PACIFIC JOURNAL OF MATHEMATICS
Vol. 49, No. 2, 1973

ON THE MAXIMAL NUMBER OF LINEARLY INDEPENDENT REAL VECTORS ANNIHILATED SIMULTANEOUSLY BY TWO REAL QUADRATIC FORMS

FRANK UHLIG

For a nonsingular pair of real symmetric (r.s.) matrices S and T the maximal number m of lin. ind. vectors simultaneously annihilated by the associated quadratic forms is computed as a function of the real Jordan normal form of $S^{-1}T$. Conversely one can deduce which real Jordan normal form $S^{-1}T$ must have, if a specific m is the maximal number of such vectors. Furthermore, two new conditions are found that assure S and T to be simultaneously diagonalizable by a real congruence transformation.

First we introduce the notions of Jordan blocks, real Jordan normal form and the canonical pair form for pairs of r.s. matrices.

DEFINITION 0.1. A square matrix of the form

$$M = \begin{pmatrix} \lambda & e & & 0 \\ & \ddots & \ddots & \\ & & \ddots & e \\ 0 & & & \lambda \end{pmatrix}_{k \times k}$$

is called a *Jordan block of type* (A), if for $k \geqq 2$ we have $\lambda \in \boldsymbol{R}$ and $e = 1$, while for $k = 1$ we have $M = (\lambda)$ with $\lambda \in \boldsymbol{R}$. Such a matrix M is called a *Jordan block of type* (B), if for $k \geqq 4$ we have $\lambda = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $a, b \in \boldsymbol{R}$, $b \neq 0$ and $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, while for $k = 2$ we have $M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $a, b \in \boldsymbol{R}$, $b \neq 0$. Jordan blocks will also be denoted by $J(\lambda, k)$ and $J(a, b, k)$, respectively.

Now we can state the real Jordan normal form theorem (see, e.g., Kowalski [2], p. 248).

THEOREM 0.1. *Every real square matrix A is similar over the reals to a matrix $J = \operatorname{diag}(A_1, \cdots, A_l)$, in which each square block A_j corresponds to an eigenvalue λ_j of A. If this eigenvalue λ_j is real, the associated A_j is a Jordan block of type* (A); *if $\lambda_j = a + bi \notin \boldsymbol{R}$, then A_j is a Jordan block of type* (B). *This J is called the real Jordan normal form of A. It is uniquely determined by A, except for the order of its Jordan blocks.*

The final result to be quoted will be the canonical pair form theorem for nonsingular pairs of r.s. matrices, that is pairs S and T where S is nonsingular.

THEOREM 0.2. *Let S and T be a nonsingular pair of* r.s. *matrices. Let $S^{-1}T$ have real Jordan normal form* $\operatorname{diag}(J_1, \cdots, J_r, J_{r+1}, \cdots, J_m)$, *where $J_1, \cdots, J_r$ are Jordan blocks of type* (A) *corresponding to real eigenvalues of $S^{-1}T$ and $J_{r+1}, \cdots, J_m$ are Jordan blocks of type* (B) *for pairs of complex conjugate roots of $S^{-1}T$.*

Then S and T are simultaneously congruent by a real congruence transformation to

$$\operatorname{diag}(\varepsilon_1 E_1, \cdots, \varepsilon_r E_r, E_{r+1}, \cdots, E_m)$$

and

$$\operatorname{diag}(\varepsilon_1 E_1 J_1, \cdots, \varepsilon_r E_r J_r, E_{r+1} J_{r+1}, \cdots, E_m J_m) ,$$

respectively, where $\varepsilon_i = \pm 1$ and E_i denotes the square matrix $\begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$ *of the same size as J_i for $i = 1, \cdots, m$.*

Canonical forms for a pair of r.s. matrices go back to Weierstraß and Kronecker. A list of references can be found in Uhlig [3], Theorem 0.4.

NOTATION. For S symmetric we define $Q_S = \{x \in \boldsymbol{R}^n \mid x'Sx = 0\}$.

We will now state the main theorems that relate $m = \max\{k \mid$ there exist k lin. ind. vectors in $Q_S \cap Q_T\}$ to the real Jordan normal form of $S^{-1}T$.

THEOREM 1. *Let S and T be a nonsingular pair of* r.s. *$n \times n$ matrices. Let J be the real Jordan normal form of $S^{-1}T$. If*

(i) *J contains a Jordan block of dimension greater than* 3, *or*

(ii) *J contains two Jordan blocks of dimension* 3 *each, or*

(iii) *J contains one Jordan block of dimension* 3 *and one of dimension* 2, *or*

(iv) *$n > 3$ and J contains a Jordan block of dimension* 3 *and* 1-*dimensional blocks else, but not all eigenvalues of $S^{-1}T$ are the same, or*

(v) *J contains two* 2-*dimensional Jordan blocks which correspond to different eigenvalues of $S^{-1}T$ if both blocks are of type* (A), $\cdots$, *then $Q_S \cap Q_T$ contains n linearly independent vectors.*

THEOREM 2. *Let S and T be a nonsingular pair of* r.s. *matrices of dimension n. Let J be the real Jordan normal form of $S^{-1}T$. If*

(vi) *$n > 3$, J contains one 3-dimensional Jordan block, linear blocks else and all eigenvalues of $S^{-1}T$ are the same while inertia $S \neq (n-1, 1, 0), (1, n-1, 0)$; or*

(vii) *$n > 3$ and J contains $k \geqq 1$ identical 2-dimensional Jordan blocks $J(\lambda, 2)$ of type* (A), *linear blocks else for eigenvalues $\mu_i (i = 2k+1, \cdots, n)$ and the set*

$$\{\varepsilon_1, \cdots, \varepsilon_k, \varepsilon_i(\mu_i - \lambda) \mid i > 2k\}$$

contains positive as well as negative numbers, where the $\varepsilon_j = \pm 1$ are the constants in the canonical pair form of S and T (see Theorem 0.2), *or*

(viii) *$n > 3$, J contains one 2-dimensional block $J(a, b, 2)$ of type* (B) *and linear blocks else for eigenvalues μ_i, where not all μ_i are the same or*[1] *inertia $S \neq (n-1, 1, 0), (1, n-1, 0)$,*
then $Q_S \cap Q_T$ contains n linearly independent vectors. If

(vi) (a) *condition* (vi) *holds, except that inertia $S = (n-1, 1, 0)$ or $(1, n-1, 0)$, or*

(viii) (a) *condition* (viii) *holds, except that all real eigenvalues μ_i as defined in* (viii) *are the same and inertia $S = (n-1, 1, 0)$ or $(1, n-1, 0)$. Then $Q_S \cap Q_T$ contains a maximum of $n-1$ lin. indep. vectors only. If*

