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METRIC CHARACTERIZATIONS OF EUCLIDEAN SPACES

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In a metric space an arc which is isometric to a real interval is called a segment. In this paper it is shown that, for $1 \leqq n \leqq 3$, n-dimensional Euclidean space (E^n) is topologically characterized, among locally compact, n-dimensional spaces, by admitting a metric with the following properties: (1) every two points of the space are endpoints of a unique segment, (2) if two segments have an endpoint and one other point in common then one is contained in the other and (3) every segment can be extended, at either end, to a larger segment. This follows from the more general result that, for $1 \leqq n \leqq 3$, a locally compact, n-dimensional space which admits a metric with properties (1) and (2) is homeomorphic to an n-manifold lying between the closed n-ball and its interior.

Property (1) suffices to characterize E^n, for $n = 1$ or 2, among locally compact, locally homogeneous, n-dimensional spaces. For $n > 3$, properties (1), (2), and (3) characterize E_n among locally compact, n-dimensional spaces that contain a homeomorph of an n-ball.

1. Introduction. A metric space (X, d) is said to be *convex* provided that every pair of points of X has a midpoint—m is a *midpoint* of x and y if $d(x, m) = d(y, m) = 1/2d(x, y)$. (X, d) is *strongly convex* if every pair of points has a unique midpoint and is *without ramifications* provided that no midpoint of x and y is a midpoint of z and y unless $z = x$. Convex subsets of Euclidean spaces with their inherited metrics are examples of metric spaces with these properties. Lelek and Nitka [8] and Rolfsen [11] have (topologically) characterized the 2-cell and 3-cell, respectively, among compact 2 and 3 dimensional metric spaces by the last two properties. White [12] has shown that a 2-complex is collapsible if and only if it can be given a metric which is strongly convex. Numerous other results have been obtained for metric spaces with the above properties when the underlying space is compact or when the metric is also complete.

In the present paper a number of these results are shown to hold when the underlying space is locally compact. Principally, it is shown that having a strongly convex metric without ramifications (topologically) characterizes n-manifolds that lie between the n-cell and its interior among locally compact, n-dimensional spaces for $n \leqq 3$. This reduces to Lelek and Nitka's or Rolfsen's result when the space is compact and yields a characterization of E^n under various homogeneity conditions.

2. **Existence of segments.** In a metric space (X, d) a set S is said to be a *segment* for the points x and y of X if x and y are elements of S and S is isometric with the real interval $[0, d(x, y)]$. It is well known that in a convex, complete metric space every pair of points has a segment between them. It is shown now that for locally compact spaces the requirement of completeness can be relaxed.

THEOREM 2.1. *Let (X, d) be a locally compact, convex metric space. If, for each pair of points of X, the set of midpoints of the pair is compact, then each pair of points is joined by a segment.*

Proof. Let p, q be two points of X and let $A_0 = \{p, q\}$. Order A_0 by distance from p. In general, if A_n has been defined and ordered by distance from p, then for each $x \in (A_n - \{q\})$ let m_x be a midpoint for x and the next point of A_n. Define $A_{n+1} = A_n \cup \{m_x : x \in A_n - \{q\}\}$. Let $A = \bigcup_{n=0}^{\infty} A_n$, $a = d(p, q)$ and $f = d(p, -) | \bar{A}$. Clearly f maps A isometrically onto the set of real numbers of the form $r \cdot a$ where r is a dyadic rational in $[0, 1]$, and so f maps $\bar{A}$ isometrically into the interval $[0, a]$. To show that $\bar{A}$ is a segment from p to q it is sufficient to show that $f[\bar{A}] = [0, a]$.

Since the dyadic rationals are dense in $[0, 1]$, the image of A is dense in $[0, a]$. Also, the image of $\bar{A}$ is open in $(0, a)$. To verify this observe that if $x \in (\bar{A} - \{p, q\})$ and D is a compact distance neighborhood of x then $D \cap \bar{A}$ is compact and thus $f[\bar{A} \cap D]$ is closed. Since $f[A \cap D]$ is dense in an interval containing $f(x)$ in its interior, $f(x)$ is an interior point of $f[\bar{A}]$. For the final step let t be any point of $(0, a)$, and let T be a subinterval of $(0, a)$, symmetric about t. $T \cap f[\bar{A}]$ and U, the reflection of $T \cap f[\bar{A}]$ in t, are both open and dense in T and hence their intersection is dense in T. Let $\{t_n\}_{n=1}^{\infty}$ be an increasing sequence of points of $T \cap f[\bar{A}] \cap U$ that converges to t, so if, for each n, $t'_n = 2t - t_n$, then $t'_n \in f[\bar{A}]$ and t is midway between t_n and t'_n. Let $x_n = f^{-1}(t_n)$, $x'_n = f^{-1}(t'_n)$ and $M_n = \{y \in X : y \text{ is midpoint for } x_n \text{ and } x'_n\}$. If $y \in M_{n+1}$, then y is between x_n and x'_n and also $d(x_n, y) = d(x_n, x_{n+1}) + d(x_{n+1}, y) = d(y, x'_{n+1}) + d(x'_{n+1}, x'_n) = d(y, x'_n)$ so $y \in M_n$. Since each M_n is compact there is a point x in $\bigcap_{n=1}^{\infty} M_n$. Now $d(x_n, x) = 1/2 d(x_n, x'_n) = 1/2 | t_n - t'_n |$ and so $\lim_{n \to \infty} d(x_n, x) = 0$. Thus $x \in \bar{A}$ and clearly $f(x) = t$. Evidently $f[\bar{A}] = [0, a]$ and so $\bar{A}$ is a segment from p to q.

COROLLARY 2.2. *If (X, d) is a locally compact, strongly convex metric space, then the segment between two points is unique and contains* *all the points between them.*

The proof for the case where (X, d) is also complete carries over without change in light of 2.1.

When segments are unique the segment between p and q is denoted $\overline{pq}$.

3. Strongly convex metrics.

DEFINITION 3.1. A topological space Y is said to be *contractible* if there exists a mapping $f: Y \times I \to Y$ such that $f(-, 1)$ is the identity map and $f(-, 0)$ is constant. Y is *locally contractible* if every neighborhood, U, of any point contains a neighborhood, V, of the point and a map $f: V \times I \to U$ such that $f(-, 1)$ is the identity map and $f(-, 0)$ is constant.

Throughout this section let (X, d) be a locally compact strongly convex metric space. We aim first at showing that X is contractible and locally contractible.

