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THE TYPE OF SOME C^* AND W^*-ALGEBRAS ASSOCIATED WITH TRANSFORMATION GROUPS

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Let (G, Z) be a second countable locally compact topological transformation group, $\mathscr{U}(G, Z)$ the associated C^*-algebra and L a certain naturally constructed representation of $\mathscr{U}(G, Z)$ on $L^2(G \times Z, dg \times d\alpha)$, dg being left Haar measure on G and α a quasi-invariant ergodic probability measure on Z. Representations of $\mathscr{U}(G, Z)$ constructed from positive-definite measures on $G \times Z$ are used to prove that $\mathscr{U}(G, Z)$ is type I if and only if all the isotropy subgroups are type I and Z/G is T_0, and, under the assumption of a common central isotropy subgroup, that L has no type I component if α is nontransitive. By means of quasi-unitary algebras, necessary and sufficient conditions are derived for L to be semi-finite under the weaker assumption of a common type I unimodular isotropy subgroup.

After establishing notation and discussing preliminary material in §2, we prove in §3 that $\mathscr{U}(G, Z)$ is type I if and only if Z/G is T_0 and all isotropy subgroups are type I. This result, proven by Glimm [9, Theorem 2.2] for the special case in which isotropy subgroups can be chosen "continuously", is not surprising in light of Mackey's Imprimitivity Theorem and the correspondence between representations of $\mathscr{U}(G, Z)$ and systems of imprimitivity based on (G, Z) (see §2). Our general proof, based on the fact that isotropy subgroups can always be chosen "measurably" [1, Proposition 2.3], follows by construction of a direct integral of certain representations which, by being defined in terms of positive-definite measures, are easily specified and shown to form an integrable family.

In §§4 and 5 we consider the type of a W^*-algebra $\mathscr{A}$ constructed via an ergodic quasi-invariant probability measure α on Z (see §4 for the construction). This algebra was studied by Murray and von Neumann in [14], [15], and [16] for the case of G discrete (see also [4, pp. 127–137]), by Dixmier in [3, §§10–12] for the case of G acting freely on Z and by Kallman in [10] for the case in which α is transitive. In §4 we first show that $\mathscr{A}$ is the von Neumann algebra generated by the representation of $\mathscr{U}(G, Z)$ determined by the positive-definite measure $\delta_e \times d\alpha$ on $G \times Z$. Then assuming that almost all $(d\alpha)$ points in Z have the same isotropy subgroup H, we use a direct integral decomposition of $\mathscr{A}$ arising naturally from a consideration of the measure $\delta_e \times d\alpha$ to prove that if α is nontransitive and if H is in addition central in G then $\mathscr{A}$ has no type I component. In §5 we use different methods, namely the theory of quasi-unitary algebras, to derive necessary and sufficient conditions that $\mathscr{A}$ be semi-finite, under the weaker assumption that almost all $(d\alpha)$ points in Z have the same isotropy subgroup H and that H is type I and unimodular.

The results of §3 are contained in the author's Doctoral Dissertation written at the Massachusetts Institute of Technology under the direction of Professor Roe W. Goodman.

2. Notation and preliminaries. If X is a second countable locally compact Hausdorff space, we denote by $\mathscr{K}(X)$ the continuous functions on X of compact support, with the inductive limit topology, and by $M(X)$ the dual space of Radon measures on X with the weak *-topology. For $x \in X$, $\delta_x \in M(X)$ is the probability measure on X concentrated at x. For a locally compact group G, $d_G g$, or simply dg, denotes left Haar measure on G and Δ_G the corresponding modular function. We assume throughout this paper that both G and Z are second countable locally compact Hausdorff spaces, that all Hilbert spaces are separable and that all representations of algebras are nondegenerate.

Although we refer to [6], primarily §§1, 3, and 4, for the construction of and basic results concerning $\mathscr{U}(G, Z)$, we list for convenience some facts, and establish more notation. $\mathscr{K}(G \times Z)$ is a topological *-algebra and is dense in $\mathscr{U}(G, Z)$ [6, pp. 32–35]. The correspondence $L = \langle V, M \rangle$ between representations L of $\mathscr{U}(G, Z)$ on a Hilbert space $\mathscr{H}$ and systems of imprimitivity $\langle V, M \rangle$ based on (G, Z) and acting on $\mathscr{H}$ is completely determined [6, pp. 34–37] by

$$\begin{aligned} &\langle L(f)x, y\rangle \\ (2.1)\qquad &= \int_G \langle Mf(g, \cdot) V(g)x, y\rangle dg,\ f \in \mathscr{K}(G \times Z),\ x, y \in \mathscr{H}\ . \end{aligned}$$

If there is no possibility of confusion, we shall use the same symbol M for the representation of $\mathscr{K}(Z)$, its extension to the algebra $L^\infty(Z)$ of bounded Borel functions on Z, the corresponding projection-valued measure, and the generated W^*-algebra in $\mathscr{B}(\mathscr{H})$. We denote by $D(G \times Z)$ the set of positive-definite measures on $G \times Z$, that is, $\{p \in M(G \times Z): p(f^* * f) \geqq 0\ \forall f \in \mathscr{K}(G \times Z)\}$. $p \in D(G \times Z)$ determines a representation L^p of $\mathscr{U}(G, Z)$ on $\mathscr{H}^p$, and there is a canonical continuous map of $\mathscr{K}(G \times Z)$ onto a dense subspace of $\mathscr{H}^p$ [6, §4].

Blattner's results on induced positive-definite measures and their connection with induced representations [2, Theorem 1] can be extended from the group to the transformation group context. Let H be a closed subgroup of G and $L = \langle V, M \rangle$ a representation of $\mathscr{U}(H, Z)$.

As a special case of [18, §3], one can construct an induced system of imprimitivity $\langle \operatorname{ind}(V), \operatorname{ind}(M)\rangle$ based on (G, Z) and thus by (2-1) an induced representation $\operatorname{ind}(L)$ of $\mathscr{U}(G, Z)$. $\operatorname{Ind}(V)$ is the usual representation of G induced from the representation V of H. If $p \in D(H \times Z)$ define $\tilde{p} \in M(G \times Z)$ by

$$(2.2) \qquad \tilde{p}(f) = p(f \Delta_G^{1/2} \Delta_H^{-1/2}|_{H \times Z}),\ f \in \mathscr{K}(G \times Z)\,.$$

LEMMA 2.3. *If p in $D(H \times Z)$ determines a representation L* of *$\mathscr{U}(H, Z)$, then $\tilde{p} \in D(G \times Z)$ and determines a representation of $\mathscr{U}(G, Z)$ unitarily equivalent to* $\operatorname{ind}(L)$.