(vii) (a) *condition* (vii) *holds except that the set $\{\varepsilon_1, \cdots, \varepsilon_k, \varepsilon_i(\mu_i - \lambda) \mid i > 2k\}$ as defined in* (vii) *contains $r \geqq 0$ zeroes $\mu_{2k+1} - \lambda = \cdots = \mu_{2k+r} - \lambda = 0$ and only positive or only negative numbers else, and $\varepsilon_{2k+1} = \cdots = \varepsilon_{2k+r}$, then $Q_S \cap Q_T$ contains a maximum number of k lin. ind. vectors. If*

(vii) (b) *condition* (vii) (a) *holds except that not all ε_i are the same for $2k+1 \leqq i \leqq 2k+r$, then $Q_S \cap Q_T$ contains a maximum of $k+r$ lin. ind. vectors. If*

(ix) *$n > 1$ and S and T can be simultaneously diagonalized by a real congruence transformation, then the maximal number k of* lin. ind. *vectors in $Q_S \cap Q_T$ can be $k = 0, 2, \cdots, n$ depending on S and T.*

Theorem 3 will treat n-dimensional r.s. matrix pairs for $n \leqq 3$: The following lemma is useful for the proofs of Theorems 1 and 2.

LEMMA 1. *Let S and T be real symmetric matrices and A be a real nonsingular matrix.*
Then

$$\max\{k \mid \text{there exist } k \text{ lin. ind. vectors in } Q_S \cap Q_T\}$$
$$= \max\{k \mid \text{there exist } k \text{ lin. ind. vectors in } Q_{A'SA} \cap Q_{A'TA}\}\,.$$

[1] This "or" does not mean "either ... or".

This is obvious if one observes that $Q_{A'SA} = A^{-1}(Q_S)$ for nonsingular A.

Proof. (Theorem 1) In view of the above lemma we may assume without loss of generality that S and T are in canonical pair form:

$$S = \operatorname{diag}(\pm E_1, \cdots, \pm E_r, E_{r+1}, \cdots, E_m) ,$$
$$T = \operatorname{diag}(\pm E_1J_1, \cdots, \pm E_rJ_r, E_{r+1}J_{r+1}, \cdots, E_mJ_m) ,$$

where $J_1, \cdots, J_r$ are Jordan blocks of type (A) and $J_{r+1}, \cdots, J_m$ are of type (B).

Having S and T in this form is very advantageous. For then we have $(e_i'Se_i)^2 + (e_i'Te_i)^2 \neq 0$ for at most $r + 2(m - r)$ unit vectors e_i. The reason is as follows:

For the Jordan block $J_1 = J(\lambda, k)$ of type (A) we have: if k is even:

$$e'_{k/2+1}\, Ee_{k/2+1} = 0$$

and

$$e'_{k/2+1}\, EJ(\lambda, k)e_{k/2+1} = \lambda ,$$

while for all other $i \leqq k$: $e_i'Ee_i = e_i'EJ(\lambda, k)e_i = 0_j$, if k is odd:

$$e'_{k+1/2}Ee_{k+1/2} = 1$$

and

$$e'_{k+1/2}EJ(\lambda, k)e_{k+1/2} = \lambda ,$$

while for all other $i \leqq k$: $e_i'Ee_i = e_i'EJ(\lambda, k)e_i = 0$. For the Jordan block $J_1 = J(a, b, k)(b \neq 0)$ of type (B) we have: if k is divisible by 4: $e_i'Ee_i = e_i'EJ(a, b, k)e_i = 0$ for all $i \leqq k$; while for a k not divisible by 4 we have

$$e'_{k/2}Ee_{k/2} = 0,\ e'_{k/2}EJ(a, b, k)e_{k/2} = b;\ e'_{k/2+1}Ee_{k/2+1} = 0 ,$$
$$e'_{k/2+1}EJ(a, b, k)e_{k/2+1} = -b \quad \text{and} \quad e_i'Ee_i = e_i'EJ(a, b, k)e_i = 0$$

for all other $i \leqq k$.

The same argument holds for each of the Jordan blocks. So there are at most $r + 2(m - r)$ unit vectors not simultaneously annihilated by the two quadratic forms $x'Sx$ and $x'Tx$ if S and T are in canonical pair form. For all i such that $e_i \notin Q_S \cap Q_T$ we will exhibit lin. ind. vectors $y_i \in Q_S \cap Q_T$ that have a nonzero ith component and hence are also lin. ind. of all e_i with $e_i \in Q_S \cap Q_T$. Then Theorem 1 is proved: There are n lin. ind. vectors in $Q_S \cap Q_T$.

The remainder of this proof will consist of finding these vectors y_i, one for each Jordan block of type (A), two for each Jordan block of type (B) of dimension not divisible by 4 in each of the cases (i), $\cdots$, (v).

From now on we will in general assume that the Jordan blocks

of $S^{-1}T$ mentioned in (i), $\cdots$, (v) appear in the first diagonal positions. Before starting on the individual cases we express the quadratic forms corresponding to S and T by only singling out the first block here: If a Jordan block $J(\lambda, k) = J_1$ of type (A) appears first, let us look at the two quadratic forms $F(x) = x'Sx$ and $G(x) = x'Tx$:
For

$$S = \operatorname{diag}(\pm E_1, \cdots, \pm E_m) \quad \text{and} \quad x = (x_1, \cdots, x_n)$$

we have

$$F(x) = \pm h(x) + f(x) ,$$

where

$$h(x) = x' \operatorname{diag}(E_1, 0, \cdots, 0)x = \sum_{i+j=k+1} x_i x_j$$

and

$$f(x) = x' \operatorname{diag}(0, \pm E_2, \cdots, \pm E_m)x$$

is a quadratic form involving $x_{k+1}, \cdots, x_n$ only.
For

$$T = \operatorname{diag}(\pm E_1 J_1, \cdots, \pm E_m J_m)$$

we have

$$G(x) = \pm (\lambda h(x) + e(x)) + g(x)$$

where h is as above,

$$e(x) = \sum_{i+j=k+2} x_i x_j \quad \text{for} \quad i, j \leqq k$$

and

$$g(x) = x' \operatorname{diag}(0, \pm E_2 J_2, \cdots, \pm E_m J_m)x$$

involves $x_{k+1}, \cdots, x_n$ only.