Fix $p \in X$ and define the map $Q: X \times I \to X$ as follows: for $t \in I$, $Q(p, t) = p$ for $t \in I, x \in X - \{p\}$, $Q\{x, t)$ is the point z in $\overline{px}$ such that $d(p, z) = t \cdot d(p, x)$.

Q is the contracting homotopy and in showing Q is continuous the fact that the limit of a sequence of segments is again a segment is used. This fact is the content of 3.3.

NOTATION 3.2. Throughout the paper $D(p, r)$ denotes the set $\{x \in X: d(p, x) \leqq r\}$ and $S(p, r)$ denotes the set $\{x \in X: d(p, x) < r\}$ where $p \in X$ and r is a positive real number.

PROPOSITION 3.3. *Let $x_0, x_1, x_2, \cdots$ be points of X such that $\lim_{i\to\infty} x_i = x_0 \neq p$. Then if*

(1) $y_i \in \overline{px_i}$, $i = 0, 1, 2, \cdots$ *and*

(2) $\lim_{i\to\infty} d(p, y_i) = d(p, y_0)$ *then*

(3) $\lim_{i\to\infty} y_i = y_0$.

Proof. Let $A = \{y_0 \in \overline{px_0}$: conditions (1) and (2) imply (3)$\}$. Clearly $x_0 \in A$ because if $\lim_{i\to\infty} d(p, y_i) = d(p, x_0)$, then $\lim_{i\to\infty} d(x_i, y_i) = 0$ since $d(x_i, y_i) = d(p, x_i) - d(p, y_i)$. Let A' be the component of A containing x_0 and assume that q is a boundary point of A' relative to $\overline{px_0}$. Choose $r > 0$ so that $D(q, 5r)$ is compact. Let $y_0, y_1, y_2, \cdots$ be a sequence that satisfies conditions (1) and (2) but not (3) and chosen such that $d(y_0, q) < r$. Let $q_0 \in A \cap D(q, r)$. For each $i = 1, 2, \cdots$ let $t_i = \min\{d(p, x_i), d(p, q_0)\}$ and let q_i be a point of $\overline{px_i}$ such that $d(p, q_i) = t_i$, $i = 1, 2, \cdots$. Since $\lim_{i\to\infty} t_i = d(p, q_0)$ it follows that $\lim_{i\to\infty} q_i = q_0$.

Since both of y_i, and q_i belong to $\overline{px_i}$, $i = 0, 1, 2, \cdots$ we have $d(q_i, y_i) = |d(p, q_i) - d(p, y_i)|$ and so $\lim_{i\to\infty} d(q_i, y_i) = d(q_0, y_0) \leqq 2r$. Now $d(q, y_i) \leqq d(q, q_0) + d(q_0, q_i) + d(q_i, y_i)$ and so is eventually less than $5r$. Thus the sequence $\{y_i\}_{i=1}^{\infty}$ is eventually in the compact set $D(q, 5r)$. If z is a limit point of the sequence $\{y_i\}_{i=1}^{\infty}$, then it follows from the continuity of the distance function d that $d(p, z) + d(z, x_0) = d(p, x_0)$ and so $z \in \overline{px_0}$. Since z is the same distance from p as y_0 it follows that $z = y_0$. Thus $\lim_{i\to\infty} y_i = y_0$ and this contradicts our choice of y_0, $y_1, y_2, \cdots$. It follows that A' has no boundary point relative to $\overline{px_0}$ so must be all of $\overline{px_0}$.

PROPOSITION 3.4. *$Q: X \times I \to X$ is continuous.*

Proof. Follows from Proposition 3.3 and continuity of distance function.

In contracting X to the point p the map, Q, moves every point closer to p so any distance neighborhood of p is contracted in itself. The point p was chosen without restriction so X is also locally contractible.

It follows trivially that X is also connected and locally connected and these conditions for a locally compact metric space imply separability [1].

THEOREM 3.5. *A locally compact, strongly convex metric space is contractible, locally contractible, connected, locally connected and separable.*

THEOREM 3.6. *An n-dimensional, locally compact, strongly convex metric space is an n-manifold if it is locally homogeneous and contains an n-ball.*

Proof. Since a locally compact space is second category this follows immediately from a theorem of Bing and Borsuk [3].

THEOREM 3.7. *For $n = 1$ or 2, an n-dimensional, locally homogeneous, locally compact metrizable space can be given a strongly convex metric if and only if it is homeomorphic to E^n.*

Proof. The usual metric for E^n is strongly convex.

It follows from another theorem of Bing and Borsuk [3] that such a space is an n-manifold. Since it is contractible as well it must be E^n, n being 1 or 2.

4. Strongly convex metrics without ramifications. Preliminaries. Throughout this section (X, d) will be a locally compact metric space with d a strongly convex metric without ramifications, briefly an SC-WR metric.

DEFINITION 4.1. For p and q two points of X the set $\{x \in X : x \in \overline{pq}$ or $q \in \overline{px}\}$ is called the *ray* from p through q and is denoted $\overline{pq}\rangle$.

PROPOSITION 4.2. *If* $y \in (\overline{px}\rangle - \{p\})$, *then* $\overline{py}\rangle = \overline{px}\rangle$.

Proof. Clearly $x \in \overline{py}\rangle$ so it suffices to show that if $z \in \overline{px}\rangle$ then $z \in \overline{py}\rangle$. We consider four cases:

(1) $y \in \overline{px}$ and $z \in \overline{px}$. In this case it follows immediately from the uniqueness of segments that either $y \in \overline{pz}$ or $z \in \overline{py}$ so $z \in \overline{py}\rangle$.

(2) $y \in \overline{px}$ and $x \in \overline{pz}$. The convexity of the metric yields $y \in \overline{pz}$ and so $z \in \overline{py}\rangle$.

(3) $x \in \overline{py}$ and $z \in \overline{px}$. Same as (2).

(4) $x \in \overline{py}$ and $x \in \overline{pz}$. Unless $\overline{py} \subset \overline{pz}$ or $\overline{pz} \subset \overline{py}$ there would be a ramification point in $\overline{pz} \cap \overline{py}$. Thus $z \in \overline{py}\rangle$.

PROPOSITION 4.3. $\overline{px}\rangle$ *is isometric to a real interval of one of the following forms*: $[0, \infty)$, $[0, a)$, *or* $[0, a]$.

Proof. This is evident from the previous proposition and the fact that rays are arc connected.

If $\overline{px}\rangle$ is isometric to the closed interval $[0, a]$, then we say $\overline{px}\rangle$ is a ray with endpoint or a compact ray and the point of $\overline{px}\rangle$ a distance a from p is the endpoint.

