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COHOMOLOGICAL DIMENSION OF DISCRETE MODULES OVER PROFINITE GROUPS

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The main purpose of this note is to show that the finiteness of the cohomological dimension of a discrete module is closely related to the finiteness of its injective dimension. Moreover, a sufficient condition for the finiteness of the cohomological dimension is given. Both results are proved making a heavy use of the theory of cohomological triviality for finite groups.

The reader is referred to [3] for a treatment of profinite cohomology.

Throughout this note, G is a profinite group. As usual, the cohomology of G is denoted by $H(G, \ )$.

Recall that, if A is a discrete G-module, the infimum of the (set of) nonnegative integers r such that $H^n(S, A) = 0$, for any integer $n > r$ and any closed subgroup S of G, is called the *cohomological dimension* of A, and is denoted by $cd(G, A)$. If S is a closed subgroup of G, $H^n(S, A) \cong \varinjlim H^n(V, A)$, where V runs through all open subgroups of G containing S [3, Chap. I, Proposition 8, p. I-9]. Hence, if $H^n(V, A) = 0$ for every open subgroup V of G, then $H^n(S, A) = 0$ for every closed subgroup S of G.

In this paper, a discrete module is called *injective* only when it is injective in the corresponding category of discrete modules. If A is injective, it is well-known that $cd(G, A) = 0$, because, for instance, A is V-injective for all open subgroups V of G. Finally, recall that the *injective dimension* of A, denoted by $id(G, A)$, is the least length of an injective resolution of A.

The connection between cohomologically trivial modules over finite groups [2, Chap. IX, § 3, p. 148] and discrete modules of cohomological dimension zero over profinite groups was observed, and used, by Tate in his duality theory for profinite cohomology [3, Annexe au Chapitre I, p. I-79]. Tate's observation is quoted, for future reference, in the following.

LEMMA 1. *Let A be a discrete G-module. Then, $cd(G, A) = 0$ if, and only if, for every open, normal subgroup U of G, the G/U-module A^U is cohomologically trivial.*

Proof. See [3, Annexe au Chapitre I, Lemme 1, p. I-82]. Notice that G/U is a finite group, because G is compact and U is open.

The Nakayama-Tate criterion for cohomological triviality takes the following form, in the cohomology theory of profinite groups.

PROPOSITION 2. *Let A be a discrete G-module. If there exists a positive integer q such that $H^q(V, A) = H^{q+1}(V, A) = 0$ for all open subgroups V of G, then $cd(G, A) < q$.*

Proof. Since A embeds in an injective, whose cohomological dimension is zero, by repeated applications of dimension-shifting it suffices to consider the case $q = 1$. Let U be an open, normal subgroup of G. If V is any subgroup of G containing U, the Hochschild-Serre spectral sequence of the V/U-module A^U yields the exact sequence for low degrees

$$0 \longrightarrow H^1(V/U, A^U) \longrightarrow H^1(V, A) \longrightarrow H^1(U, A)^{V/U}$$
$$\longrightarrow H^2(V/U, A^U) \longrightarrow H^2(V, A) .$$

Since U is open, so is V, and thus, $H^1(U, A) = H^1(V, A) = H^2(V, A) = 0$. Therefore, $H^1(V/U, A^U) = H^2(V/U, A^U) = 0$, and applying the Nakayama-Tate criterion [2, Chap. IX, Théorème 8, p. 152], the G/U-module A^U is cohomologically trivial. By (1), the proof is complete.

The main result of this paper can be stated as follows.