Now $F(x) = 0$ iff $f(x) = \mp h(x)$. And by definition $x \in Q_S \cap Q_T$ iff $F(x) = G(x) = 0$ hence iff

$$(1) \qquad \pm e(x) + g(x) - \lambda f(x) = 0 \quad \text{and} \quad F(x) = 0 .$$

If a Jordan block $J(a, b, k) = J_1 (b \neq 0)$ of type (B) appears first in $S^{-1}T$, then we define $F(x) = x'Sx = h(x) + f(x)$ with h and f as above and $G(x) = x'Tx = ah(x) + bt(x) + u(x) + g(x)$, where h and g are as above and

$$u(x) = \sum_{\substack{i+j=k+3 \\ i,j \leqq k}} x_i x_j, \quad \text{while} \quad t(x) = \sum_{\substack{i+j=k \\ i,j \text{ odd}}} x_i x_j - \sum_{\substack{i+j=k+2 \\ i,j \text{ even} \\ i,j \leqq k}} x_i x_j .$$

Thus in this case $x \in Q_S \cap Q_T$ iff $F(x) = G(x) = 0$, hence iff

$$(2) \qquad bt(x) + u(x) + g(x) - af(x) = 0 \quad \text{and} \quad F(x) = 0 \, .$$

(i): Assume (i) holds with a Jordan block $J(\lambda, k)$ of type (A) for $k \geqq 4$. Then from above there is an i, $2 < i < k$, such that $e_i \notin Q_S \cap Q_T$. For this index i we define $\alpha_i, \beta_i \in \boldsymbol{R}$ and $y_i = \alpha_i e_1 + \beta_i e_2 + e_i + e_k$ such that (1) holds: $\pm\, e(y_i) + g(y_i) - \lambda f(y_i) = \pm\,(2\beta_i + e(e_i)) = 0$ determines β_i and $F(y_i) = 0$ determines α_i.

For $i > k$ such that $e_i \notin Q_S \cap Q_T$ and $g(e_i) - \lambda f(e_i) = 0$, we define the vector $y_i = \alpha_i e_1 + e_k + e_i$, where α_i is such that $F(y_i) = 0$. In the case that $g(e_i) - \lambda f(e_i) \neq 0$ we define $y_i = \alpha_i e_1 + \beta_i e_2 + e_k + e_i$, where $\alpha_i, \beta_i \in \boldsymbol{R}$ are such that (1) holds: $+\, 2\beta_i + g(e_i) - \lambda f(e_i) = 0$ defines β_i and $F(y_i) = 0$ defines α_i.

Next assume (i) holds for a Jordan block $J(a, b, k)$ of type (B) for $k = 2l \geqq 4$.

First assume $k = 2l$ is divisible by 4. Then $e_i \notin Q_S \cap Q_T$ implies $i > k$ as pointed out above. For such an i define $y_i = \alpha_i e_{l-1} + \beta_i e_l + e_{l+1} + e_i$ where $\alpha_i, \beta_i \in \boldsymbol{R}$ are such that (2) holds. When checking (2), note that l is even, if k is divisible by 4. $2b\alpha_i + g(e_i) - af(e_i) = 0$ defines α_i and $2\beta_i + h(e_i) = 0$ defines β_i.

Now assume $k = 2l$ is not divisible by 4. Then l is odd and we know that both $e_l, e_{l+1} \notin Q_S \cap Q_T$ from the above. If we define

$$y_l = e_l - \frac{b}{2} e_{l+3} \quad \text{and} \quad y_{l+1} = e_{l+1} + \frac{b}{2} e_{l+2} \, ,$$

then (2) holds for these two vectors. For $i > k$ such that $e_i \notin Q_S \cap Q_T$ we define as before for the real case $y_i = \alpha_i e_1 + e_k + e_i$ if $g(e_i) - \lambda f(e_i) = 0$ and $y_i = \alpha_i e_1 + \beta_i e_2 + e_k + e_i$ otherwise. This proves (i) of Theorem 1.

(ii): Assume J contains two Jordan block of dimensions 3 each. Then these must be Jordan blocks of type (A); $J(\lambda, 3)$ and $J(\mu, 3)$ for $\lambda, \mu \in \boldsymbol{R}$. Define for $x = (x_1, \cdots, x_n)$:

$$F(x) = x'Sx = \varepsilon(2x_1x_3 + x_2^2) + \delta(2x_4x_6 + x_5^2) + f(x)$$

and

$$(3) \qquad \begin{aligned} G(x) = x'Tx &= \varepsilon(\lambda(2x_1x_3 + x_2^2) + 2x_2x_3) \\ &\quad + \delta(\mu(x_4x_6 + x_5^2) + 2x_5x_6) + g(x) \, , \end{aligned}$$

where f and g are quadratic forms not involving $x_1, \cdots, x_6$ and $\varepsilon, \delta = \pm 1$, independently from the canonical pair form.

Now $e_2, e_5 \notin Q_S \cap Q_T$. And for these indices define the vectors $y_2 = -1/2e_1 - \delta\varepsilon e_2 + e_3 - 1/2e_4 + e_5 + e_6$ and $y_5 = -1/2e_1 - \delta\varepsilon e_2 + e_3 + 1/2e_4 - e_5 - e_6$. They are lin. ind. and satisfy $F(y_i) = G(y_i) = 0$ in (3). For $i > 6$ such that $e_i \notin Q_S \cap Q_T$ we define

$$y_i = \alpha_i e_1 + \beta_i e_2 + e_3 + e_i ,$$

where β_i is chosen such that

$$2\varepsilon\beta_i + g(e_i) - \lambda f(e_i) = 0 ,$$

and α_i is such that

$$F(y_i) = \varepsilon(2\alpha_i + \beta_i^2) + f(e_i) = 0 .$$

Then $G(y_i) = 0$, too.

This completes (ii).

(iii): Here again the 3-dimensional Jordan block has to be of type (A): $J(\lambda, 3)$, while the 2-dimensional block can be of either type. Let for $x = (x_1, \cdots, x_n)$,

$$F(x) = x'Sx = \varepsilon(2x_1x_3 + x_2^2) + \delta(2x_4x_5) + f(x)$$

and

$$\begin{aligned} G(x) = x'Tx &= \varepsilon(\lambda(2x_1x_3 + x_2^2) + 2x_2x_3) \\ &+ \delta\begin{Bmatrix} (2\mu x_4x_5 + x_5^2) \\ (2ax_4x_5 + b(x_4^2 - x_5^2)) \end{Bmatrix} + g(x) \quad \begin{matrix} \text{in case of (A)} \\ \text{in case of (B)} \end{matrix} \end{aligned}$$

where $\delta, \varepsilon = \pm 1$ from the canonical pair form and f and g do not involve the first five components. If the 2-dimensional Jordan block in question is of type (A), then for $i \leqq 5$ we have $e_i \notin Q_S \cap Q_T$ exactly for $i = 2, 5$, while for a Jordan block of type (B) those indices are $i = 2, 4, 5$.

In case of (A) define

$$y_2 = \delta\varepsilon e_1 + e_2 - \frac{\delta\varepsilon}{2}e_3 + e_5$$

$$y_5 = \delta\varepsilon e_1 + e_2 - \frac{\delta\varepsilon}{2}e_3 - e_5$$

and one has $y_2, y_5 \in Q_S \cap Q_T$.