DEFINITION 4.4. A metric space (Y, ρ) is said to be *externally convex* if given p and q in Y there is a point $y \in Y$ such that $\rho(p, y) = \rho(p, q) + \rho(q, y)$.

Note that (X, d) is externally convex if and only if no ray has an endpoint.

For rays a result analogous to Proposition 3.3 holds and the proof carries over as well.

PROPOSITION 4.5. *Let* $p, x_0, x_1, x_2, \cdots$ *be points of* X *such that* $\lim_{i\to\infty} x_i = x_0 \neq p$. *Then if*

(1) $y_i \in \overline{px_i}\rangle$, $i = 0, 1, 2, \cdots$ *and*

(2) $\lim_{i\to\infty} d(p, y_i) = d(p, y_0)$, *then*

(3) $\lim_{i\to\infty} y_i = y_0$.

We now define a map which moves points along rays similar to Q in §3. Fix p in X and let

$$D^x = \begin{cases} \{x\} \times [0, \infty) & \text{if } \overline{px}\rangle \text{ is isometric to } [0, \infty) \\ \{x\} \times \left[0, \dfrac{a}{d(p, x)}\right) & \text{if } \overline{px}\rangle \text{ is isometric to } [0, a) \\ \{x\} \times \left[0, \dfrac{a}{d(p, x)}\right] & \text{if } \overline{px}\rangle \text{ is isometric to } [0, a] \,. \end{cases}$$

Now let $D = \bigcup_{x \in X - \{p\}} D^X$ be the domain of P and define $P\colon D \to X$ by the following rule:

$$P(p, t) = p$$

$P(x, t) =$ the point z of $\overline{px}\rangle$ such that $d(p, z) = t \cdot d(p, x)$ for $x \neq p$.

PROPOSITION 4.6. *P is continuous.*

Proof. Follows from 4.5 and continuity of distance function.

REMARK 4.7. Since the function P depends on the choice of the basepoint p, P will be denoted P_p any time confusion might arise. Likewise D is denoted D_p.

Our next goal is to show that every open subset of X contains a homeomorphic copy of X. The first step is to show that if a sequence of rays converges to a ray with endpoint, then all but finitely many of the sequence of rays have endpoints.

LEMMA 4.8. *Let $p, x_0, x_1, x_2, \cdots$ be points of X such that $\overline{px_0}\rangle = \overline{px_0}$ and $\lim_{i\to\infty} x_i = x_0$. Then, for i sufficiently large there is a point $z_i \in \overline{px_i}\rangle$ such that $\overline{pz_i} = \overline{px_i}\rangle$ and $\lim_{i\to\infty} z_i = x_0$.*

Proof. Clearly $\underline{\lim}_{i\to\infty} \operatorname{diam}(\overline{px_i}\rangle) \geqq d(p, x_0)$. On the other hand, if

$$\overline{\lim_{i\to\infty}} \operatorname{diam}(\overline{px_i}\rangle) = d(p, x_0) + \delta > d(p, x_0)$$

then there is an infinite set of integers, M, such that for $i \in M$ there is a $y_i \in \overline{px_i}\rangle$ such that $d(p, y_i) = d(p, x_0) + \min\{\delta, r\}$ where $r > 0$ is chosen to make $D(x_0, 2r)$ compact. The set $\{y_i\colon i \in M\}$ has a limit point, y, in $D(x_0, 2r)$. Since y satisfies $d(p, y) = d(p, x_0) + d(x_0, y)$ we have $y \in \overline{px_0}\rangle = \overline{px_0}$. This is a contradiction because $d(p, y) > d(p, x_0)$. Thus $\lim_{i\to\infty} \operatorname{diam}(\overline{px_i}\rangle) = d(p, x_0)$.

Now choose n_0 to be an integer such that $x_i \in D(x_0, r)$ and

$$|\operatorname{diam} \overline{px_i}\rangle - d(p, x_0)| \leqq r$$

whenever $i \geqq n_0$. Then the set $((\overline{px_i}\rangle - \overline{px_i}) \cup \{x_i\})$ is in $D(p, 2r)$ when $i \geqq n_0$, and since each of the sets is closed it must be compact. Each of the rays $\overline{px_i}$ with $i \geqq n_0$ is then compact and letting z_i be the endpoint the lemma is proved.

THEOREM 4.9. *Let p be a point of X and f be a map from $X - \{p\}$ into $(0, 1]$. Then the function $G: X \to X$, defined by the formula*

$$g(p) = p$$
$$g(x) = P_p(x, f(x)) \text{ for } x \neq p$$

is a homeomorphism if it is one-to-one.

Proof. On $X - \{p\}$, g is the composition of continuous functions so is continuous. It is continuous at p as well because

$$d(p, P_p(x, f(x)) = f(x)\cdot d(p, x) \leqq d(p, x) .$$

Assume that g is one-to-one. It remains only to show that g^{-1} is continuous. Note that g^{-1} is continuous at p because if $D(p, r)$ is a compact neighborhood of p and $a = \inf\{f(z): z \in D(p, r)\}$, then $g[D(p, r)]$ contains $D(p, a\cdot r)$ which is a neighborhood of p.

The map g restricted to any segment $\overline{px}$ is a homeomorphism of $\overline{px}$ into itself with p remaining fixed so if points are ordered by distance from p, g preserves that order. Let $x_0, x_1, x_2, \cdots$ be points of $g[X]$ such that $\lim_{i\to\infty} x_i = x_0$. Let $y_i = g^{-1}(x_i)$ for $i = 0, 1, 2, \cdots$. Consider first the case where $d(p, y_i) \geqq d(p, y_0) + \delta$ for some $\delta > 0$, and all $i = 1, 2, 3, \cdots$. In view of Lemma 4.8, y_0 cannot be the endpoint of $\overline{py_0}\rangle$ so we may choose a point w_0 in $(\overline{py_0}\rangle - \overline{py_0})$ and within $\delta/2$ of y_0. Let w_i be a point of $\overline{px_i}\rangle$ such that $d(p, w_i) = d(p, w_0)$ for $i = 1, 2, 3, \cdots$, and note that $\lim_{i\to\infty} w_i = w_0$. Since w_0 is farther than y_0 from p, $g(w_0)$ is farther than $g(y_0) = x_0$ from p. On the other hand, for $i = 1, 2, 3, \cdots$, w_i is closer than y_i to p and so $g(w_i)$ is closer than $x_i = g(y_i)$. From the continuity of g, $g(w_0)$ is at least as close to p as x_0. This case is ruled out and if no sequence can belong to this case, no infinite subsequence of a sequence can either, so the remaining possibility is that $d(p, y_i) \leqq d(p, y_0)$ for all $i \geqq n_0$, for some integer n_0. But then $\bigcup_{i=0}^{\infty} \overline{py_i}$ is compact and g^{-1} is continuous on $g[\bigcup_{i=0}^{\infty} \overline{py_i}] = \bigcup_{i=0}^{\infty} \overline{px_i}$ and so $\lim_{i\to\infty} g^{-1}(x_i) = g^{-1}(x_0)$.