THEOREM 3. *Let A be a discrete G-module, and let q be a positive integer. Then, $id(G, A) \leqq q$ if, and only if, $cd(G, A) \leqq q$ and $H^q(U, A)$ is a divisible abelian group for every open, normal subgroup U of G.*

Proof. Assume the assertion true for $q - 1$, with $q > 1$. If $id(G, A) \leqq q$, A has an injective resolution of length $\leqq q$, say

$$0 \longrightarrow A \xrightarrow{e} X_0 \xrightarrow{d_0} X_1 \longrightarrow \cdots \longrightarrow X_{q-1} \xrightarrow{d_{q-1}} X_q \longrightarrow 0 .$$

If $B = \operatorname{Coker} e$ and $f\colon X_0 \to B$ is the canonical morphism, the sequence of discrete G-modules

$$0 \longrightarrow A \xrightarrow{e} X_0 \xrightarrow{f} B \longrightarrow 0$$

is exact. Since $cd(G, X_0) = 0$ (injectivity of X_0), from the corresponding cohomology sequence it follows that

$$H^n(S, B) \cong H^{n+1}(S, A)$$

for any positive integer n and any closed subgroup S of G. Therefore, it is enough to prove that $cd(G, B) \leqq q - 1$, and that $H^{q-1}(U, B)$ is divisible for all open, normal subgroups U of G. By the induction hypothesis, this follows from showing that $id(G, B) \leqq q - 1$. In fact, if $e'\colon B \to X_1$ is the morphism induced by $d_0\colon X_0 \to X_1$, then $\operatorname{Ker} e' = 0$ and $\operatorname{Im} e' = \operatorname{Im} d_0$. Thus, the sequence

$$0 \longrightarrow B \xrightarrow{e'} X_1 \xrightarrow{d_1} X_2 \longrightarrow \cdots \longrightarrow X_{q-1} \xrightarrow{d_{q-1}} X_q \longrightarrow 0$$

is exact.

Reciprocally, if $cd(G, A) \leqq q$, let

$$0 \longrightarrow A \xrightarrow{g} Q \xrightarrow{h} C \longrightarrow 0$$

be an exact sequence of discrete G-modules, with Q injective. Then, $cd(G, C) \leqq q - 1$, because

$$H^n(S, C) \cong H^{n+1}(S, A)$$

for all positive integers n and all closed subgroups S of G. By the same reason, if $H^q(U, A)$ is divisible for every open, normal subgroup U of G, then so is $H^{q-1}(U, C)$. Hence, by induction, C admits an injective resolution of length $\leqq q - 1$, say

$$0 \longrightarrow C \xrightarrow{i} Y_0 \xrightarrow{d_0} Y_1 \longrightarrow \cdots \longrightarrow Y_{q-2} \xrightarrow{d_{q-2}} Y_{q-1} \longrightarrow 0 .$$

Since $\operatorname{Ker} ih = \operatorname{Ker} h$ and $\operatorname{Im} ih = \operatorname{Im} i$, the sequence

$$0 \longrightarrow A \xrightarrow{g} Q \xrightarrow{ih} Y_0 \xrightarrow{d_0} Y_1 \longrightarrow \cdots \longrightarrow Y_{q-2} \xrightarrow{d_{q-2}} Y_{q-1} \longrightarrow 0$$

is exact, and so $id(G, A) \leqq q$.

It remains to prove the assertion for $q = 1$.

Let

$$0 \longrightarrow A \longrightarrow X_0 \longrightarrow X_1 \longrightarrow 0$$

be an exact sequence of discrete G-modules, where X_0 and X_1 are injectives. Since $cd(G, X_0) = cd(G, X_1) = 0$, passing to cohomology it follows that $cd(G, A) \leqq 1$, and that the connecting operator $\partial_S \colon X_1^S \to H^1(S, A)$ is an epimorphism for all closed subgroups S of G. But, if D is any injective, discrete G-module and U is any open, normal subgroup of G, it is easy to check that D^U is an injective G/U-module, whence [2, Chap. IX, Lemme 7, p. 153] implies D^U is divisible. Therefore, as the image of a divisible group, $H^1(U, A)$ is divisible for all open, normal subgroups U of G.