Proof. The proof of Theorem 1 of [2] can be repeated, with obvious modifications, and we omit the details.

LEMMA 2.4. *If $x \to L^x$ is an integrable family of representations of $\mathscr{U}(H, Z)$, then $x \to \operatorname{ind}(L^x)$ is an integrable family of representations of $\mathscr{U}(G, Z)$ and $\int \operatorname{ind}(L^x)$ is unitarily equivalent to* $\operatorname{ind}\left(\int L^x\right)$.

Proof. We sketch the argument. Let $L^x = \langle V^x, M^x\rangle$ on $\mathscr{H}^x$. By using the approximate identity in $\mathscr{K}(H \times Z)$ and the two formulas in [6, Lemma 3.26] one sees that $x \to V^x(s)$ and $x \to M^x(h)$ are measurable operator fields for $s \in H$, $h \in \mathscr{K}(Z)$. By Theorem 10.1 of [12], $x \to \operatorname{ind}(V^x)$ is a measurable field of representations of G on the induced Hilbert spaces $\operatorname{ind}(\mathscr{H}^x)$ and $\int \operatorname{ind}(V^x)$ on $\int \operatorname{ind}(\mathscr{H}^x)$ is unitarily equivalent to $\operatorname{ind}\left(\int V^x\right)$ on $\operatorname{ind}\left(\int \mathscr{H}^x\right)$. A similar argument verifies that $x \to (\operatorname{ind}(M^x))(h)$ is measurable for $h \in \mathscr{K}(Z)$ and that the unitary operator implementing the above equivalence for the representations of G transforms $\int(\operatorname{ind}(M^x))(h)$ into $\left(\operatorname{ind}\left(\int M^x\right)\right)(h)$. From the fact $x \to \operatorname{ind}(V^x)$ and $x \to \operatorname{ind}(M^x)$ are measurable it follows that $x \to \operatorname{ind}(L^x)$ is measurable. To see this, note that any $u \in \mathscr{U}(G, Z)$ can be approximated in norm by finite sums of the form $\sum f_i \otimes h_i$, $f_i \in \mathscr{K}(G)$ and $h_i \in \mathscr{K}(Z)$, and then apply (2.1). To finish the proof we note that from the 2 formulas in [6, Lemma 3.26] again, it is clear that for any measurable field of representations $x \to L^x = \langle V^x, M^x\rangle$ the system of imprimitivity corresponding to $\int L^x$ is $\left\langle \int V^x, \int M^x \right\rangle$. Thus the representations $\int \operatorname{ind}(L^x)$ and $\operatorname{ind}\left(\int L^x\right)$ are unitarily equivalent because their respective systems of imprimitivity $\left\langle \int \operatorname{ind}(V^x), \int \operatorname{ind}(M^x) \right\rangle$ and $\left\langle \operatorname{ind}\left(\int V^x\right), \operatorname{ind}\left(\int M^x\right) \right\rangle$ are.

For $\nu \in D(H)$ the measure $\tilde{\nu}$, defined by (2.2) with Z ignored, lies in $D(G)$. If H is the isotropy subgroup of $\varphi \in Z$ then $\nu \times \delta\hat{\varphi} \in D(H \times Z)$ and the induced measure on $G \times Z$ is exactly $\tilde{\nu} \times \delta\hat{\varphi}$. If $L = \langle V, M\rangle$ is the representation of $\mathscr{U}(H, Z)$ determined by $\nu \times \delta\hat{\varphi}$, V is unitarily equivalent to the representation of H determined by ν, $M(k) = k(\varphi)I$ for $k \in \mathscr{K}(Z)$, $\operatorname{ind}(M)$ is concentrated on the orbit $G\varphi$, and the commutants of $\operatorname{ind}(L)$ and V are algebraically isomorphic (see [6, §4] for details).

3. The type of $\mathscr{U}(G, Z)$.

For $\varphi \in Z$ let H_φ denote the isotropy subgroup of φ, d_{H_φ} a left Haar measure on H_φ and ν_φ the induced measure $\tilde{d}_{H_\varphi}$.

LEMMA 3.1. *There is a choice of left Haar measures on the isotropy subgroups of G so that for each $f \in \mathscr{K}(G \times Z)$, the function $\theta: Z \to \boldsymbol{C}$ defined by $\theta(\varphi) = (\nu_\varphi \times \delta_\varphi)(f)$ is bounded and Borel.*

Proof. Let $\mathscr{S}(G)$ denote the family of all closed subgroups of G, endowed with the compact Hausdorff topology described by Fell in [7]. The map $\varphi \to H_\varphi$ of Z into $\mathscr{S}(G)$ is Borel [1, Proposition 2.3] and left Haar measures d_H can be chosen on the subgroups H of G so that the map $H \to \tilde{d}_H$ of $\mathscr{S}(G)$ into $M(G)$ is continuous (this follows from [9, appendix] and the proof of Theorem 4.2 of [8]). Thus for $g \in \mathscr{K}(G)$ the composite map $\varphi \to \nu_\varphi(g)$ is Borel. To show that θ is bounded and Borel, we need the following estimate (see [9, Lemma 1.1]). Let K be a compact subset of G and $\ell \in \mathscr{K}(G)$ with $\ell \geqq 0$ and $\ell \equiv 1$ on K. Since $H \to \tilde{d}_H(\ell)$ is a continuous function on the compact set $\mathscr{S}(G)$, it is bounded by a positive constant α. For any $k \in \mathscr{K}(G \times Z)$ with $\operatorname{supp} k \subseteqq K \times Z$, and for any $H \in \mathscr{S}(G)$, $\varphi \in Z$, we have

$$(*) \qquad |\tilde{d}_H \times \delta_\varphi(k)| = |\tilde{d}_H(k(\cdot, \varphi))| \leqq \|k\|_\infty |\tilde{d}_H(\ell)| \leqq \alpha \|k\|_\infty .$$

Thus θ is bounded. Let A and B be compact subsets of G and Z contained, respectively, in relatively compact open sets U and V. If $f \in \mathscr{K}(G \times Z)$ with $\operatorname{supp} f \subseteqq A \times B$, f can be uniformly approximated by finite sums of the form $\sum g_i \otimes h_i$, $g_i \in \mathscr{K}(G)$, $\operatorname{supp} g_i \subseteqq U$, $h_i \in \mathscr{K}(Z)$, $\operatorname{supp} h_i \subseteqq V$. The estimate $(*)$, applied to the compact set $K = \bar{U}$, implies that θ is the uniform limit on Z of the Borel functions $\varphi \to (\nu_\varphi \times \delta_\varphi)(\sum g_i \otimes h_i) = \sum \nu_\varphi(g_i) h_i(\varphi)$, and is thus Borel.