Note that if $f(x)$ is a nondecreasing function of $d(p, x)$, then g is a homeomorphism.

COROLLARY 4.10. *Let p be a point of X and U a neighborhood of p. Then there is a homeomorphism, g, of X into U leaving p fixed and for $x \in X$, $g(x) \in \overline{px}$.*

Proof. Let $0 < r < 1$ be chosen so that $S(p, r) \subset U$, and define the map $h\colon (0, \infty) \to (0, r)$ by the formula $h(t) = 2r/\pi \tan^{-1} t$. Observe that h is one-to-one and $h(t) \leqq t$. Define $f\colon X - \{p\} \to (0, 1]$ by the rule $f(x) = h(d(p, x))/d(p, x)$. Let g be defined as in Theorem 4.9 in terms of f and P_p. The map g could fail to be one-to-one only by mapping two points of some ray, $\overline{px}\rangle$, to the same point. But $d(p, g(y)) = d(p, y) \cdot f(y) = h(d(p, y))$, so this cannot happen. Moreover, the last expression must be less than r so $g[X] \subset S(p, r) \subset U$.

5. Endpoints of rays are sparse.

Next we develop some contractibility conditions for X and certain subsets, then show that in an SC-WR metric space of finite dimension the endpoints of rays are contained in a nowhere dense set.

Throughout this section let (X, d) be an SC-WR metric space.

PROPOSITION 5.1. *Let p be a point of X and r be a positive real. Then there is a map $h\colon X \times [0, 1] \to X$ with the following properties*:

(1) *$h(-, 0)$ is the identity on X;*

(2) *$h(-, 1)[X] \subset D(p, r)$;*

(3) *for $t \in [0, 1]$ $h(-, t) \mid D(p, r)$ is the identity on $D(p, r)$ and*

(4) *for $(x, t) \in (X - D(p, r)) \times [0, 1]$ $h(x, t) \notin S(p, r)$.*

Proof. Difine the function $m\colon X \to [0, 1]$ by the formulas

$$m(x) = \begin{cases} 1 \text{ if } x = p \\ \min\left\{1, \dfrac{r}{d(p, x)}\right\} \text{ in } x \neq p \end{cases}$$

and $g\colon X \times [0, 1] \to [0, 1]$ by $g(x, t) = (1 - t) + t \cdot m(x)$. Let P be the function defined in § 4 with p as its base point. Define $h\colon X \times [0, 1] \to X$ by $h(x, t) = P(x, g(x, t))$ and it is routine to verify h satisfies conditions (1) through (4).

REMARK 5.2. By virtue of h satisfying conditions (1), (2), and (3), $D(p, r)$ is said to be a *strong deformation retract* of X. A subset A of X is a *retract* if there is a map from $X \to A$ which is the identity on A.

PROPOSITION 5.3. *Let $p \in X$ and r a positive real. Then for $y \in S(p, r)$, $X - \{y\}$ is contractible (in itself) if and only if $(D(p, r) - \{y\})$ is contractible (in itself).*

Proof. Let $y \in S(p, r)$ be given and take h to be the deformation map defined in the previous proposition relative to $D(p, r)$. From properties (3) and (4) of h it is clear that $h[(X - \{y\}) \times [0, 1]] \subset X - \{y\}$. Thus h retracts $X - \{y\}$ onto $(D(p, r) - \{y\})$. A retract of a contractible space is contractible [4, p. 26], so $D(p, r) - \{y\}$ is contractible if X-(y) is. The converse is obvious.

PROPOSITION 5.4. *Let $p \in X$ and $r > 0$. Then $D(p, r)$ is contractible and locally contractible.*

Proof. In §3 it was shown that $D(p, r)$ is contractible.

Fix $y \in D \equiv D(p, r)$ and $\delta > 0$. Letting h be the deformation map from 5.1 and setting $f = h(-, 1)$ gives us that f is a retraction of X onto D. Define the map $g: D \times I \to D$ by the formula $g(x, t) = f(P_y(x, 1 - t))$. Clearly, g is continuous, $g(-, 0)$ is the identity and $g(-, 1)$ is constantly y. Choose $r > 0$ so that $D(y, r)$ is compact and it follows that $C_n \equiv D \cap D(y, r/n)$ is compact for each $n = 1, 2, 3, \cdots$. The nested sequence of sets $C_n \times I$ converge to $\{y\} \times I$ and since g is continuous there exists n_0 such that $g[C_{n_0} \times I] \subset S(y, \delta) \cap D$. It follows that $g \mid C_{n_0} \times I$ contracts C_{n_0} to y inside $S(y, \delta)$ and thus D is locally contractible.

DEFINITION 5.5. For a set A in a topological space Y the space $A \times I/A \times \{0\}$, i.e., the upper-semi-continuous decomposition of $A \times I$ whose only nondegenerate element is $A \times \{0\}$, is called the *cone* over A.

PROPOSITION 5.6. *Let (X, d) be a locally compact SC-WR metric space. If $A \subset X$ is compact and $p \in (X - A)$ such that, for $x \in A$, $\overline{px} \cap A = \{x\}$, then the set $B = \bigcup_{x \in A} \overline{px}$ is homeomorphic to the cone over A.*

Proof. The proof of this proposition appears in [8, 6.2] for X compact. The proof carries over for X locally compact in light of the properties shown in the preceding propositions.

The following theorem generalizes a result of D. Rolfsen [11] which was for compact spaces. The proof is identical except that it relies on earlier propositions in this paper for properties of locally compact spaces with SC-WR metrics.

THEOREM 5.7. *Let (X, d) be a locally compact SC-WR metric space with $\dim X = n$ and $0 < n < \infty$. Then the set $U = \{x: X - \{x\}$ fails to be contractible in itself$\}$ contains a dense, open subset of X.*

COROLLARY 5.8. *If (X, d) and U are as in Theorem 5.7, then no point of U is the endpoint of a ray.*

Proof. Let $x \in X$ and $p \neq x$ such that $\overline{px}\rangle = \overline{px}$. The map P_p: $X \times I \to X$ defined earlier when restricted to $(X - \{x\}) \times I$ clearly contracts $X - \{x\}$ to p missing x so $X - \{x\}$ is contractible (in itself). By Theorem 5.7 $x \notin U$.