In case of a 2-dimensional block $J(a, b, 2)$, $b \neq 0$ of type (B), define

$$y_2 = -\frac{\varepsilon}{b}e_1 + e_2 + \frac{b}{2}\varepsilon e_3 + e_5$$

$$y_4 = +\frac{\varepsilon}{b}e_1 + e_2 - \frac{b}{2}\varepsilon e_3 + e_4$$

$$y_5 = -\frac{\varepsilon}{b}e_1 + e_2 + \frac{b}{2}\varepsilon e_3 - e_5 .$$

Then $y_2, y_4, y_5 \in Q_S \cap Q_T$.

For $i > 5$ such that $e_i \notin Q_S \cap Q_T$, define $y_i = \alpha_i e_1 + \beta_i e_2 + e_3 + e_i$, where $\alpha_i, \beta_i \in \boldsymbol{R}$ are such that $2\varepsilon\beta_i + g(e_i) - \lambda f(e_i) = 0$ and $F(y_i) = 0$. This concludes part (iii).

(iv): Here we have $F(x) = x'Sx = \varepsilon(2x_1x_3 + x_2^2) + f(x)$ and $G(x) = x'Tx = \varepsilon(\lambda(2x_1x_3 + x_2^2) + 2x_2x_3) + g(x)$ and $F(x) = G(x) = 0$ iff

$$(4) \qquad \varepsilon 2x_2x_3 + g(x) - \lambda f(x) = 0 \quad \text{and} \quad F(x) = 0 \; .$$

By assumption all but the first Jordan block $J(\lambda, 3)$ in $S^{-1}T$ are 1-dimensional blocks $J(\mu_i, 1)$. We assume $n > 3$, so there exists an $i_0 > 3$ such that $g(e_{i_0}) - \lambda f(e_{i_0}) \neq 0$, for $g(e_i) - \lambda f(e_i) = \pm(\mu_i - \lambda) = 0$ for all $i > 3$ contradicts our assumption.

Now $e_2 \notin Q_S \cap Q_T$ and we define $y_2 = \alpha_2 e_1 + \beta_2 e_2 + e_3 + e_{i_0}$, where $\beta_2 \neq 0$ is such that $2\varepsilon\beta_2 + g(e_{i_0}) - \lambda f(e_{i_0}) = 0$ and α_2 is such that $F(y_2) = 0$. For all $i > 3$ we have $e_i \notin Q_S \cap Q_T$ and we define $y_{i_0} = -\alpha_2 e_1 - \beta_2 e_2 - e_3 + e_{i_0}$ and $y_i = \alpha_i e_1 + \beta_i e_2 + e_3 + e_i$ for all other $i > 3$, where the α's and β's are chosen such that (4) holds for all y_i. These n vectors y_i are lin. ind.

(v): Now only (v) remains to be proved. Let us first assume that the two 2-dimensional Jordan blocks in question are both of type (A): $J(\lambda, 2), J(\mu, 2)$, where by assumption $\lambda \neq \mu$. Then $F(x) = x'Sx = \varepsilon 2x_1x_2 + \delta 2x_3x_4 + f(x)$ and $G(x) = x'Tx = \varepsilon(2\lambda x_1x_2 + x_2^2) + \delta(2\mu x_3x_4 + x_4^2) + g(x)$ where $\varepsilon, \delta = \pm 1$ and f and g do not involve the first four components of x. Then $F(x) = G(x) = 0$ is equivalent to

$$(5) \quad F(x) = 0 \quad \text{and} \quad 2\delta(\mu - \lambda)x_3x_4 + \varepsilon x_2^2 + \delta x_4^2 + g(x) - \lambda f(x) = 0 \; .$$

Now if $e_i \notin Q_S \cap Q_T$, then $i = 2$ or $i = 4$, unless $i > 4$. We define

$$y_2 = \alpha e_1 + 2e_2 + \beta e_3 - e_4$$
$$y_4 = \alpha e_1 + 2e_2 + \beta e_3 + e_4$$

and

$$y_i = \alpha_i e_1 + \gamma_i e_2 + \beta_i e_3 + e_4 + e_i$$

for all $i > 4$ with $e_i \notin Q_S \cap Q_T$. Here $\gamma_i \neq 0$ are chosen such that $\varepsilon\gamma_i^2 + \delta + g(e_i) - \lambda f(e_i) \neq 0$ while the α's and β's are chosen such that (5) holds.

Next assume, the two 2-dimensional blocks are both of type (B): $J(a, b, 2), J(c, d, 2)$ where $b, c \neq 0$.

Then $F(x)$ is as above with $\varepsilon = \delta = 1$ while

$$G(x) = x'Tx = 2ax_1x_2 + 2cx_3x_4 + b(x_1^2 - x_2^2) + d(x_3^2 - x_4^2) + g(x) \; ,$$

and $F(x) = G(x) = 0$ is equivalent to

$$(6)\quad \begin{aligned} F(x) = 0 \quad \text{and} \quad 2(c-a)x_3x_4 + b(x_1^2 - x_2^2) \\ + d(x_3^2 - x_4^2) + g(x) - af(x) = 0 \,. \end{aligned}$$

Here we have $e_i \notin Q_S \cap Q_T$ for all $i \leqq 4$.

If $bd > 0$ we define the following four lin. ind. vectors

$$\begin{aligned} y_1 &= \alpha e_1 + \beta e_4 \,, & y_2 &= \alpha e_1 - \beta e_4 \,, \\ y_3 &= \alpha e_2 + \beta e_3 \,, & y_4 &= \alpha e_2 - \beta e_3 \,, \end{aligned}$$

where $\alpha, \beta \neq 0$ are such that $b\alpha^2 - d\beta^2 = 0$ and thus (6) holds for all y_i, $i \leqq 4$.

In $bd < 0$, we define y_i as follows:

$$\begin{aligned} y_1 &= \alpha e_1 + \beta e_3 \,, & y_2 &= \alpha e_1 - \beta e_3 \,, \\ y_3 &= \alpha e_2 + \beta e_4 \,, & y_4 &= \alpha e_2 - \beta e_4 \,, \end{aligned}$$

where $\alpha, \beta \neq 0$ satisfy $b\alpha^2 + d\beta^2 = 0$ such that all four y_i satisfy (6) again.

For indices $i > 4$ such that $e_i \notin Q_S \cap Q_T$ we define the corresponding vector y_i as follows:

If $f(e_i) = 0$ and $bd > 0$, let $y_i = \alpha_i e_1 + \beta_i e_4 + e_i$, where α_i, β_i are chosen such that $b\alpha_i^2 - d\beta_i^2 = -g(e_i)$. If $f(e_i) = 0$ and $bd < 0$, let $y_i = \alpha_i e_1 + \beta_i e_3 + e_i$, where $\alpha_i, \beta_i \in \boldsymbol{R}$ such that $b\alpha_i^2 + d\beta_i^2 = -g(e_i)$. If $f(e_i) \neq 0$ and $g(e_i) - af(e_i) = 0$, then let $y_i = \alpha_i e_1 + \beta_i e_2 + e_i$ where $|\alpha_i| = |\beta_i|$ such that y_i satisfies (6). If $f(e_i) \neq 0$ and $(g(e_i) - af(e_i))d > 0$, let $y_i = \alpha_i e_1 + \beta_i e_2 + \gamma_i e_4 + e_i$, where $|\alpha_i| = |\beta_i|$ and γ_i are chosen such that (6) holds. If $f(e_i) \neq 0$ and $(g(e_i) - af(e_i))d < 0$, let $y_i = \alpha_i e_1 + \beta_i e_2 + \gamma_i e_3 + e_i$, with $\alpha_i, \beta_i, \gamma_i$ chosen to satisfy (6).