Fix a "measurable" choice of left Haar measures on the isotropy subgroups as allowed by Lemma 3.1 and for $\varphi \in Z$ let L^φ denote the representation of $\mathscr{U}(G, Z)$ on the Hilbert space $\mathscr{H}^\varphi$ determined by $\nu_\varphi \times \delta_\varphi$.

LEMMA 3.2. *For every positive Radon measure α on Z the direct* *integral representation* $L = \int_Z L^\varphi d\alpha(\varphi)$ *exists.*

Proof. For $f \in \mathscr{K}(G\times Z)$ let $f'(\varphi)$ denote the canonical image of f in $\mathscr{H}^\varphi$. The map $f \to f'(\varphi)$ is continuous with respect to the inductive limit topology on $\mathscr{K}(G \times Z)$ and the norm topology on $\mathscr{H}^\varphi$ (this follows from [6, Lemma 3.7]). Since G and Z are second countable, $\mathscr{K}(G \times Z)$ contains a countable dense set $\{f_i\}$ [6, proof of Corollary 4.12], and by the preceding remarks and Lemma 3.1, it follows that the $f'_i(\varphi)$ are a fundamental sequence of measurable vector fields and thus the direct integral $\mathscr{H} = \int_Z \mathscr{H}^\varphi d\alpha(\varphi)$ exists [4, Chapter II, §1, n^0 4]. Each $u \in \mathscr{U}(G, Z)$ is the limit in norm of a sequence $h_n \in \mathscr{K}(G \times Z)$ and thus $\langle L^\varphi(u)f'_i(\varphi), f'_j(\varphi)\rangle = \lim_n (\nu_\varphi \times \delta_\varphi)(f_j^* * h_n * f_i)$ is a measurable function on Z, again by Lemma 3.1, and the direct integral $L = \int_Z L^\varphi d\alpha(\varphi)$ exists.

It follows from [9, Theorem 2.1] and [6, Theorem 4.29 and Lemma 4.30] that each L^φ is an irreducible representation of $\mathscr{U}(G, Z)$ and that $L^\varphi \cong L^\eta$ if and only if φ and η lie in the same G-orbit.

THEOREM 3.3. *$\mathscr{U}(G, Z)$ is type I if and only if the orbit space Z/G is T_0 and all the isotropy subgroups are type I.*

Proof. If Z/G is not T_0, there exists an ergodic positive Borel measure α on Z which is not concentrated on any orbit [5, Theorem 2.6]. By Lemma 3.2 and [5, Lemma 4.2], $L = \int_Z L^\varphi d\alpha(\varphi)$ is a factor representation of $\mathscr{U}(G, Z)$ not of type I. Also, since a factor representation W of an isotropy subgroup induces a factor representation L of $\mathscr{U}(G, Z)$ of the same type, the commutants of L and W being algebraically isomorphic, $\mathscr{U}(G, Z)$ is not type I if there is a nontype I isotropy subgroup. Conversely if Z/G is T_0 every factor representation $L = \langle V, M\rangle$ is induced from an isotropy subgroup by the Imprimitivity Theorem, since the projection-valued measure M is ergodic and thus concentrated on an orbit. If in addition all the isotropy subgroups are type I, so therefore is $\mathscr{U}(G, Z)$.

4. On the type of $\mathscr{A}$.

Let α be an ergodic quasi-invariant probability measure on Z, $g\cdot\alpha$ the measure defined by $g\cdot\alpha(A) = \alpha(g^{-1}A)$, $g \in G$, A Borel $\subseteq Z$, and $\lambda_g(\cdot)$ the Radon-Nikodym derivative $d(g\cdot\alpha)/d\alpha$. Let $\langle W, P\rangle$ be the system of imprimitivity based on (G, Z) and acting on $L^2(Z, d\alpha)$ by

$$(W(g)f)(\varphi) = \lambda_g(\varphi)^{1/2}f(g^{-1}\varphi)\ , \quad (P(h)f)(\varphi) = h(\varphi)f(\varphi)\ ,$$

$g \in G$, $\varphi \in Z$, $h \in L^\infty(Z)$ and $f \in L^2(Z, d\alpha)$. Denoting by U the left regular representation of G on $L^2(G)$, we consider the type of the W^*-algebra $\mathscr{A}$ on $L^2(G) \otimes L^2(Z, d\alpha)$ generated by the operators $U(g) \otimes W(g)$ and $I \otimes P(h)$, $g \in G$, $h \in L^\infty(Z)$. Our definition of $\mathscr{A}$ is the same as Kallman's [10] except for modifications due to our preference for left rather than right action of G on Z.

LEMMA 4.1. *$\mathscr{A}$ is spatially isomorphic to the W^*-algebra generated by the representation L^α of $\mathscr{U}(G, Z)$ determined by $\delta_e \times d\alpha$ in $D(G \times Z)$.*