6. Retract properties.

DEFINITION 6.1. Let Y be a topological space and A a subset of Y. A is said to be a *neighborhood retract* of Y provided there exists an open set, O, of Y such that $A \subset O$ and A is a retract of O.

DEFINITION 6.2. A metric space Y is said to be an *absolute retract for metrizable spaces*, or an AR (M)-space, if for any metric space Z and a closed subset A of Z with A homeomorphic to Y, A is a retract of Z. Y is said to be an *absolute neighborhood retract for metrizable spaces*, or an ANR (M)-space, if for any metric space Z and closed subset A of Z, with A homeomorphic to Y, A is a neighborhood retract of Z.

DEFINITION 6.3. A metric space Y is said to be an *absolute retract* or AR-space if Y is an AR (M)-space and Y is compact, Y is said to be an *absolute neighborhood retract*, or ANR-space, if Y is an ANR (M)-space and Y is compact.

PROPOSITION 6.4. *Let (X, d) be a locally compact* SC-WR *metric space of finite dimension. If $D(p, r) \subset X$ is compact, then it is an absolute retract and if no ray from p ends inside $D(p, r)$, then the set $Sh\,(p, r) = \{x \in X: d(x, p) = r\}$ is an absolute neighborhood retract.*

Proof. As is evident from Proposition 5.4, $D(p, r)$ is contractible in itself and locally contractible and since it is compact and finite dimensional it is an absolute retract [4, 10.5, p. 122].

To show that $\mathrm{Sh}\,(p, r)$ is an ANR it is sufficient to show that it is a neighborhood retract of the absolute retract $D(p, r)$ [4, 2.4, p. 101]. Since no rays end inside $D(p, r)$ we can retract $D(p, r) - \{p\}$ onto $\mathrm{Sh}\,(p, r)$ by pushing outward along rays from p.

THEOREM 6.5. *If (X, d) is a locally compact* SC-WR *metric space of finite dimension, then $X \in$ AR (M).*

Proof. For a point p of X there is a positive number r_p so that $D(p, r_p)$ is compact and by Proposition 6.3, $D(p, r_p) \in$ AR (M). As noted in Theorem 3.5, X is separable and since each point of X has a neighborhood which is an ANR (M)-space, $X \in$ ANR (M) [4, 10.4,

p. 99]. However, since X is contractible, $X \in \mathrm{AR}(M)$ [4, 9.1, p. 96].

7. Existence of cells in low dimension spaces.

LEMMA 7.1. *Let (X, d) be a locally compact,* SC-WR *metric space. Then if p, x, and y are three non-colinear points of X, then $\bigcup_{z \in \overline{xy}} \overline{pz}$ is a 2-cell and $\bigcup_{z \in \overline{xy}} \overline{pz}\rangle$ is 2-dimensional and closed.*

Proof. Let $A = \bigcup_{z \in \overline{xy}} \overline{pz}$ and $\hat{A} = \bigcup_{z \in \overline{xy}} \overline{pz}\rangle$.

In light of Proposition 5.6 Lelek and Nitka's proof [8] that A is a 2-cell carries over from compact to locally compact spaces.

To establish the second part of the lemma, let $r = \inf\{d(p, z): z \in \overline{xy}\}$. Clearly, $r > 0$, and by Corollary 4.10 there is a homeomorphism of X into $S(p, r)$ that moves points along rays. Under this map, $\hat{A}$ is carried into A and so is 2-dimensional. Moreover, if q is a point of the closure of $\hat{A}$, then the image of q is in the compact set A, so $\overline{pq}\rangle$ meets $\overline{xy}$ at a point z_0. It follows that $\overline{pq}\rangle = \overline{pz_0}\rangle \subset \hat{A}$ and $q \in \hat{A}$.

THEOREM 7.2. *Let (X, d) be a locally compact* SC-WR *metric space of dimension n with $1 \leqq n \leqq 3$. Then there is a dense, open set V of X such that points of V have closed distance neighborhoods homeomorphic to I^n.*

Proof. For the case $n = 1$ the theorem follows directly from the lemma. Since dimension $X = 1$, X has two points p and q. Since X cannot contain 3 noncolinear points (lemma), $X = \overline{pq}\rangle \cup \overline{qp}\rangle$. Letting $V = X - \{$endpoints of $\overline{pq}\rangle$ and $\overline{qp}\rangle$, if any$\}$ the proof is complete.

The case $n = 2$ or $n = 3$. The argument that Rolfsen [11] gives for a similar theorem with X compact and $\dim X = 3$ carries over to locally compact spaces and, with a small addition, works for $\dim X = 2$ as well. That argument is outlined below with references to results of this paper needed to carry through various of the steps.

Let $U = \{x \in X: X - \{x\}$ fails to be contractible in itself$\}$ and let $V = \operatorname{int} U$. Fix $p \in V$ and choose $\varepsilon > 0$ so that $\bar{N} = D(p, \varepsilon)$ is compact and contained in V. Let $S = \{x \in X: d(p, x) = \varepsilon\}$.

(1) V is open and dense in X (Proposition 5.7).

(2) S is compact, $(n - 1)$-dimensional and $\bar{N}$ is homeomorphic to the (abstract) cone over S [11, (4), p. 218], (Proposition 6.4).

(3) S is an ANR-space [11, (6), p. 218], (Corollary 4.10).

(4) S does not have the fixed point property [11, (8), p. 218].

(5) For $s \in S$, $S - \{s\}$ is contractible in itself [11, (7), p. 218].

(6) S is connected and if $n = 3$, then no finite set separates S [11, (9), p. 218], (Lemma 7.1).

(7) If $n = 3$, then S is a 2-sphere [11, (10), p. 219].

(8) If $n = 2$, then S is a 1-sphere.

Since S does not have the fixed point property, it follows from a theorem of Lefschetz [5] that for some $k \geqq 0$ the (reduced) singular homology group (integral coefficients), $H_k(S)$ is nontrivial. S is connected so $H_0(S) = 0$ and $\dim(S) = 1$ so $H_k(S) = 0$ for $k \geqq 2$, hence $H_1(S) \neq 0$. Because of (5), $H_k(S - \{s\}) = 0$ for all $k \geqq 0$. It follows from a theorem of McCord's [9] that S is a 1-sphere.

