = \infty$ and Y_1, Z are quasi-complements.*

(b) *If X/Y has a weakly compact fundamental subset (in particular, if X does), then there exists a subspace Y_2 of X with $Y_2 \supset Y$, dim $Y_2/Y = \infty$, and Y_2, Z are quasi-complements.*

Proof. Pick positive numbers (a_n) less than 1 so that $a_1 + a_1a_2 + a_1a_2a_3 + \cdots < \infty$. Let p be a bijection of $N \times N$ onto N (N is the set of natural numbers) so that for each n and j, $p(n, j) \geqq j$.

To prove (a), we use the fact that $Y + Z$ is not closed to select unit vectors (y_n) in Y with $d(y_n, Z) \equiv \inf\{\|y_n + z\|: z \in Z\} \to 0$. Since $Y \cap Z = \{0\}$, 0 is the only possible weak cluster point of (y_n), and hence either $y_n \xrightarrow{w} 0$ or the weak closure of (y_n) is not weakly compact. Thus, by either [2] or [11], (y_n) has a basic subsequence, which we also denote by (y_n).

Let (y_n^*) be a bounded sequence of functionals in Y^* biorthogonal to (y_n). Since Y admits a weakly compact fundamental set, the unit ball of Y^* is weak* sequentially compact (cf. [1]), so we may assume, by passing to a subsequence, that $y_n^* \xrightarrow{w^*} y^*$. $(y_n^* - y^*)$ converges w^* to 0 and is bounded away from zero, so it has a w^*-basic subsequence. Thus by passing to a subsequence of $(y_n, y_n^* - y^*)$, we have that there exists a biorthogonal sequence (x_n, x_n^*) in Y with $\|x_n\| = 1$, $(\|x_n^*\|)$ bounded, $d(x_n, Z) \leqq n^{-1}a_1a_2a_3 \cdots a_n$, (x_n) is basic, and (x_n^*) is w^*-basic.

Let $Y_1 = [(x_i^*)^\tau \cup (a_i x_{p(n,i)} - x_{p(n,i+1)})_{i,n=1}^\infty]$. (The annihilator of (x_i^*) is of course taken in Y.) We claim that $Y_1 \cap [x_{p(n,1)}] = \{0\}$. To see this, first note that $w_n^* = x_{p(n,1)}^* + a_1 x_{p(n,2)}^* + a_1 a_2 x_{p(n,3)}^* + \cdots$ is absolutely convergent, $w_n^*(x_{p(n,1)}) = 1$, while $w_n^*(x_{p(m,1)}) = 0$ when $n \neq m$. By construction, $Y_1 \subset (w_n^*)^\tau$, and $(w_n^*)^\tau \cap [(x_{p(n,1)})] = \{0\}$ because $(x_{p(n,1)})$ is basic under some ordering and $(x_{p(n,1)}, w_n^*)$ is biorthogonal. Hence, $Y_1 \cap [x_{p(n,1)}] = \{0\}$, whence dim $Y/Y_1 = \infty$.

We complete the proof by showing that $Y_1 + Z$ is dense in X. Now $(x_n^*)^\tau + [x_n]$ is dense in Y because (x_n^*) is w^*-basic, so we need show only that $(x_{p(n,1)}) \subset \overline{Y_1 + Z}$. But

$$\begin{aligned} x_{p(n,1)} &- a_1^{-1}(a_1 x_{p(n,1)} - x_{p(n,2)}) - (a_1 a_2)^{-1}(a_2 x_{p(n,2)} - x_{p(n,3)}) \\ &- \cdots - (a_1 a_2 \cdots a_j)^{-1}(a_j x_{p(n,j)} - x_{p(n,j+1)}) \\ &= (a_1 a_2 \cdots a_j)^{-1} x_{p(n,j+1)} \,. \end{aligned}$$

Since $d(x_{p(n,j+1)}, Z) \leqq p(n, j+1)^{-1} a_1 a_2 \cdots a_{p(n,j+1)} \leqq (j+1)^{-1} a_1 a_2 \cdots a_j$, it follows that $d(x_{p(n,1)}, Y_1 + Z) \leqq (j+1)^{-1}$. Since j is arbitrary, this completes the proof of (a).