Proof. The natural map of the algebraic tensor product $\mathscr{K}(G) \otimes \mathscr{K}(Z)$ onto a dense subspace of $\mathscr{K}(G \times Z)$ clearly extends to an isometry of $L^2(G) \otimes L^2(Z, d\alpha)$ onto $L^2(G \times Z, dg \times d\alpha)$. By the proof of Theorem 5.3 of [13], λ can be chosen to be jointly measurable on $G \times Z$ and it is then clear that under the above isometry the system of imprimitivity $\langle U \otimes W, I \otimes P \rangle$ is transformed into the system $\langle V', M' \rangle$ given by

$$(V'(g)f)(t, \varphi) = \lambda_g(\varphi)^{1/2} f(g^{-1}t, g^{-1}\varphi)$$

and

$$(M'(h)f)(t, \varphi) = h(\varphi) f(t, \varphi) ,$$

$g, t \in G$, $\varphi \in Z$, $h \in L^\infty(Z)$ and $f \in L^2(G \times Z, dg \times d\alpha)$. For $f \in \mathscr{K}(G \times Z)$ define $(Rf)(g, \varphi) = f(g, \varphi)\lambda_g(\varphi)^{1/2}$. Rf is measurable on $G \times Z$. Since

$$\int_G \int_Z | f(g, \varphi) |^2 \lambda_g(\varphi) d\alpha(\varphi) dg = \int_G \int_Z | f(g, g\varphi) |^2 d\alpha(\varphi) dg$$

and $k(g, \varphi) = f(g, g\varphi)$ lies in $\mathscr{K}(G \times Z)$, Rf is square-integrable. Routine calculations verify that R extends from $\mathscr{K}(G \times Z)$ to an isometry of $\mathscr{H}^\alpha$, the Hilbert space of L^α, onto $L^2(G \times Z, dg \times d\alpha)$ which transforms the system of imprimitivity given by

$$(V(g)f)(t, \varphi) = f(g^{-1}t, g^{-1}\varphi) \quad \text{and} \quad (M(h)f)(t, \varphi) = h(\varphi) f(t, \varphi) ,$$

t, $g \in G$, $\varphi \in Z$, $h \in L^\infty(Z)$ and $f \in \mathscr{K}(G \times Z)$ into $\langle V', M' \rangle$. To check that V transforms into V' requires use of the identity

$$\lambda_{st}(\varphi) = \lambda_s(\varphi)\lambda_t(s^{-1}\varphi) \text{ a.e. } (d\alpha)$$

for each $s, t \in G$. As $\langle V, M \rangle$ is precisely the system of imprimitivity on $\mathscr{H}^\alpha$ determined by $\delta_e \times d\alpha$ (see formulas 4.4 and 4.6 of [6]) and as $\langle V, M \rangle$ generates exactly the same W^*-algebra as the corresponding representation L^α of $\mathscr{U}(G, Z)$, we are done.

Now let $\mathscr{A}$ denote the W^*-algebra generated by the representation

L^α. Henceforth, we assume that α is concentrated on a G-invariant Borel set in Z all of whose points have the same isotropy group H, which is a priori normal in G. The more general case in which it is assumed merely that all isotropy subgroups are conjugate can be reduced to the above case [1, Chapter II, §2]. If π is a representation of H, we denote by $g \cdot \pi$ the representation $(g \cdot \pi)(h) = \pi(g^{-1}hg)$. We shall obtain a direct integral decomposition of $\mathscr{A}$ and then use the following lemma to prove that, under additional hypotheses on H, $\mathscr{A}$ has no type I component if α is nontransitive. We denote by $[\mathscr{B}, \mathscr{C}]$ the W^*-algebra generated by operator algebras $\mathscr{B}$ and $\mathscr{C}$, by $\mathscr{B}'$ the commutant of $\mathscr{B}$ and by $\mathscr{Z}\mathscr{B}$ the center $\mathscr{B} \cap \mathscr{B}'$ of $\mathscr{B}$.

LEMMA 4.2. *Let $\mathscr{B}$ be a W^*-algebra on a Hilbert space $\mathscr{H}$ and $\mathscr{C}$ a commutative subalgebra of $\mathscr{B}'$. If $\mathscr{B}$ has a type I component then so does $\mathscr{D} = [\mathscr{B}, \mathscr{C}]$.*

Proof. We use the notation of [4, Chapter I, §2, $n^0 1$] for induced and reduced algebras. $\mathscr{B}$ has a type I component if and only if there is a nonzero projection F in $\mathscr{Z}\mathscr{B}$ and an abelian projection E in $\mathscr{B}_F$ whose central support is the identity (relative to $\mathscr{B}_F$ on the Hilbert space $F\mathscr{H}$) [4, Chapter II, §8, $n^0 1$, Corollary 1 and $n^0 2$, Theorem 1]. We shall show that the projections F and E satisfy the same properties for $\mathscr{D}$ as they do for $\mathscr{B}$. Since $\mathscr{Z}\mathscr{B} = \mathscr{B} \cap (\mathscr{B} \cap \mathscr{B}') \subseteqq \mathscr{D} \cap (\mathscr{C}' \cap \mathscr{B}') = \mathscr{Z}\mathscr{D}$, $F \in \mathscr{Z}\mathscr{D}$. $\mathscr{C}_F$ is clearly a commutative algebra commuting with $\mathscr{B}_F$ and by [4, Chapter I, §2 $n^0 1$, Proposition 1], $\mathscr{D}_F$ is generated by $\mathscr{C}_F$ and $\mathscr{B}_F$, and $(\mathscr{D}_F)_E$ is generated by elements of the form $EBCE$, $B \in \mathscr{B}_F$ and $C \in \mathscr{C}_F$. This is because products of the form BC, $B \in \mathscr{B}_F$, $C \in \mathscr{C}_F$, form a generating subset of $\mathscr{D}_F$ closed under involution and multiplication. That $(\mathscr{D}_F)_E$ is abelian follows easily now from the hypothesis that $(\mathscr{B}_F)_E$ is abelian and from the fact that E lies in $\mathscr{B}_F$ and thus commutes with $\mathscr{C}_F$. Since $E \in \mathscr{B}_F \subseteqq \mathscr{D}_F$, E clearly has central support equal to the identity with respect to the larger algebra $\mathscr{D}_F$, and we are done.