Part (2) along with (7) and (8) yield that $\bar{N}$ is homeomorphic to I^n.

8. Topological characterizations.

DEFINITION 8.1. A point y in a topological space Y has a *Euclidean neighborhood* if for some neighborhood V of y and some natural number, n, V is homeomorphic to E^n.

Throughout this section let (X, d) be a locally compact SC-WR metric space.

PROPOSITION 8.2. *If some point of X has a Euclidean neighborhood, then the set, $M \equiv \{x \in X: x$ is the endpoint of some ray$\}$, is closed in X and every point of $(X - M)$ has a Euclidean neighborhood.*

Proof. Let p be a point of X with a Euclidean neighborhood V, V homeomorphic to E^n. There is a homeomorphism of X into V so we may consider X, as a topological space, to be imbedded in E^n. Let $\operatorname{int} X$ and $\operatorname{Bd} X$ denote the interior and boundary of X as a subset of E^n.

For any subset, Y, of E^n if $y \in \operatorname{int} Y$, then $Y - \{y\}$ is not contractible (in itself). It follows from proof of Corollary 5.8 that for $x \in M$, $X - \{x\}$ is contractible (in itself), so $M \subset \operatorname{Bd} X$.

Consider a point, $x \in (X - M)$. Since x is not the endpoint of the ray $\overline{px}\rangle$ there is a point q in $\overline{px}\rangle - \overline{px}$. Set $t = d(p, x)/d(p, q)$ and since $0 < t < 1$, the map $P_q(-, t)$ is a homeomorphism of X into itself (Theorem 4.9) that carries p to x. By the invariance of domain the image of V under this map is open in E^n, hence $x \in \operatorname{int} X$. It follows that $M \subset (X \cap \operatorname{Bd} X)$.

Now $M = (X \cap \operatorname{Bd} X)$ and so M is closed in X, and since $(X - M) = \operatorname{int} X$ every point of $(X - M)$ has a Euclidean neighborhood.

REMARK 8.3. Note that the set, $M_p \equiv \{x \in X: \overline{px} = \overline{px}\rangle\}$, where p is a point with Euclidean neighborhood, is contained in M. However, in the last part of the above proof it was shown that, in fact, $(X \cap \operatorname{Bd} X) \subset M_p$ so $M_p = M$.

PROPOSITION 8.4. *Let $p \in X$ with M_p closed. The function r_p: $X - \{p\} \to E^1 \cup \{+\infty\}$, defined by $r_p(x) = \operatorname{diam} \overline{px}\rangle$, is lower-semi-continuous.*

Proof. Let $x_0, x_1, x_2, \cdots$ be points of $X - \{p\}$ with $\lim_{n\to\infty} x_n = x_0$ and let t be a number less than $r_p(x_0)$. We may as well assume $\lim_{n\to\infty} r_p(x_n)$ exists, and call it s. To complete the proof it remains only to rule out the possibility that $s < t$.

If $s < t$, then there is a point $z_0 \in \overline{px_0}\rangle$ such that $d(p, z_0) = s$. We can also assume $r_p(x_n) < \infty$ for $n > 0$, so if we choose $z_n \in \overline{px_n}\rangle$ such that $d(p, z_n) \geqq (\operatorname{diam} \overline{px_n}\rangle - 1/n)$ then, by Proposition 4.5, $\lim z_n = z_0$. Let $D(z_0, r)$ be a compact neighborhood of z_0 and, clearly, $T_n \subset D(z_0, r)$ for n sufficiently large where $T_n = \overline{px_n}\rangle - \overline{pz_n}$. Thus T_n, and consequently, $\overline{px_n}\rangle$ are compact for n large. Let y_n be the endpoint of the compact $\overline{px_n}\rangle$ and observe that $\lim y_n = z_0$. Since $y_n \in M_p$ and M_p is closed $z_0 \in M_p$, hence $\overline{px_0}\rangle = \overline{pz_0}$, contradicting the choice of z_0.

THEOREM 8.5. *Let $p \in (X - M)$ have a Euclidean neighborhood. If $r > 0$ and $D(p, 4r)$ is compact and contained in $(X - M)$, then there is a subset T of $D(p, r)$ such that $S(p, r) \subset T$ and T is homeomorphic to X.*

Proof. The method of the proof will be to use a sequence of continuous functions approximating r_p to partition X into countably many subsets. These subsets will be mapped homeomorphically onto $S(p, r/2)$ and countably many annuli between $S(p, r/2)$ and $S(p, r)$ along with a subset of $D(p, r) - S(p, r)$.

The map r_p is lower-semi-continuous and has range contained in $[4r, +\infty]$ since no ray ends in $D(p, 4r)$. Let $S = \{x \in X: D(p, x) = r\}$ and $r_p | S$ is lower-semi-continuous. A lower-semi-continuous function on a separable, finite dimensional metric space which is bounded below can be pointwise approximated by a (strictly) increasing sequence of continuous functions [2]. Let $\hat{f}_1, \hat{f}_2, \hat{f}_3, \cdots$ be such a sequence approximating $r_p | S$ and we can assume range of $\hat{f}_n$, all n, is contained in $[2r, \infty)$. Extend each $\hat{f}_n$ to all of $X - \{p\}$ by letting $f_n(x) = \hat{f}_n(y)$ where y is the unique point of S in the ray $\overline{px}\rangle$. Clearly the extended functions are continuous on $X - \{p\}$.

Define

$$A_0 \equiv \{x \in X - \{p\}: d(p, x) \leqq f_1(x)\}$$
$$A_n \equiv \{x \in X - \{p\}: f_n(x) \leqq d(p, x) \leqq f_{n+1}(x)\} \text{ for } 0 < n < \infty$$
$$A_\infty \equiv \{x \in X - \{p\}: f_n(x) < d(p, x) \text{ all } n\} .$$

The desired homeomorphism $h: X \to D(p, r)$ is defined by the formulas

$$\begin{aligned} h(p) &= p \\ h(x) &= P_p(x, m(x)) \text{ for } x \neq p \end{aligned}$$

where $m\colon X - \{p\} \to (0, 1]$ is defined as follows:

$$m(x) = \begin{cases} \dfrac{r}{f_1(x)} \cdot \left(\dfrac{2^1 - 1}{2^1}\right) \text{ if } x \in A_0 \\ \left[1 + \dfrac{d(p, x) - f_n(x)}{f_{n+1}(x) - f_n(x)} \cdot \left(\dfrac{1/2}{2^n - 1}\right)\right] \cdot \dfrac{r}{d(p, x)} \cdot \dfrac{2^n - 1}{2^n}, \text{ if } x \in A_n, \\ 0 < n < \infty \\ \dfrac{r}{d(p, x)}, \text{ if } x \in A_\infty \ . \end{cases}$$

If $x \in A_n \cap A_{n+1}$ for $0 \leqq n < \infty$, then $m(x)$ has two definitions but since $d(p, x) = f_{n+1}(x)$ in that case it is routine to verify that

$$m(x) = \frac{r}{f_{n+1}(x)} \cdot \frac{2^{n+1} - 1}{2^{n+1}}$$

from both definitions. It is also evident that m is continuous on each A_n, $0 \leqq n \leqq \infty$ and on $\bigcup_{n<\infty} A_n$ as well.