Finally we prove (v) for a Jordan block of type (A) and one of type (B): $J(\lambda, 2), J(a, b, 2)$. Then $F(x)$ is as above with $\varepsilon = \pm 1$, $\delta = 1$ while $G(x) = x'Tx = \varepsilon(2\lambda x_1x_2 + x_2^2) + 2ax_3x_4 + b(x_3^2 - x_4^2) + g(x)$ where $g(x)$ does not involve $x_1, \cdots, x_4$. And $F(x) = G(x) = 0$ is equivalent to:

$$(7)\quad \begin{aligned} F(x) = 0 \quad \text{and} \quad 2(a - \varepsilon\lambda)x_3x_4 + \varepsilon x_2^2 + b(x_3^2 - x_4^2) \\ + g(x) - \lambda f(x) = 0 \,. \end{aligned}$$

If $e_i \notin Q_S \cap Q_T$, then $i = 2, 3, 4$ or $i > 4$. We define y_2 and y_3 first:

$$\begin{aligned} y_2 &= e_2 + \beta e_3 + \gamma e_4 \\ y_3 &= e_2 - \beta e_3 - \gamma e_4 \,, \end{aligned}$$

where $\beta = \sqrt{-\varepsilon/b}$, $\gamma = 0$, if $\varepsilon \cdot b < 0$ and $\beta = 0$, $\gamma = \sqrt{\varepsilon/b}$, if $\varepsilon b > 0$. Then e_1, y_2, and y_3 are all lin. ind. and satisfy (7).

If y_4 has all of its first four components nonzero it will be lin. ind. of e_1, y_2, y_3 and all e_i for $i > 4$. So let $y_4 = \alpha e_1 + \beta e_2 + \gamma e_3 +$

ηe_4 where $\alpha, \beta, \gamma, \eta$ are chosen as follows:

If $a - \varepsilon\lambda = 0$, take $\gamma = 1, \eta = 2, \beta = \sqrt{3b\varepsilon}$, if $\varepsilon b > 0$ and $\alpha \neq 0$ such that $F(y_4) = 0$; but if $\varepsilon b < 0$, choose $\gamma = 2, \eta = 1, \beta = \sqrt{-3b\varepsilon}$ and α as above, and y_4 satisfies (7).

If $a - \varepsilon\lambda \neq 0$, choose $\eta \neq 0, \gamma = 1/\eta$ such that

$$2(a - \varepsilon\lambda) + b\left(\frac{1}{\eta^2} - \eta^2\right) < 0 \qquad \text{if } \varepsilon = 1$$

and

$$2(a - \varepsilon\lambda) + b\left(\frac{1}{\eta^2} - \eta^2\right) > 0 \qquad \text{if } \varepsilon = -1 .$$

Then choose $\beta \neq 0$ such that the second equation in (7) holds and after letting $\alpha = -\varepsilon/\beta$ the vector y_4 again satisfies (7). For $i > 4$ define $y_i = \alpha e_1 + e_2 + \beta e_3 + \gamma e_4 + e_i$ where $\alpha \in \boldsymbol{R}$ and either $\beta = 0$ or $\gamma = 0$ as before in such a way that (7) holds for each y_i, $i > 4$. This completes the proof of Theorem 1.

We now go on to prove Theorem 2

Proof. (Theorem 2) We use the notation of the previous proof

(vi), (vi)(a): Let (vi) or (vi)(a) hold. Then the 3 dimensional Jordan block is of type (A): $J(\lambda, 3)$. And we have with $x = (x_1, \cdots, x_n)$

$$F(x) = x'Sx = \varepsilon_1(2x_1x_3 + x_2^2) + \sum_{i=4}^{n} \varepsilon_i x_i^2$$

and

$$G(x) = x'Tx = \varepsilon_1(\lambda(2x_1x_3 + x_2^2) + 2x_2x_3) + \sum_{i=4}^{n} \varepsilon_i \lambda x_i^2 ,$$

where $\varepsilon_i = \pm 1$. Hence $F(x) = G(x) = 0$ is equivalent to $F(x) = 0$ and $x_2x_3 = 0$.

If all ε_i are the same, then, since inertia $E_{3\times3} = (2, 1, 0)$, we have inertia $S = (n - 1, 1, 0)$ or $(1, n - 1, 0)$ and (vi)(a) would hold. But let us first assume (vi) holds. Then for some $l \geqq 4$ we must have $\varepsilon_1 \cdot \varepsilon_l < 0$. Clearly $e_1, e_3 \in Q_S \cap Q_T$ and for the other indices we define:

$$y_2 = e_2 + e_l$$

$$y_i = \frac{\varepsilon_1\varepsilon_i}{2}e_1 - e_3 + e_i \qquad \text{for } i \geqq 4 .$$

Then e_1, e_3, y_2 and $y_i (i \geqq 4)$ are in $Q_S \cap Q_T$ and are lin. ind.

If (vi) (a) holds, then $F(x) = 0$ and $x_2 x_3 = 0$ implies $x_2 = 0$, such that we cannot find a vector $y_2 \in Q_S \cap Q_T$ with a nonzero second component. Hence $Q_S \cap Q_T$ contains at most $(n-1)$ lin. ind. vectors. But e_1, e_3, $y_i (i \geqq 4)$ defined above are linearly independent and belong to $Q_S \cap Q_T$. This proves (vi) and (vi)(a).