THEOREM 4.3. *Let α be a nontransitive ergodic quasi-invariant probability measure on Z, and assume that almost all ($d\alpha$) points of Z have the same isotropy subgroup H. If the left regular representation T of H can be decomposed as a direct integral $T = \int_\Gamma T^\gamma d\gamma$ of irreducibles T^γ on $\mathscr{H}^\gamma$, so that* a.e. *($d\gamma$), $g \cdot T^\gamma$ is unitarily equivalent to T^γ for all $g \in G$, then $\mathscr{A}$ has no type I component.*

Proof. We note first that the hypotheses on T are certainly satisfied if H is central in G. In any case, since H is normal in G, $\Delta_G|_H = \Delta_H$ and $\delta_e \times d\alpha \in D(G \times Z)$ is induced from the measure $\delta_e \times d\alpha \in D(H \times Z)$ (see formula (2.2)). Thus L^α is induced from the representation R^α of $\mathscr{U}(H, Z)$ determined by $\delta_e \times d\alpha$ in $D(H \times Z)$. By applying Lemma 4.1 to R^α one obtains a unitary equivalence between R^α and $\pi^\alpha = \langle T \otimes I, I \otimes Q \rangle$ on $L^2(H) \otimes L^2(Z, d\alpha)$, where Q is the natural projection-valued measure from Z to $L^2(Z, d\alpha)$. As H leaves almost all $(d\alpha)$ points of Z fixed, each $\langle T^\gamma \otimes I, I \otimes Q \rangle$ is a system of imprimitivity, based on (H, Z) and acting on $\mathscr{H}^\gamma \otimes L^2(Z, d\alpha)$. Denoting by σ^γ the corresponding representation of $\mathscr{U}(H, Z)$ and by $\operatorname{ind} \sigma^\gamma$ the induced representation of $\mathscr{U}(G, Z)$, we have by Lemma 2.4 and its proof a unitary equivalence $L^\alpha \cong \int_\Gamma \operatorname{ind} \sigma^\gamma d\gamma$. If $\mathscr{A}$ had a type I component so would $[\mathscr{A}, L^\infty(\Gamma, d\gamma)]$ by Lemma 4.2, and therefore [4, Chapter II, §3, Exercise 1] so would the representations $\operatorname{ind} \sigma^\gamma$ for γ in a set of positive measure on Γ. We shall use Lemma 4.2 of [5] to verify that in fact $\operatorname{ind} \sigma^\gamma$ is a.e. $(d\gamma)$ a nontype I factor representation, and the theorem will be proven. Q has a natural direct integral decomposition $Q(h) = \int_Z Q^\varphi(h) d\alpha(\varphi)$, where $Q^\varphi(h)$ is multiplication by $h(\varphi)$ on $\boldsymbol{C}$, $h \in L^\infty(Z)$. Fix $\gamma \in \Gamma$. The system of imprimitivity $\langle T^\gamma \otimes I, I \otimes Q^\varphi \rangle$, or simply $\langle T^\gamma, Q^\varphi \rangle$, on $\mathscr{H}^\gamma \otimes \boldsymbol{C} = \mathscr{H}^\gamma$ determines a representation τ^φ of $\mathscr{U}(H, Z)$ and again by Lemma 2.4, $\operatorname{ind} \sigma^\gamma \cong \int_Z \operatorname{ind} \tau^\varphi d\alpha(\varphi)$. It follows from Theorem 2.1 of [9] and the discussion preceding that theorem that each $\operatorname{ind} \tau^\varphi$ is an irreducible representation of $\mathscr{U}(G, Z)$, since T^γ is an irreducible representation of H, and furthemore that $\operatorname{ind} \tau^\varphi$ is unitarily equivalent to $\operatorname{ind} \tau^\eta$ if and only if $\varphi = g \cdot \eta$ and $T^\gamma \cong g \cdot T^\gamma$ for some $g \in G$. If $g \cdot T^\gamma \cong T^\gamma$ for all $g \in G$, $\operatorname{ind} \tau^\varphi \cong \operatorname{ind} \tau^\eta$ if and only if φ and η lie in the same G-orbit. α is thus ergodic with respect to the relation of unitary equivalence among the components $\operatorname{ind} \tau^\varphi$ of $\operatorname{ind} \sigma^\gamma$, and by Lemma 4.2 of [5] $\operatorname{ind} \sigma^\gamma$ is a nontype I factor representation. By hypothesis, this is true a.e. $(d\gamma)$ and we are done.

5. On the type of $\mathscr{A}$ (continued).

We derive necessary and sufficient conditions for $\mathscr{A}$ to be semi-finite, under the assumption of a common isotropy group H which is type I and unimodular. Our proof is modelled on Dixmier's in [3, §§10–12], where the case of free action is considered. As there, we assume that the Radon-Nikodym derivative $d(g \cdot \alpha)/d\alpha = \lambda_g(\cdot)$, considered as a function on $G \times Z$, is continuous and strictly positive. With no loss of generality, we also assume that support $\alpha = Z$. We start with the realization of $\mathscr{A}$ as the W^*-algebra on $L^2(G \times Z, dg \times d\alpha)$ generated by $\{V(g), M(h): g \in G, h \in L^\infty(Z)\}$, where

$$(5.1) \quad \begin{aligned} (V(g)f)(t, \varphi) &= \lambda_g(\varphi)^{1/2} f(g^{-1}t, g^{-1}\varphi) \quad \text{and} \\ (M(h)f)(t, \varphi) &= h(\varphi) f(t, \varphi), \quad f \in L^2(G \times Z, dg \times d\alpha). \end{aligned}$$

(See the proof of Lemma 4.1, where V and M are denoted by V' and M'.)

For $f \in \mathscr{K}(G \times Z)$, define

$$(5.2) \quad \begin{aligned} f^j(g, \varphi) &= \Delta(g)^{-1/2} \lambda_g(\varphi)^{1/2} f(g, \varphi) \quad \text{and} \\ f^s(g, \varphi) &= \Delta(g)^{-1/2} \lambda_g(\varphi)^{1/2} \bar{f}(g^{-1}, g^{-1}\varphi). \end{aligned}$$

Also, let $\langle W, N \rangle$ be the system of imprimitivity on $L^2(G \times Z, dg \times d\alpha)$ given by

$$(5.3) \quad \begin{aligned} (W(g)f)(t, \varphi) &= \Delta(g)^{1/2} f(tg, \varphi) \quad \text{and} \\ (N(h)f)(t, \varphi) &= h(t^{-1}\varphi) f(t, \varphi). \end{aligned}$$

Our definitions differ from Dixmier's due essentially to our preference for left action of G on Z. Denote by L and R the representations of $\mathscr{U}(G, Z)$ corresponding, respectively, to $\langle V, M \rangle$ and $\langle W, N \rangle$.