Observe that for $x \in A_n$, $0 < n < \infty$, then

$$0 \leqq \frac{d(p, x) - f_n(x)}{f_{n+1}(x) - f_n(x)} \leqq 1$$

which yields

$$\frac{r}{d(p, x)}\left(\frac{2^n - 1}{2^n}\right) \leqq m(x) \leqq \frac{r}{d(p, x)}\left(\frac{2^{n+1} - 1}{2^{n+1}}\right) .$$

Thus if $x_1, x_2, x_3, \cdots$ is a sequence of points in $\bigcup_{n<\infty} A_n$ with limit $x_0 \in A_\infty$, then

$$\lim_{k \to \infty} m(x_k) = \frac{r}{d(p, x_0)} = m(x_0) .$$

Thus m is continuous on X and the above bounds on m shows that m has range $[1/4, 1/2] \subset (0, 1]$.

In order to show that h is a homeomorphism it only remains to show that h is one-to-one, and because h moves points along rays from p, it is sufficient to consider one such ray. Fix $x_0 \in X - \{p\}$ and let $b_1, b_2, b_3, \cdots$ be points of $\overline{px_0}\rangle$ chosen so $d(p, b_n) = f_n(x_0) = f_n(b_n)$. Let $a_n = h(b_n)$ and note

$$d(p, a_n) = m(b_n) \cdot d(p, b_n) = r \cdot \frac{2^n - 1}{2^n} .$$

The function m is constant on $\overline{pb_1} - \{p\}$ so h is one-to-one on $\overline{pb_1}$. In general, $\overline{b_n b_{n+1}} = A_n \cap \overline{px_0}\rangle$, so on $\overline{b_n b_{n+1}}$

$$m(x) = \frac{\alpha_n}{d(p, x)} + \beta_n \text{ where} \begin{cases} \alpha_n = r\cdot\left(\frac{2^n - 1}{2^n}\right)\left[1 + \frac{-f_n(x_0)}{f_{n+1}(x_0) - f_n(x_0)}\cdot\frac{1/2}{2^n - 1}\right] \\ \beta_n = r\left(\frac{2^n - 1}{2^n}\right)\left(\frac{1}{f_{n+1}(x_0) - f_n(x_0)}\right)\left(\frac{1/2}{2^n - 1}\right). \end{cases}$$

Thus $d(p, h(x)) = \alpha_n + \beta_n d(p, x)$ and since $\beta_n > 0$, h is one-to-one on $\overline{b_n b_{n+1}}$, carrying $\overline{b_n b_{n+1}}$ onto $\overline{a_n a_{n+1}}$. $A_\infty \cap \overline{px_0}\rangle$ consists of at most one point whose image lies a distance r from p. It follows that h is one-to-one on $\overline{px_0}\rangle$ and also that the image of $\overline{px_0}\rangle$ under h contains $(\overline{px_0}\rangle \cap S(p, r))$.

Let $T = h[X]$ and the theorem is proved.

COROLLARY 8.6. *For $1 \leqq n \leqq 3$ an n-dimensional, locally compact metrizable space, X, admits an* SC-WR *metric if and only if X is an n-manifold (with boundary) and is homeomorphic to a subset of closed unit n-ball and which contains the interior of the n-ball.*

Proof. The necessity is obvious because the usual metric for E^n restricted to such a subset is SC-WR.

To show the sufficiency let d be an SC-WR metric for X. By Theorem 7.2 there exists a point p of X and a positive number t such that $D(p, t)$ is homeomorphic to I^n. X is homeomorphic to a set T with $S(p, r) \subset T \subset D(p, r)$ where $r = t/4$. $D(p, r)$ is homeomorphic to I^n and thus to B, the unit ball in E^n. Let $T' \subset B$ be the image of T under the last homeomorphism. Since $T \supset S(p, r)$, $T' \supset \operatorname{int} B$ and T' being locally compact yields that T' contains a relatively open subset of $\operatorname{Bd} B$. T' is a n-manifold and consequently X is as well.

PROPOSITION 8.7. *Let (X, d) be a locally compact* SC-WR *space. If X is of finite dimension, the following are equivalent:*

(a) *(X, d) is externally convex*
(b) *no ray has an endpoint*
(c) *X is homogeneous*
(d) *X is locally homogeneous.*

Proof. The pattern of the proof is (a) $\leftrightarrow$ (b) $\rightarrow$ (c) $\rightarrow$ (d) $\rightarrow$ (b).

(a) $\leftrightarrow$ (b). This equivalence was noted in § 4.

(b) $\rightarrow$ (c). Assume (b) holds. We first establish that if $D(p, r)$ is compact and $q \in S(p, r)$, then for $w \in \overline{pq} - \{p\}$ there is a homeomorphism of X onto itself that carries q to w.

Let $a = d(p, q)$ and $b = d(p, w)$. Define $A_1 \equiv D(p, a) - \{p\}$, $A_2 \equiv$

$D(p, r) - S(p, a)$ and $A_3 \equiv X - S(p, r)$ and define the map $f: X - \{p\} \to (0, 1]$ by the formula:

$$f(x) = \begin{cases} \dfrac{b}{a} & \text{if } x \in A_1 \\ \dfrac{b}{a} + \left(1 - \dfrac{b}{a}\right)\dfrac{d(p, x) - a}{r - a} & \text{if } x \in A_2 \\ 1 & \text{if } x \in A_3 . \end{cases}$$

Since f is continuous on each of the three closed sets A_1, A_2, and A_3 and uniquely defined on their intersections, it is continuous. Let $h(x) = P_p(x, f(x))$ for $x \neq p$ and $f(p) = p$. P_p moves points along rays and since $f(x)$ is a nondecreasing function of $d(p, x)$, h is one-to-one and therefore a homeomorphism (4.9). Note also that $d(p, h(q)) = d(p, q) \cdot f(q) = a \cdot b/a = b = d(p, w)$, so $h(q) = w$.