The proof of (b) is very similar to the above: Since Y, Z are not complements, $Y^\perp + Z^\perp$ is not closed in X^*. Thus there exists a sequence (y_n^*) of unit vectors in $Y^\perp$ with $d(y_n^*, Z^\perp) \to 0$. Of necessity, $y_n^* \xrightarrow{w^*} 0$. Now $Y^\perp = (X/Y)^*$ in the canonical way, so (y_n^*) has a w^*-basic subsequence. Hence for an appropriate subsequence (x_n^*) of (y_n^*), we have that there exists a biorthogonal sequence (x_n, x_n^*) in X with $(\|x_n\|)$ bounded, $\|x_n^*\| = 1$, $(x_n^*) \subset Y^\perp$, (x_n^*) w^*-basic, and $d(x_n^*, Z^\perp) \leqq n^{-1} a_1 a_2 \cdots a_n$.

We define $Y_2^\perp$ to be the weak*-closure of $[Y^\perp \cap (x_n)^\perp \cup (a_i x_{p(n,i)}^* - x_{p(n,i+1)}^*)_{n,i=1}^\infty]$. Since $Y_2^\perp \subset Y^\perp$, we have $Y_2 \supset Y$. To show that dim $Y_2/Y = \infty$, it clearly suffices to prove that $Y_2^\perp \cap \widetilde{[x_{p(n,1)}^*]} = \{0\}$. But note that $y_n \equiv x_{p(n,1)} + a_1 a_2 x_{p(n,2)} + a_1 a_2 a_3 x_{p(n,3)} + \cdots$ is absolutely convergent, $x_{p(n,1)}^*(y_n) = 1$, while $x_{p(m,1)}^*(y_n) = 0$ when $m \neq n$. By construction, $(y_n)^\perp \supset (a_i x_{p(n,i)}^* - x_{p(n,i+1)}^*)_{n,i=1}^\infty$ and $(y_n)^\perp \supset (x_n)^\perp$, hence $(y_n)^\perp \supset Y_2^\perp$. But $(y_n)^\perp \cap \widetilde{[x_{p(n,1)}^*]} = \{0\}$ because $(x_{p(n,1)}^*)$ is w^*-basic in some ordering and $(y_n, x_{p(n,1)}^*)$ is biorthogonal.

Since $Y_2^\perp \cap Z^\perp \subset Y^\perp \cap Z^\perp = \{0\}$, we have that $Y_2 + Z$ is dense in X. To show that $Y_2 \cap Z = \{0\}$, we prove the equivalent fact that $Y_2^\perp + Z^\perp$ is w^* dense in X^*. But $Y^\perp \cap (x_n)^\perp + [x_n^*]$ is w^* dense in $Y^\perp$ because (x_n^*) is w^*-basic, so we need only show that each $x_{p(n,1)}^*$ is in the closure of $Y_2^\perp + Z$. To see that this last statement is true, write

$$x^*_{p(n,1)} - a_1^{-1}[a_1 x^*_{p(n,1)} - x^*_{p(n,2)}] - (a_1 a_2)^{-1}[a_2 x^*_{p(n,2)} - x^*_{p(n,3)}] - \cdots$$
$$- (a_1 a_2 \cdots a_j)^{-1}[a_j x^*_{p(n,j)} - x^*_{p(n,j+1)}]$$
$$= (a_1 a_2 \cdots a_j)^{-1} x^*_{p(n,j+1)} .$$

Since $d(x^*_{p(n,j+1)}, Z) \leqq p(n, j+1)^{-1} a_1 a_2 \cdots a_{p(n,j+1)} \leqq (j+1)^{-1} a_1 \cdots a_j$, we have $d(x^*_{p(n,1)}, Y_2^\perp + Z) \leqq (j+1)^{-1}$ for arbitrary j.

Next we prove the result of Lindenstrauss and Rosenthal.

Proof of Theorem 2. Since X/Y has a separable quotient, there exists a biorthogonal sequence (x_n, x_n^*) in X with $(x_n^*) \subset Y^\perp$, (x_n^*) w^*-basic, and normalized so that $\|x_n\| = 1$. Since Y^* is w^*-separable, a biorthogonalization argument (cf., e.g., [8] or [7]) shows that there exists a biorthogonal sequence (y_n, y_n^*) for Y with $(y_n^*) \subset X^*$, $Y \cap (y_n^*)^\tau = \{0\}$, and normalized so that $\|y_n^*\| = 1$.

Define $T: X \to X$ by $Tx = \sum_{n=1}^\infty 2^{-n-1} y_n^*(x) x_n$. Then $\|T\| \leqq 1/2$, so $I + T$ is an isomorphism. Hence $(I + T)^*$ is a weak*-isomorphism on X^*, whence $(x_n^* + T^* x_n^*)$ is a w^*-basic sequence w^*-equivalent to (x_n^*).