LEMMA 5.4. *$\mathscr{K}(G \times Z)$, with f^j and f^s as in (5.2), convolution as multiplication and inner product as in $L^2(G \times Z, dg \times d\alpha)$, is a quasi-unitary algebra with underlying Hilbert space $L^2(G \times Z, dg \times d\alpha)$. Its left algebra $\mathscr{R}^l$ is $\mathscr{A}$ and its right algebra $\mathscr{R}^r = (\mathscr{R}^l)'$ is the algebra generated by $\langle W, N \rangle$.*

Proof. That the conditions on [3, p. 277] are satisfied can be verified as in [3, Proposition 9] and we omit the computations. For $f \in \mathscr{K}(G \times Z)$, denote by $\pi^l(f)$ and $\pi^r(f)$, respectively, the bounded operators on $L^2(G \times Z, dg \times d\alpha)$ of left and right convolution by f. $\mathscr{R}^l$ and $\mathscr{R}^r$ are, respectively, the W^*-algebras generated by all $\pi^l(f)$, $\pi^r(f)$, $f \in \mathscr{K}(G \times Z)$ (see [3, p. 278]). The remainder of the lemma follows by use of (2.1) to verify that $L(f) = \pi^l(f \cdot \lambda^{1/2})$ and $R(f) = \pi^r(\bar{f}^s \cdot \lambda^{-1/2})$ for all $f \in \mathscr{K}(G \times Z)$.

We denote by J the positive self-adjoint extension of $f \to f^j$, by S the isometric extension of $f \to f^s$ [3, p. 278], by $P^l(P^r)$ the set of operators in $\mathscr{R}^l(\mathscr{R}^r)$ commuting with J, and by $Q^l(Q^r)$ the operators in $\mathscr{R}^l(\mathscr{R}^r)$ commuting with all of $P^l(P^r)$. Theorem 2 of [3] and Theorem 1 of [17] yield the following: $\mathscr{R}^l$ is semi-finite if and only if there exist (unbounded) positive invertible self-adjoint operators A and A' belonging to $\mathscr{R}^l$ and $\mathscr{R}^r$, respectively, so that $A' = SAS$ and J is the minimal closed extension of $A(A')^{-1}$, and if this is the case, then A and A' belong to Q^l and Q^r, respectively, and $Q^l \subsetneqq P^l$, $Q^r \subsetneqq P^r$. As in [3], we derive necessary and sufficient conditions for A, A' as above to exist in terms of the action of G on some measure space (B, db) by investigating how A and A' correspond to the operators of multiplication by certain elements of $L^\infty(B, db)$. In the case of a nontrivial isotropy subgroup H, this necessitates an examination of various direct integral decompositions. We assume familiarity with the notation and results of [4, Chapter II, §§1–3]. If π is a representation of a group K, we denote by $\pi(K)$ the W^*-algebra generated by $\{\pi(k): k \in K\}$.

$L^2(G \times Z, dg \times d\alpha)$ is naturally isometric with the direct integral over $(Z, d\alpha)$ of the constant field of Hilbert spaces $\varphi \to \mathscr{H}(\varphi) = L^2(G)$, with the algebra M corresponding naturally to the algebra of diagonalizable operators. Denote by $\mathscr{L}$ the algebra on $L^2(G \times Z, dg \times d\alpha)$ generated by multiplication by bounded Borel functions on $G \times Z$ and by $\mathscr{L}_1$ the subalgebra generated by the bounded Borel functions on $G/H \times Z$, considered as functions on $G \times Z$. Let $\bar{V}$ and $\bar{W}$ denote, respectively, the left and right regular representations of G on $L^2(G)((\bar{W}(g)f)(t) = \Delta(g)^{1/2}f(tg))$, and $M(G)(M(G/H))$ the algebra on $L^2(G)$ generated by multiplication by bounded Borel functions on $G(G/H)$. Then clearly (see (5.1) and (5.3)) the W^*-algebras $\mathscr{L}, \mathscr{L}_1, [\mathscr{L}, V(H)]$, $[\mathscr{L}, \mathscr{Z}\, V(H)]$ and $[M, \mathscr{Z}\, V(H)]$ are all the direct integrals, respectively, of the constant fields of W^*-algebras $\varphi \to M(G), M(G/H), [M(G), \bar{V}(H)]$, $[M(G), \mathscr{Z}\, \bar{V}(H)]$ and $\mathscr{Z}\, \bar{V}(H)$ on $L^2(G)$. Also, each operator $W(g)$ decomposes as $\int_Z \bar{W}(g) d\alpha(\varphi)$.

Lemma 5.5. *If H is unimodular, then $Q^l \subseteqq [M, \mathscr{Z}\, V(H)]$.*

Proof. It follows from (5.1) and (5.2) that $M \subseteqq P^l$ and that $V(H) \subseteqq P^l$ for H unimodular. If $A \in Q^l \subseteqq \mathscr{R}^l = (\mathscr{R}^r)'$, then $A \in [W(G), N]'$ by Lemma 5.4, and $A \in [M, V(H)]'$ by the preceding remark. By modifying the proof of [3, Lemme 26] so that instead of dealing with compact subsets K, K' of G one deals with subsets of the form KH, $K'H$, K and K' compact, it follows that $[M, N] = \mathscr{L} \cap V(H)' = \mathscr{L}_1$. As $(\mathscr{L} \cap V(H)')' = [\mathscr{L}', V(H)] = [\mathscr{L}, V(H)]$ we have $A \in [M, N]' = [\mathscr{L}, V(H)]$. Thus $A = \int_Z A(\varphi) d\alpha(\varphi)$, $A(\varphi) \in [M(G), \bar{V}(H)]$ a.e. $(d\alpha)$, and we must show $A(\varphi) \in \mathscr{Z}\, \bar{V}(H)$ a.e. $(d\alpha)$. From the fact that $A \in V(H)'$ it follows that $A(\varphi) \in \bar{V}(H)'$ a.e. $(d\alpha)$, and from the fact that $A \in (W(G))' \cap \mathscr{L}_1'$ it follows that $A(\varphi) \in (\bar{W}(G))' \cap (M(G/H))'$ a.e. $(d\alpha)$. By a commutation theorem of Takesaki [19, Theorem 3] the latter algebra is exactly $\bar{V}(H)$ (note that the left and right coset spaces $G/H, H\backslash G$ are identical) and we are done.