On $A_3 \cup \{p\}$, h is the identity and if $x \in X - (A_3 \cup \{p\})$, then the ray $\overline{px}\rangle$ is not contained in $D(p, r)$ so there is a point y in $\overline{px}\rangle$ a distance r from p. The segment $\overline{py}$ maps into itself under h and both y and p are fixed so x is the image under h of some point. Thus $h[X] = X$.

Moreover, there is a homeomorphism carrying q to p because there is a point p' in $\overline{qp}\rangle - \overline{qp}$ and close to p which has a compact distance neighborhood contained in $D(p, r)$ and containing q in its interior.

For x and y two points of X there is a finite, simple chain of open distance neighborhoods with the first centered at x and the last at y. The above homeomorphisms and their inverses allow us to push x into the second distance neighborhood and then into center of it. Continuing this process a finite number of sets pushes x to y.

(c) $\to$ (d). Obvious.

(d) $\to$ (b). Assume that (d) holds and there is a point $q \in X$ such that q is the endpoint of a ray. Since X has finite dimension, there is a point $p \in X$ such that $X - \{p\}$ is not contractible (5.7). Let U and V be neighborhoods of q and p, respectively, and h be a homeomorphism of U onto V carrying q to p. The point q has arbitrarily small deleted distance neighborhoods that are contractible, so let D be one contained in U. $F[D]$ is a neighborhood of p, so there exists an r such that $D(p, r) \subset f[D]$, and $D(p, r)$ compact. $D(p, r) - \{p\}$ is a retract of $X - \{p\}$, so is a retract of $f[D] - \{p\} = f[D - \{q\}]$. Since $f[D - \{q\}]$ is contractible and since a retract of a contractible space is contractible [4, p. 26, 13.2], it follows that $D(p, r) - \{p\}$ is contractible. This is a contradiction because $X - \{p\}$ is then contractible (5.3).

THEOREM 8.8. *Let* (X, d) *be a locolly compact* SC-WR *metric space* *of dimension n and let $M \equiv \{x \in X: x$ is the endpoint of a ray$\}$. Then if $1 \leqq n \leqq 3$ or some point of X has an E^n-like neighborhood, then $(X - M)$ is homeomorphic to E^n.*

Proof. If $n \leqq 3$, then some points of X have an E^n-like neighborhood, so we may, in any case, choose $p \in X$ with an E^n-like neighborhood. There exist $r > 0$ and a set T such that $S(p, r) \subset T \subset D(p, r)$ and T homeomorphic to X. Under this homeomorphism, if x is not the endpoint of the $\overline{px}\rangle$, then x maps into $S(p, r)$, and if x is the endpoint, it maps into $D(p, r) - S(p, r)$. Thus $(X - M)$ is homeomorphic to $S(p, r)$. By Proposition 5.6, $D(p, r)$ is the cone over $S \equiv \{x \in X: d(p, x) = r\}$. M. Brown has shown [5] that if the cone over a set A is E^n-like at the vertex, then the (cone over A) $- A$ is homeomorphic to E^n. Thus, $S(p, r) = D(p, r) - S$ is homeomorphic to E^n and the theorem follows.

COROLLARY 8.9. *Let (X, d) be a locally compact* SC-WR *metric space of dimension n. X is homeomorphic to E^n if and only if* (1) *any condition of Proposition* 8.7 *holds, and* (2) $1 \leqq n \leqq 3$ *or some point of X has an E^n-like neighborhood.*

Note that if any condition of 8.7 holds, then X is locally homogeneous and by Theorem 3.6, X is an n-manifold if it contains an n-ball. We can change 8.9 slightly as follows:

COROLLARY 8.10. *A locally compact space of dimension n is homeomorphic to E^n if and only if it admits an* SC-WR, *externally convex metric and, for $n \geqq 4$, contains an n-ball.*

9. Compact spaces. Rolfsen [10] proved that a compact n-manifold (with boundary) which admits an SC-WR metric is homeomorphic to I^n when $n \geqq 6$. In this section it is shown that the result holds for $n = 4$ or 5 whenever there is a terminal point in the space.

DEFINITION 9.1. In a metric space (X, d), a point p is said to be a *terminal* point if for $x, y \in X$, $d(x, y) = d(x, p) + d(p, y)$ implies $p = x$ or $p = y$.

THEOREM 9.2. *Let (X, d) be a compact* SC-WR *metric space. If X is an n-manifold and has a terminal point, then X is homeomorphic to I^n.*

Proof. Let p be a point of $(X - \partial X)$ (∂X is the boundary of X), and let $M = \{x \in X: \overline{px}\rangle = \overline{px}\}$. As is evident from the proof of Proposition 8.2, M is the boundary of X in an embedding of X in

E^n, so $M = \partial X$. First, note that if $x \in M$ then the segment, $\overline{px}$, meets M only at x.

Take q to be a terminal point of X and since $q \in M$, q has a neighborhood, V, relative to M which is homeomorphic to E^{n-1}. Let $p_1, p_2, p_3, \cdots$ be a sequence of points of $\overline{pq} - \{q\}$ which converges to q. For each $i = 1, 2, 3, \cdots$ define the function $h_i: M \to M$ by the rule: $h_i(x)$ is the endpoint of the ray $\overline{xp_i}\rangle$. If $y = h_i(x)$, then $x \in \overline{yp_i}\rangle$ and by our earlier observation, $h_i(y) = x$. Thus h_i is one-to-one and onto for each i, and the continuity of h_i is easily established, so h_i is a homeomorphism of M onto itself.

Suppose that for each integer, i, there is a point $x_i \in M - (V \cup h_i[V])$. M is compact, so there is a point $x_0 \in M$ which is a limit of some subsequence of $x_1, x_2, \cdots$. We may assume that $\lim_{i\to\infty} x_i = x_0$. Let z be a limit point of $\{h_i(x_i): i = 1, 2, \cdots\}$ and note $z \notin V$ hence $z \neq q$. But since $\lim p_i = q$, $d(x_0, z) = d(x_0, q) + d(q, z)$ contradicting the choice of q. For some i then, $M = V \cup h_i[V]$.

The compact Hausdorff space M being the union of two open $(n-1)$-cells is an $(n-1)$-sphere. The set $\bigcup_{x \in M} \overline{px}$ is homeomorphic to I^n and is all of X, so the theorem is proved.

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