(vii), (vii)(a), (vii)(b): We define

$$F(x) = x'Sx = 2 \sum_{i=1}^{k} \varepsilon_i x_{2i-1} x_{2i} + \sum_{i=2k+1}^{n} \varepsilon_i x_i^2$$

and

$$G(x) = x'Tx = 2\lambda \sum_{i=1}^{k} \varepsilon_i x_{2i-1} x_{2i} + \sum_{i=1}^{k} \varepsilon_i x_{2i}^2 + \sum_{i=2k+1}^{n} \varepsilon_i \mu_i x_i^2 \, ,$$

where $\varepsilon_i = \pm 1$. Thus $F(x) = G(x) = 0$ is equivalent to $F(x) = 0$ and

$$\sum_{i=1}^{k} \varepsilon_i x_{2i}^2 + \sum_{i=2k+1}^{n} \varepsilon_i (\mu_i - \lambda) x_i^2 = 0 \, . \tag{8}$$

Assuming (vii) holds, then the quadratic form in (8) is indefinite, so there must exist an index l such that

$$\varepsilon_1 \varepsilon_l (\mu_l - \lambda) < 0 \qquad \text{for some } l \geqq 2k+1$$

or such that

$$\varepsilon_1 \varepsilon_{l/2} < 0 \qquad \text{for even } l \leqq 2k \, .$$

We define

$$\begin{aligned} y_1 &= e_1 \\ y_2 &= \alpha_2 e_1 + \beta_2 e_2 + e_l \qquad \text{for } \beta_2 \neq 0 \, , \\ y_l &= \alpha_2 e_1 + \beta_2 e_2 - e_l \end{aligned}$$

and

$$y_i = \alpha_i e_1 + \beta_i e_2 + \gamma_i e_l + e_i \qquad \text{for } i \neq 1, 2, l, \beta_i \neq 0 \, ,$$

where β_i and γ_i are chosen such that y_i satisfies (8), while α_i are chosen such that $F(y_i) = 0$. This proves (vii).

To prove (vii)(a) and (vii)(b) assume now that the quadratic form in (8) is semidefinite and that the symmetric matrix corresponding to the quadratic form in (8) has rank $n - k - r$, where the r zeroes among the $\varepsilon_i(\mu_i - \lambda)$ occur for the indices $i = 2k+1, \cdots, 2k+r$. Then the only unit vectors satisfying (8) are $e_1, e_3, \cdots, e_{2k-1}, e_{2k+1}, \cdots, e_{2k+r}$. And clearly $e_1, e_3, \cdots, e_{2k-1} \in Q_S \cap Q_T$ in either of the cases (vii)(a) or (vii)(b).

In case of (vii)(a) exactly $e_1, e_3, \cdots, e_{2k-1} \in Q_S \cap Q_T$, because the quadratic form in r variables

$$\sum_{i=2k+1}^{2k+r} \varepsilon_i x_i^2$$

appearing in F is definite and $F(x) \neq 0$ for all x with $x_i \neq 0$ for $2k < i \leqq 2k + r$. So in this case we conclude that $Q_S \cap Q_T$ contains a maximum of k lin. ind. vectors.

In case of (vii)(b)

$$\sum_{i=2k+1}^{2k+r} \varepsilon_i x_i^2 \tag{9}$$

is an indefinite quadratic form and besides $e_1, e_3, \cdots, e_{2k-1}$, r more lin. indep. vectors $y_1, \cdots, y_r$ can be found that satisfy $F(x) = 0$ and (8): Choose y_i as follows. Since (9) is indefinite, there are indices $2k < l, j \leqq 2k + r$ with $\varepsilon_l = 1, \varepsilon_j = -1$. Then define for $2k < i \leqq 2k + r$, $i \neq l, j$:

$$y_i = e_j + e_i \qquad \text{if } F(e_i) = 1$$

and

$$y_i = e_l + e_i \qquad \text{if } F(e_i) = -1$$

while we set

$$y_l = e_j + e_l$$

and

$$y_j = e_j - e_l \,.$$

This proves (vii)(b).

(viii): Here we define

$$F(x) = x'Sx = 2x_1x_2 + \sum_{i=3}^{n} \varepsilon_i x_i^2$$

and

$$G(x) = x'Tx = 2ax_1x_2 + b(x_1^2 - x_2^2) + \sum_{i=3}^{n} \varepsilon_i \mu_i x_i^2 \,.$$

So $F(x) = G(x) = 0$ is equivalent to

$$F(x) = 0 \quad \text{and} \quad b(x_1^2 - x_2^2) + \sum_{i=3}^{n} \varepsilon_i(\mu_i - a)x_i^2 = 0 \,. \tag{10}$$

Now unless (viii)(a) holds, not all μ_i or[2] not all ε_i are the same for $i \geqq 3$. So for some pair of indices $i, j \geqq 3$ we must have $\mu_i \neq \mu_j$ or[2] $\varepsilon_i \neq \varepsilon_j$. After a suitable index permutation we may start the proof assuming that $\mu_3 \neq \mu_4$ or[2] $\varepsilon_3 \neq \varepsilon_4$ already.

We define $y_1 = \alpha_3 e_1 + \beta_3 e_2 - e_3$, $y_2 = \alpha_4 e_1 + \beta_4 e_2 - e_4$ and $y_i = \alpha_i e_1 + \beta_i e_2 + e_i$ for $i \geqq 3$, where the α_i, β_i are chosen to satisfy

(11) $b(\alpha_i^2 - \beta_i^2) + \varepsilon_i(\mu_i - a) = 0$ and $2\alpha_i\beta_i + \varepsilon_i = 0$ for each i.

Then the vectors y_i for $i \leqq n$ are lin. ind. iff

$$\det(y_1, \cdots, y_n) = \begin{vmatrix} \alpha_3 & \beta_3 & -1 & 0 \\ \alpha_4 & \beta_4 & 0 & -0 \\ \alpha_3 & \beta_3 & 1 & 0 \\ \alpha_4 & \beta_4 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2\alpha_3 & 2\beta_3 & 0 & 0 \\ 2\alpha_4 & 2\beta_4 & 0 & 0 \\ \alpha_3 & \beta_3 & 1 & 0 \\ \alpha_4 & \beta_4 & 0 & 1 \end{vmatrix} \neq 0 .$$

So the n vectors y_i are lin. dep. iff for the 2-vectors we have

$$(\alpha_3, \beta_3) = d(\alpha_4, \beta_4) \tag{12}$$

for some real coefficient d. Now (12) holds only if $d = \pm 1$, for (12) implies $\alpha_3\beta_3 = d^2\alpha_4\beta_4$ and we know $\alpha_3\beta_3 = -\varepsilon_3/2$, since $F(y_3) = 0$ and $\alpha_4\beta_4 = -\varepsilon 4/2$, since $F(y_4) = 0$ and thus $d = \pm 1$.

If $d = 1$, then by (11) we have $\varepsilon_3 = \varepsilon_4$ and hence by assumption $\mu_3 \neq \mu_4$ which contradicts (11). If $d = -1$, then $\varepsilon_3 = -\varepsilon_4$ and $\alpha_3\beta_3 = -\alpha_4\beta_4$ by (11), contradicting (12).

Thus we found that the n vectors y_i in $Q_S \cap Q_T$ are lin. ind. in case of (viii).