Computing $T^* x_n^*$, we have $T^* x_n^*(x) = x_n^* Tx = x_n^* \sum_{m=1}^\infty 2^{-m-1} y_m^*(x) x_m = 2^{-n-1} y_n^*(x)$; i.e., $T^* x_n^* = 2^{-n-1} y_n^*$.

We claim that $(x_n^* + 2^{-n-1} y_n^*)^\tau$ is a quasi-complement to Y. First we show that $Y^\perp \cap \widetilde{[x_n^* + 2^{-n-1} y_n^*]} = \{0\}$ (so that $Y + (x_n^* + 2^{-n-1} y_n^*)^\tau$ is dense). But if $x^* \in \widetilde{[x_n^* + 2^{-n-1} y_n^*]}$, then, since $(x_n^* + 2^{-n-1} y_n^*)$ is w^*-equivalent to (x_n^*), we can write $x^* = w^*\text{-}\lim_{n\to\infty} \sum_{i=1}^n \alpha_i x_i^* + \sum_{i=1}^\infty 2^{-i-1} \alpha_i y_i^*$ for some sequence (α_i) of scalars. Thus for each n, $x^*(y_n) = 2^{-n-1}\alpha_n$, hence, since $x^* \in Y^\perp$, $\alpha_n = 0$.

We complete the proof by showing that $Y \cap (x_n^* + 2^{-n-1} y_n^*)^\tau = \{0\}$. For suppose y is in this intersection. Since $y \in Y$, $x_n^*(y) = 0$ for each n. Hence $y_n^*(y) = 0$ for each n, whence $y \in (y_n^*)^\tau \cap Y = \{0\}$.

THEOREM 3. *Suppose X^* is separable and Y is a subspace of X^* with dim $X^*/Y = \infty$. Then there exists a weak*-closed subspace Z of X^* with Y, Z quasi-complements and $Z = [z_n]$ for some boundedly complete, w^*-basic sequence (z_n).*

Proof. Mackey [8] showed that Y has a quasi-complement, say, W. Let (w_n, w_n^*) be a biorthogonal sequence in W with $\|w_n\| = 1$ and $[w_n] = W$ (cf. [9]). By Theorem III. 2 of [6], there exists a biorthogonal sequence (x_n, x_n^*) in X with $(x_n^*) \subset Y$, (x_n^*) boundedly complete and w^*-basic, normalized so that $||x_n|| = 1$.

Define $T: X \to X$ by $Tx = \sum_{n=1}^{\infty} 2^{-n-1} w_n(x) x_n$. Then $||T|| \leqq 1/2$, so $I + T$ is an isomorphism and hence $(I + T)^*$ is a weak*-isomorphism. One checks that $T^* x_n^* = 2^{-n-1} w_n$, so that $(x_n^* + 2^{-n-1} w_n)$ is a w^*-basic sequence w^*-equivalent to (x_n^*). Letting $Z = [x_n^* + 2^{-n-1} w_n]$, we have by Proposition 1 of [6] that Z is weak*-closed.

Certainly $Z + Y \supset (w_n)$, so $Z + Y \supset Y + W$ and thus is dense. Suppose that $z \in Z \cap Y$. Then $z = \sum_{n=1}^{\infty} \alpha_n (x_n^* + 2^{-n-1} w_n)$ for some scalars (α_n) because $(x_n^* + 2^{-n-1} w_n)$ is basic. Hence also $\sum_{n=1}^{\infty} \alpha_n x_n^*$ converges, whence $z - \sum_{n=1}^{\infty} \alpha_n x_n^* = \sum_{n=1}^{\infty} \alpha_n 2^{-n-1} w_n$ is again in Y. Certainly $\sum_{n=1}^{\infty} \alpha_n 2^{-n-1} w_n$ is also in W so that $\sum_{n=1}^{\infty} \alpha_n 2^{-n-1} w_n = 0$. Thus $\alpha_n 2^{-n-1} = w_n^* (\sum_{m=1}^{\infty} \alpha_m 2^{-m-1} w_m) = 0$, so that $z = 0$.

REMARK. Separability of X^* in Theorem 3 is essential to get that Z is weak*-closed. Indeed, regard $m = l_1^*$. Rosenthal [12] showed that c_0 is quasi-complemented in m. However, if Z is a quasi-complement for c_0 in m, then Z cannot be weak*-closed. For if Z were w^*-closed, then m/Z would be isomorphic to $(Z^\tau)^*$. But m/Z is separable, hence reflexive (cf. [4]). Thus Z^τ would be a reflexive subspace of l_1, a contradiction (cf., e.g., [10]).

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