Reciprocally, suppose $cd(G, A) \leqq 1$, and let

$$0 \longrightarrow A \longrightarrow Y_0 \longrightarrow Y_1 \longrightarrow 0$$

be an exact sequence of discrete G-modules, with Y_0 injective. Since $cd(G, Y_0) = 0$, taking cohomology it follows that $cd(G, Y_1) = 0$, and that the sequence of abelian groups

$$Y_0^S \longrightarrow Y_1^S \xrightarrow{\partial_S} H^1(S, A) \longrightarrow 0$$

is exact for all closed subgroups S of G. If U is an open, normal subgroup of G, $\operatorname{Ker}\partial_U$ is divisible, because so is Y_0^U. Therefore, if $\operatorname{Im}\partial_U = H^1(U, A)$ is divisible, then $\operatorname{Dom}\partial_U = Y_1^U$ is also divisible, and the proof is complete applying to Y_1 the following.

PROPOSITION 4. *Let A be a discrete G-module. If $cd(G, A) = 0$, and A^U is a divisible abelian group for every open, normal subgroup U of G, then A is injective.*

Proof. Recall that the category of discrete G-modules has injective envelopes for each of its objects. Since $(\boldsymbol{Z}[G/U])_U$, where U runs through all open, normal subgroups of G, is a family of generators, this result can be obtained by using a general theorem from category theory, due to Mitchell [1, Chap. III, Theorem 3.2, p. 89].

Let $f: A \to Q$ be an injective envelope of A (in the category of discrete G-modules). If $C = \operatorname{Coker} f$ and $g: Q \to C$ is the canonical morphism, the sequence of discrete G-modules

$$0 \longrightarrow A \xrightarrow{f} Q \xrightarrow{g} C \longrightarrow 0$$

is exact. Thus, if U is an open, normal subgroup of G, the sequence of G/U-modules

$$0 \longrightarrow A^U \xrightarrow{f^U} Q^U \xrightarrow{g^U} C^U \longrightarrow 0$$

is exact, because $cd(G, A) = 0$. Since Q^U is an injective G/U-module and $R \cap \operatorname{Im} f^U = R \cap \operatorname{Im} f$ for any sub-G/U-module R of Q^U (because, regarding R as a G-module, U operates trivially on R), $f^U: A^U \to Q^U$ is an injective envelope of A^U (in the category of G/U-modules). On the other hand, since $cd(G, A) = 0$, A^U is a cohomologically trivial G/U-module, by (1). Thus, A^U is G/U-injective [2, Chap. IX, Théorème 10, p. 154], and hence, $C^U = 0$ [1, Chap. III, Proposition 2.5, p. 88]. Since $C = \bigcup C^U$, $C = 0$, whence the result.

COROLLARY 5. *Let A be a discrete G-module, and let r be a nonnegative integer. If $cd(G, A) \leqq r$, then $id(G, A) \leqq r + 1$.*

Proof. Take $q = r + 1$ in (3).

This result can be applied to profinite groups of finite dimension, as follows.

COROLLARY 6. *Let r be a nonnegative integer. The following statements are true:*

(i) *If p is a prime number and $cd_p(G) \leqq r$, then $id(G, A) \leqq r + 1$ for all discrete G-modules A which are p-primary abelian groups.*

(ii) *If* $cd(G) \leqq r$, *then* $id(G, A) \leqq r + 1$ *for all discrete G-modules A which are torsion abelian groups.*

(iii) *If* $scd(G) \leqq r$, *then* $id(G, A) \leqq r + 1$ *for all discrete G-modules A.*

(iv) *If* $cd(G) \leqq r$, *then* $id(G, A) \leqq r + 2$ *for all discrete G-modules A.*

Proof. Applying [3, Chap. I, Proposition 14, p. I-20] and [3, Chap. I, Proposition 11, p. I-17], the following three equivalences are clear:

(i) $cd_p(G) \leqq r$ if, and only if, $cd(G, A) \leqq r$ for all p-primary, discrete G-modules A.

(ii) $cd(G) \leqq r$ if, and only if, $cd(G, A) \leqq r$ for all torsion, discrete G-modules A.

(iii) $scd(G) \leqq r$ if, and only if, $cd(G, A) \leqq r$ for all discrete G-modules A.

Finally, (6, iv) is clear by [3, Chap. I, Proposition 13, p. 1-19].

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