We now decompose $L^2(G)$ explicitly with respect to the abelian W^*-algebra $\mathscr{Z}\, \bar{V}(H)$. Choose left Haar measure dh and $d\bar{g}$ on H and G/H, respectively, so that $\int_G f(g)dg = \int_{G/H}\int_H f(gh)dh\, d\bar{g}$, for all $f \in$

$\mathscr{K}(G)$. Let σ denote a Borel cross-section from G/H to G with $\sigma(\bar{e}) = e$, and let $\eta(g) = \sigma(\bar{g})^{-1}g$, so that every $g \in G$ may be written uniquely $g = \sigma(\bar{g})\eta(g)$, $\eta(g) \in H$. Define $\theta(g)$ by

$$\int_H f(ghg^{-1})dh = \theta(g)\int_H f(h)dh,\ f \in \mathscr{K}(H)\ ,$$

and denote by U_g the isometry of $L^2(H)$ into itself given by $(U_g f)(h) = \theta(g)^{1/2}f(g^{-1}hg)$. Let $\tilde{V}$ be the left regular representation of H on $L^2(H)$ and $\int_{\hat{H}} n(\gamma)R^\gamma d\gamma$ its canonical central decomposition. $(G, \hat{H})$ is a Borel transformation group [18, Theorem 2.4].

LEMMA 5.6. *$L^2(G)$ is isometric with the direct integral over $(G/H, d\bar{g})$ of the constant field of Hilbert spaces $\bar{g} \rightarrow L^2(H)$. The operator U implementing the isometry is $(Uf)(\bar{g}, h) = f(\sigma(\bar{g})h)$, $f \in L^2(G)$. For $\ell \in L^2(G/H, d\bar{g}, L^2(H))$, $(U^{-1}\ell)(g) = \ell(\bar{g}, \eta(g))$. Furthermore,*

$$U\bar{V}(h)U^{-1} = \int_{G/H}(\sigma(\bar{g})\cdot\tilde{V})(h)d\bar{g}$$

and $(\sigma(\bar{g})\cdot\tilde{V})(h) = U^{-1}_{\sigma(\bar{g})}\tilde{V}(h)U_{\sigma(\bar{g})}$, so that $\mathscr{Z}\bar{V}(H)$ is transformed by U into

$$\left\{\int_{G/H} U^{-1}_{\sigma(\bar{g})}AU_{\sigma(\bar{g})}d\bar{g}\colon A \in \mathscr{Z}\tilde{V}(H)\right\}.$$

$\mathscr{Z}\tilde{V}(H)$ is invariant under $A \rightarrow U^{-1}_{\sigma(\bar{g})}AU_{\sigma(\bar{g})}$, and if $A \in \mathscr{Z}\tilde{V}(H)$ corresponds to $f \in L^\infty(\hat{H}, d\gamma)$, $U^{-1}_{\sigma(\bar{g})}AU_{\sigma(\bar{g})}$ corresponds to the function $g^{-1}\cdot f$, given by $(g^{-1}\cdot f)(\gamma) = f(g\cdot\gamma)$.

Proof. All of the statements except the last are either standard results or can be verified easily by direct computation. We note that $\ell \in L^2(G/H, d\bar{g}, L^2(H))$ can indeed be considered as a jointly measurable function on $(G/H \times H, d\bar{g} \times dh)$ by [11, Lemma 3.1]. For the last statement of the lemma see, for example, [1, Introduction, Proposition 10.2].

REMARK 1. The automorphisms $A \rightarrow U^{-1}_{\sigma(\bar{g})}AU_{\sigma(\bar{g})}$ of $\mathscr{Z}\tilde{V}(H)$ into itself define an action of G/H on $\mathscr{Z}\tilde{V}(H)$, for if $h \in H$, U_h is the product of a left and a right translation by elements of H and thus commutes with $\mathscr{Z}\tilde{V}(H)$ [3, Theoreme 1]. Thus G/H is an automorphism group on $(\hat{H}, d\gamma)$, as indeed it is on $(Z, d\alpha)$, but we shall continue to regard G as the group acting on these spaces. Since H acts trivially and is unimodular, however, the following equalities, which we shall use shortly, hold: $\sigma(\bar{g})\varphi = g\varphi$, $\Delta(\sigma(\bar{g})) = \Delta(g)$, $\lambda_{\sigma(\bar{g})} = \lambda_g$ and $\theta(\sigma(\bar{g})) = \theta(g)$, $g \in G$, $\varphi \in Z$.

REMARK 2. We shall use Lemma 3.1 of [11], without further explicit mention, to identify $L^2(X, dx, L^2(Y, dy))$ with $L^2(X \times Y, dx \times dy)$ and the space of essentially bounded measurable functions from (X, dx) to $L^\infty(Y, dy)$ with $L^\infty(X \times Y, dx \times dy)$, where (X, dx) and (Y, dy) are each one of the spaces $(Z, d\alpha)$, $(G/H, d\bar{g})$ or $(\hat{H}, d\gamma)$.

By Lemma 5.6 and the discussion preceding Lemma 5.5, an operator $A \in [M, \mathscr{X} V(H)]$ corresponds, after direct integral decomposition of $L^2(G \times Z, dg \times d\alpha)$ over $(Z, d\alpha)$ and $(G/H, d\bar{g})$, to

$$(*) \qquad \int_Z \int_{G/H} U_{\sigma(\bar{g})}^{-1} A(\varphi) U_{\sigma(\bar{g})} d\bar{g} d\alpha(\varphi) \;, \quad A(\varphi) \in \mathscr{X} \tilde{V}(H) \;.$$

But after decomposition over $(\hat{H}, d\gamma)$, $A(\varphi)$ corresponds to multiplication by $f^\varphi \in L^\infty(\hat{H}, d\gamma)$ and $U_{\sigma(\bar{g})}^{-1} A(\varphi) U_{\sigma(\bar{g})}$ corresponds to multiplication by $g^{-1} \cdot f^\varphi$. Regarding $f(\varphi, \gamma) = f^\varphi(\gamma)$ as an element of $L^\infty(Z \times \hat{H}, d\alpha \times d\gamma)$, which may involve changing values of f on a $(d\alpha \times d\gamma)$ null set, we have A corresponding to multiplication by $m(\varphi, \bar{g}, \gamma) = f(\varphi, g \cdot \gamma)$. We now examine what SAS and J correspond to, and we shall obtain our final result.