If (viii)(a) holds, $\mu_i = \mu$ and $\varepsilon_i = \varepsilon$ for all i. We define for $x = (x_1, \cdots, x_n)$

$$F(x) = x'Sx = 2x_1x_2 + \varepsilon \sum_{i=3}^{n} x_i^2 \tag{13}$$

and

$$G(x) = x'Tx = 2ax_1x_2 + b(x_1^2 - x_2^2) + \varepsilon\mu \sum_{i=3}^{n} x_i^2 .$$

And $F(x) = G(x) = 0$ is equivalent to

$$F(x) = 0 \quad \text{and} \quad b(x_1^2 - x_2^2) + \varepsilon(\mu - a) \sum_{i=3}^{n} x_i^2 = 0 . \tag{14}$$

We define the following $n - 1$ lin. ind. vectors

$$y_i = \alpha e_1 + \beta e_2 + e_i \qquad \text{for } i \geqq 3 ,$$
$$y_2 = \alpha e_1 + \beta e_2 - e_3 ,$$

[2] This "or" does not mean "either . . . or".

where α, β are chosen such that $F(y_i) = G(y_i) = 0$ for all i. Such numbers α, β exist, since they can be chosen as the intersection of the following two hyperbolas

$$2\alpha\beta + \varepsilon = 0; \qquad \alpha^2 - \beta^2 = -\frac{\varepsilon}{b}(\mu - a) .$$

Now any $w = (\beta_1, \cdots, \beta_n) \in Q_S \cap Q_T$ satisfies (14). We are going to show that if $0 \neq w \in Q_S \cap Q_T$ then the 2-vector (β_1, β_2) can be written as $\pm \|\hat{x}\|(\alpha, \beta)$ with α, β as chosen above and $\hat{x} = (0, 0, \beta_3, \cdots, \beta_n)$. Now if $\|\hat{x}\| = 0$, i.e., $\beta_i = 0$ for all $i \geqq 3$, then by (14) $\beta_1\beta_2 = 0 = \beta_1^2 - \beta_2^2$ so that $w = 0$. If $\beta_1 = 0$, then by (13) we get $w = 0$.

So if $w \neq 0$ belongs to $Q_S \cap Q_T$, then $\|\hat{x}\| \neq 0$ and we define d as $d = \beta_1/\alpha$ with α as introduced above. Using the equations $F(w) = F(y_i) = 0$ we get $2\alpha\beta = -\varepsilon = 2\beta_1\beta_2/\|\hat{x}\|^2$ and hence $\beta_2 = \|\hat{x}\|^2\beta/d$. The second equation in (14), written out for y_i and w, reads like

$$b(\alpha^2 - \beta^2) + \varepsilon(\mu - a) = 0 = b(d^2\alpha^2 - \|\hat{x}\|^4\beta^2/d^2) + \varepsilon(\mu - a)\|\hat{x}\|^2 ,$$

and hence

$$\alpha^2 - \beta^2 = d^2\alpha^2/\|\hat{x}\|^2 - \|\hat{x}\|^2\beta^2/d^2$$

or

$$d^4\alpha^2 + d^2\|\hat{x}\|^2(\beta^2 - \alpha^2) - \beta^2\|\hat{x}\|^4 = 0 .$$

This last equation in d has only two real roots, namely $d = \pm\|\hat{x}\|$. Hence $\beta_2 = \pm\|\hat{x}\|\beta$, while $\beta_1 = \pm\|\hat{x}\|\alpha$.

So the equation $w = (\beta_1, \cdots, \beta_n) = d(\alpha e_1 + \beta e_2) + (0, 0, \beta_3, \cdots, \beta_n) = \sum_{i=2}^n \eta_i y_i$ can be solved for real coefficients η_i, namely by $\eta_i = \beta_i$ for $i > 3$,

$$\eta_2 = \left(d - \sum_{i=3}^n \beta_i\right)/2$$

and $\eta_3 = \beta_3 + \eta_2$, where

$$d = \pm\left(\sum_{i=3}^n \beta_i^2\right)^{1/2}$$

as we have seen above.

So every $w \in Q_S \cap Q_T$ is lin. dep. of $y_2, \cdots, y_n$ and in this case $n - 1$ is the maximal number of lin. ind. vectors in $Q_S \cap Q_T$. This proves (viii)(a).

(ix): It only remains to show (ix): Let S and T be simultaneously diagonalizable.

Assume S is positive definite, then $Q_S = \{0\}$ and hence for any symmetric T we have $Q_S \cap Q_T = \{0\}$, hence the case $k = 0$ occurs.

If $S = \mathrm{diag}\,(1, -1, \cdots, -1, 1, \cdots, 1)$ and $T = \mathrm{diag}\,(\lambda, -\lambda, \cdots, -\lambda, 0, \cdots, 0)$ with $(l-1)$ numbers -1 and $-\lambda$ appearing on the diagonals of S and T, then $Q_S \cap Q_T$ contains a maximum of l lin. indep. vectors for $\lambda \neq 0$, $2 \leqq l \leqq n$ as can be seen by inspection. Finally if $x \in Q_S \cap Q_T$, then x can be written as $x = \alpha e_l + \beta e_k + y$ for two indices l, k, nonzero constants α, β, and y orthogonal to e_l and e_k, because x has to satisfy

$$F(x) = x'Sx = \sum_{i=1}^{n} \varepsilon_i x_i^2 = 0$$

and

$$G(x) = x'Tx = \sum_{i=1}^{n} \varepsilon_i \mu_i x_i^2 = 0\ , \qquad \text{with } \varepsilon_i = \pm 1\ .$$

But then $\hat{x} = \alpha e_l - \beta e_k + y \in Q_S \cap Q_T$ as well and x and $\hat{x}$ are lin. ind. So in case (ix) $Q_S \cap Q_T$ cannot contain just one vector and its multiples.

This proves Theorem 2.

Next we treat nonsingular pairs of real symmetric matrices that have dimensions 2 or 3.

THEOREM 3. *Let S, T be a nonsingular pair of* r.s. *matrices of dimension n. Assume that $n = 2$ or 3. Let the Roman numerals* (vi), $\cdots$, (viii) *denote the various cases of Theorem* 2.

If (vii) *holds, then $Q_S \cap Q_T$ contains n* lin. ind. *vectors.*

In (vi)(a) *or* (viii)(a) (*with $n = 3$*) *holds, then $Q_S \cap T_T$ contains a maximum of $n - 1$* lin. ind. *vectors.*

If (vii)(a) *holds, then $Q_S \cap Q_T$ contains a maximum of k* lin. ind. *vectors, where k is defined as in Theorem* 2.

If (viii)(a) *holds with $n = 2$, then $Q_S \cap Q_T = \{0\}$.*

Proof. In view of Lemma 2 we can again assume that S and T are already in canonical pair form.