LEMMA 5.7. *Let A and f be as above. After decomposing over Z, G/H and $\hat{H}$, SAS corresponds to multiplication by $k(\varphi, \bar{g}, \gamma) = \bar{f}(g^{-1}\varphi, \gamma)$ and J corresponds to multiplication by*

$$\ell(\varphi, \bar{g}, \gamma) = \Delta(g)^{-1/2} \lambda_g(\varphi)^{1/2} \;.$$

Proof. The result for J follows directly from (5.2). Let U_1 be the isometry implementing the decomposition over Z and G/H. For $r \in L^2(Z \times G, d\alpha \times dg)$, $(U_1 r)(\varphi, \bar{g}, h) = r(\varphi, \sigma(\bar{g})h)$, and for $r \in L^2(Z \times G/H, d\alpha \times d\bar{g}, L^2(H))$, $(U_1^{-1} r)(\varphi, g) = r(\varphi, \bar{g}, \eta(g))$. $U_1 A U_1^{-1}$ is given by $(*)$. We shall compute $U_1 S U_1^{-1}$ and then

$$(**) \qquad U_1(SAS)U_1^{-1} = (U_1 S U_1^{-1})(U_1 A U_1^{-1})(U_1 S U_1^{-1}) \;.$$

Although the computation of $U_1 S U_1^{-1}$ and other operators by pointwise evaluation yields (pointwise) formulas valid only a.e., these formulas still uniquely determine the element of $L^\infty(Z \times G/H \times \hat{H}, d\alpha \times d\bar{g} \times d\gamma)$ to which SAS corresponds. Thus we may for simplicity ignore a.e. considerations. For $r \in L^2(Z \times G/H, d\alpha \times d\bar{g}, L^2(H))$, it can be verified directly that

$$(U_1 S U_1^{-1} r)(\varphi, \bar{g}, h) = \Delta(g)^{-1/2} \lambda_g(\varphi)^{1/2} \bar{r}(g^{-1}\varphi, \bar{g}^{-1}, \eta(h^{-1}\sigma(\bar{g})^{-1})) \;.$$

Now

$$\begin{aligned} \eta(h^{-1}\sigma(\bar{g})^{-1}) &= \sigma(\bar{g}^{-1})^{-1} h^{-1} \sigma(\bar{g})^{-1} \\ &= (\sigma(\bar{g}^{-1})^{-1} h^{-1} \sigma(\bar{g}^{-1}))(\sigma(\bar{g}^{-1})^{-1} \sigma(\bar{g})^{-1}) \;. \end{aligned}$$

Defining $\Phi(g) = \sigma(\bar{g}^{-1})^{-1}\sigma(\bar{g})^{-1}$ and an operator $\tilde{S}$ on $L^2(H)$ by $(\tilde{S}a)(h) = \bar{a}(h^{-1})$, one can compute directly from the above formulae that

$$\begin{aligned}&(U_1 S U_1^{-1} r)(\varphi, \bar{g}) \\ &= (\Delta(g)^{-1/2}\theta(g)^{1/2}\lambda_g(\varphi)^{1/2} U_{\sigma(\bar{g}^{-1})} \tilde{V}(\Phi(g))\tilde{S})(r(g^{-1}\varphi, \bar{g}^{-1}))\end{aligned}$$

as elements of $L^2(H)$, and again by direct computation and $(**)$ it follows that

$$\begin{aligned}&(U_1 SASU_1^{-1} r)(\varphi, \bar{g}) \\ &= (U_{\sigma(\bar{g}^{-1})} \tilde{V}(\Phi(g))\tilde{S}\, U^{-1}_{\sigma(\bar{g}^{-1})} A(g^{-1}\varphi)\, U_{\sigma(\bar{g}^{-1})} U_{\sigma(\bar{g})} \tilde{V}(\Phi(g^{-1}))\tilde{S})(r(\varphi, \bar{g}))\ .\end{aligned}$$

It is clear that $\tilde{S}\tilde{S} = I$ on $L^2(H)$, and therefore the operator on $L^2(H)$ given by the right-hand side of the above equation equals the product $T_1T_2T_3$, where

$$\begin{aligned}T_1 &= U_{\sigma(\bar{g}^{-1})}) \tilde{V}(\Phi(g))\tilde{S}\, U^{-1}_{\sigma(\bar{g}^{-1})}\tilde{S}\ , \\ T_2 &= \tilde{S}A(g^{-1}\varphi)\tilde{S} \quad \text{and} \\ T_3 &= \tilde{S}\, U_{\sigma(\bar{g}^{-1})} U_{\sigma(\bar{g})} \tilde{V}(\Phi(g^{-1}))\tilde{S}\ .\end{aligned}$$

Now $A(g^{-1}\varphi) \in \mathscr{X}\tilde{V}(H)$ and thus $\tilde{S}A(g^{-1}\varphi)\tilde{S} = A^*(g^{-1}\varphi) \in \mathscr{X}\tilde{V}(H)$ by [3, Corollaire, p. 283]. By a tedious but straightforward computation, one checks that $T_1 = \tilde{V}(\Phi(g^{-1}))$ and thus

$$\begin{aligned}T_1T_2T_3 &= T_2(T_1T_3) \\ &= A^*(g^{-1}\varphi)(U_{\sigma(\bar{g}^{-1})} \tilde{V}(\Phi(g))\tilde{S}\, U_{\sigma(\bar{g})} \tilde{V}(\Phi(g^{-1}))\tilde{S})\ .\end{aligned}$$

But T_1T_3 equals the identity (again a straightforward computation) and we have finally that

$$(U_1 SASU_1^{-1} r)(\varphi, \bar{g}) = A^*(g^{-1}\varphi)(r(\varphi, \bar{g}))\ .$$

Thus SAS corresponds to $k(\varphi, \bar{g}, \gamma) = \bar{f}(g^{-1}\varphi, \gamma)$ and we are done.

THEOREM 5.8. *$\mathscr{A}$ is semi-finite if and only if there exists a positive measurable function ψ on $(Z \times \hat{H}, d\alpha \times d\gamma)$ such that*

$$\frac{\psi(\varphi, g\gamma)}{\psi(g^{-1}\varphi, \gamma)} = \Delta(g^{-1})\lambda_g(\varphi) \text{ a.e. } (dg \times d\alpha \times d\gamma) \quad \text{on} \quad G \times Z \times \hat{H}\ .$$

Proof. See [3, Theoreme 7 and Proposition 12] for the proof. Also see [3, Remarque 1, p. 318] for a slight strengthening of the theorem and [3, Remarque 2, p. 319] for the measure-theoretic significance of the hypothesis on ψ.

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