(a) Let $n = 3$: If $J = S^{-1}T$ contains just one 3-dimensional block $J(\lambda, 3)$, then inertia $S = (2, 1, 0)$ or $(1, 2, 0)$ and we have condition (vi)(a). Then $F(x) = x'Sx = \varepsilon(2x_1x_3 + x_2^2)$ and $G(x) = x'Tx = \varepsilon(\lambda(2x_1x_3 + x_2^2) + 2x_2x_3)$ with $\varepsilon = \pm 1$. Hence the only vectors x satisfying $F(x) = G(x) = 0$ are multiples of e_1 and of e_3. Hence there are at most 2 lin. ind. vectors in $Q_S \cap Q_T$.

If $S^{-1}T$ has a complex root, then we have case (viii)(a) and the proof of Theorem 2 (viii)(a) carries over.

If $S^{-1}T$ satisfies condition (vii), then we have for $\varepsilon_i = \pm 1$, $F(x) = x'Sx = \varepsilon_1(2x_1x_2) + \varepsilon_3x_3^2$ and $G(X) = x'Tx = \varepsilon_1(\lambda 2x_1x_2 + x_2^2) + \varepsilon_3\mu x_3^2$ and thus $F(x) = G(x) = 0$ is equivalent to

$$F(x) = 0 \quad \text{and} \quad \varepsilon_1 x_2^2 + \varepsilon_3(\mu - \lambda)x_3^2 = 0\,. \tag{15}$$

If $\lambda = \mu$, then only multiples of e_1 are in $Q_S \cap Q_T$ and if $\lambda \neq \mu$, but $\varepsilon_1\varepsilon_3(\mu - \lambda) > 0$, then again only multiples of e_1 are in $Q_S \cap Q_T$. Now condition (vii)(a) encompasses exactly these two cases, hence if (vii)(a) holds, then $Q_S \cap Q_T$ is just a one dimensional space.

If (vii) holds, i.e., $\lambda \neq \mu$ and $\varepsilon_1\varepsilon_3(\mu - \lambda) < 0$, then we define

$$y_1 = e_1$$
$$y_2 = \alpha e_1 + \beta e_2 - e_3$$

and

$$y_3 = \alpha e_1 + \beta e_2 + e_3 \qquad \text{where } \alpha, \beta \neq 0$$

are such that y_2, y_3 satisfy (15).

(b) If $n = 2$, we have in case of just one Jordan block $J(\lambda, 2)$ in $J = S^{-1}T$: $F(x) = x'Sx = \varepsilon(2x_1x_2)$ and $G(x) = x'Tx = \varepsilon(2\lambda x_1x_2 + x_2^2)$ for $\varepsilon = \pm 1$. So $F(x) = G(x) = 0$ holds iff $x = \alpha e_1$. Hence (vii)(a) is proved. In case of (viii)(a) for a Jordan block $J(a, b, 2)$ of type (B), we have $F(x) = 2x_1x_2$ and $G(x) + 2ax_1x_2 + b(x_1^2 - x_2^2)$. And hence $F(x) = G(x) = 0$ holds iff $x = 0$.

Let S and T be a nonsingular pair of r.s. matrices of dimension greater than 2. In Theorems 1, 2, and 3 we have seen how the real Jordan normal form of $S^{-1}T$ determines the maximal number of lin. ind. vectors in $Q_S \cap Q_T$. Since we have dealt with all possible real Jordan normal forms, we can reverse the argument and get the following:

THEOREM 4. *Let S and T be a nonsingular pair of* r.s. $n \times n$ *matrices where* $n > 2$.

Let $m = \max\{l \mid$ *there exist* l lin. ind. *vectors in* $Q_S \cap Q_T\}$. *Let the Roman numerals* (i), $\cdots$, (viii) *denote the various conditions in Theorem* 2.

If $m = 0$, then S and T can be simultaneously diagonalized by a real congruence transformation.

If $m = 1$, *then* (vii)(a) *holds with* $k = 1$.

If $2 \leqq m \leqq [n/2]$, *then* (vii)(a) *holds with* $k = m$ *or* (vii)(b) *holds with $r = m - k$ for S and T, or S and T can be diagonalized simultaneously.*

If $[n/2] < m < n - 1$, *then* (vii)(b) *holds with* $r = m - k$ *where* $k \leqq [n/2]$ *for* S *and* T, *or* S *and* T *can be diagonalized simultaneously.*

If $m = n - 1$, *then* (vi)(a) *or* (viii)(a) *or* (vii)(b) *holds with* $r = m - k$, *where* $k \leqq [n/2]$ *for* S *and* T, *or* S *and* T *can be diagonalized simultaneously.*

If $m = n$, *then* (i), $\cdots$ *or* (viii) *holds for* S *and* T, *or* S *and* T *can be diagonalized simultaneously by a real congruence transformation.*

Here [ ] denotes the greatest integer function.

If m, the maximal number of lin. ind. vectors simultaneously annihilated by two quadratic forms $x'Sx$, $x'Tx$, lies properly between 1 and $n - 1$, and if we can rule out the cases (vii)(a) or (vii)(b), then we can conclude that S and T are simultaneously diagonalizable. For example, here are two such conditions that make (vii)(a) or (vii)(b) impossible to happen:

COROLLARY 1. *Let* S *and* T *be a nonsingular part of* r.s. $n \times n$ *matrices. Let* $m = \max\{l \mid$ *there exist* l lin. ind. *vectors in* $Q_S \cap Q_T\}$.

Assume $1 < m < n - 1$.

If (a) $S^{-1}T$ *is nonderogatory, or*

(b) *for every eigenvalue* λ *of* $S^{-1}T$ *the number of associated* lin. ind. *eigenvectors is smaller than half the algebraic multiplicity of* λ, *unless both are the same, then* S *and* T *can be diagonalized simultaneously by a real congruence transformation.*

Nonderogators matrices are those that have only one Jordan block for each different eigenvalue.

As a further corollary to Theorem 4 ($m = 0$) we get a result due to Greub and Milnor [1, p. 256]:

COROLLARY 2. *Let* S *and* T *be a nonsingular pair of* r.s. *matrices. If* $Q_S \cap Q_T = \{0\}$, *then* S *and* T *can be diagonalized simultaneously be a real congruence transformation.*

ACKNOWLEDGMENT. This paper in essence constitutes Chapter 4 of my Ph. D. Thesis at the California Institute of Technology, 1971. I am indebted to my advisor Dr. Olga Taussky-Todd and to Dr. H. F. Bohnenblust for their guidance and advice. Dr. Olga Taussky-Todd suggested that in order to generalize the theorems known about simultaneous diagonalization of symmetric matrices one ought to study the set $Q_S \cap Q_T$.

REFERENCES

1. W. Greub, *Linear Algebra*, Springer, Berlin, 3rd edition, 1967.
2. H. J. Kowalski, *Lineare Algebra*, Göschens Lehrbücherei 27, Berlin, 1963.
3. F. Uhlig, *Simultaneous block diagonalization of two real symmetric matrices*, Linear Algebra and Appl., **7** (1973), 281–289.

Received July 12, 1972.

UNIVERSITÄT WÜRZBURG