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ON RELATIONS BETWEEN NÖRLUND AND RIESZ MEANS

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Several results on relations between (absolute) Nörlund summability and (absolute) Riesz summability are known. Among them, Dikshit gives sufficient conditions for $|\bar{N}, q_n| \subseteqq |N, p_n|$ when the sequence $\{p_n\}$ is nonincreasing. The purpose of this paper is to give sufficient conditions for $|N, p_n| \subseteqq |\bar{N}, q_n|$ or $|\bar{N}, q_n| \subseteqq |N, p_n|$ when $\{p_n\}$ is monotone. The results obtained here are also absolute summability analogues of Ishiguro's theorems and Kuttner and Rhoades' theorems which state the inclusion relations between (N, p_n) and $(\bar{N}, p_n)$ summability.

1. Let $\{p_n\}$ be a sequence such that $p_n > 0$, $P_n = \sum_{k=0}^{n} p_k \neq 0$. A series $\sum_{n=0}^{\infty} a_n$ with its partial sum s_n is said to be summable (N, p_n) to sum s, if $t_n = \sum_{k=0}^{n} p_{n-k}s_k/P_n \to s$ as $n \to \infty$, and summable $(\bar{N}, p_n)$ to sum s, if $u_n = \sum_{k=0}^{n} p_k s_k/P_n \to s$ as $n \to \infty$. It is said to be absolutely summable (N, p_n), or summable $|N, p_n|$, if $\Sigma|t_n - t_{n+1}| < \infty$, and absolutely summable $(\bar{N}, p_n)$, or summable $|\bar{N}, p_n|$, if $\Sigma|u_n - u_{n+1}| < \infty$. Given two summability methods A and B, we write $(A) \subseteqq (B)$ if each series summable A is summable B. Throughout this paper, we write for a sequence $\{b_n\}$

$$b_{-n} = 0 (n \geqq 1),\ \Delta b_n = b_n - b_{n+1}\ ,$$

and for a double sequence $\{c_{mn}\}$

$$\Delta_n(c_{mn}) = c_{mn} - c_{m,n+1}\ ,$$

and K denotes an absolute constant, not necessarily the same at each occurrence.

On inclusion relations between two summability, the following results are known.

THEOREM A. [1] *If the sequence $\{p_n\}$ is nonincreasing, $Q_n \to +\infty$ and $Q_n/q_{n+1} = O(P_{n+1})$, where $q_n > 0$ and $Q_n = \sum_{k=0}^{n} q_k \neq 0$, then $|\bar{N}, q_n| \subseteqq |N, p_n|$.*

THEOREM B. [2] *If $\{p_n\}$ is the nondecreasing sequence such that $P_n \to +\infty$ and $p_n = o(P_n)$, then $(\bar{N}, p_n) \subseteqq (N, p_n)$.*

THEOREM C. [3] *If $\{p_n\}$ is the nonincreasing sequence such that $P_n \to +\infty$, then $(N, p_n) \subseteqq (\bar{N}, p_n)$.*

REMARK. Kuttner and Rhoades' theorem [3, Theorem 2] is more precise than Theorem C, but we refer to it in the above form.

THEOREM D. [3] *If* $\{p_n\}$ *is the nonincreasing sequence such that* $p_n \geqq K > 0$, *then* (N, p_n) *and* $(\bar{N}, p_n)$ *are equivalent.*

The purpose of this paper is to prove the following theorems.

THEOREM 1. *If* $\{p_n\}$ *and* $\{q_n\}$ *are positive and nondecreasing sequences and if* $\{p_{n+1}/p_n\}$ *is nonincreasing, then* $|\bar{N}, q_n| \subseteqq |N, p_n|$.

This theorem deals with the case in which $\{p_n\}$ is nondecreasing, while theorem A deals with the case in which $\{p_n\}$ is nonincreasing. In this Theorem, if we put $p_n = q_n$, then we obtain the following

COROLLARY 1. *If* $\{p_n\}$ *is the nondecreasing sequence such that* $\{p_{n+1}/p_n\}$ *is nonincreasing, then* $|\bar{N}, p_n| \subseteqq |N, p_n|$.

This is an absolute summability analogue of Theorem B.

THEOREM 2. *If* $\{p_n\}$ *and* $\{q_n\}$ *are positive and nonincreasing sequences and if* $\{p_{n+1}/p_n\}$ *is nondecreasing, then* $|N, p_n| \subseteqq |\bar{N}, q_n|$.

In this theorem, if we put $p_n = q_n$, we obtain the following

COROLLARY 2. *If* $\{p_n\}$ *is the nonincreasing sequence such that* $\{p_{n+1}/p_n\}$ *is nondecreasing, then* $|N, p_n| \subseteqq |\bar{N}, p_n|$.

This is an absolute summability analogue of Theorem C.

THEOREM 3. *If* $\{p_n\}$ *is the nonincreasing sequence such that* $p_n \geqq K > 0$, *then* $|\bar{N}, p_n| \subseteqq |N, p_n|$.

Combining Theorem 3 and Corollary 2 we have the following

COROLLARY 3. *If* $\{p_n\}$ *is the nonincreasing sequence such that* $\{p_{n+1}/p_n\}$ *is nondecreasing and* $p_n \geqq K > 0$, *then* $|N, p_n|$ *and* $|\bar{N}, p_n|$ *are equivalent.*

This is an absolute summability analogue of Theorem D. Theorems 1–3 are proved in §§3–5, respectively.

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2. We require the following lemmas.

LEMMA 1. *Let* $y_n = \sum_{k=0}^{n} c_{nk} x_k$. *If*
(i) $\sum_{j=0}^{n} |c_{nj}| \leqq K < \infty$ *for all* n, *and*
(ii) $\sum_{j=k}^{n} (c_{nj} - c_{n-1,j}) \geqq 0$ *for* $k = 0, 1, 2, \cdots, n$,
then $\sum_{n=0}^{\infty} |\Delta y_n| < \infty$ *whenever* $\sum_{n=0}^{\infty} |\Delta x_n| < \infty$.

This is easily proved by the method analogous to that of the proof of McFadden's theorem [4, Theorem (2.12)].

LEMMA 2. *For* $m, n = 0, 1, 2, \cdots$,

$$\sum_{k=0}^{m} Q_k \Delta_k\left(\frac{p_{n-k}}{q_k}\right) = P_n - P_{n-m-1} - \frac{Q_m}{q_{m+1}} p_{n-m-1} .$$

This is Lemma 2 in [1].

LEMMA 3. *If* $\{p_n\}$ *is the nondecreasing sequence such that* $\{p_{n+1}/p_n\}$ *is nonincreasing and* $p_{n-k}/P_n < p_{n-k-1}/P_{n-1}$, *then*

$$k\left(\frac{p_{n-k}}{P_n} - \frac{p_{n-k-1}}{P_{n-1}}\right) \geqq \sum_{m=0}^{k-1}\left(\frac{p_{n-m}}{P_n} - \frac{p_{n-m-1}}{P_{n-1}}\right) .$$

This is due to McFadden (see [4, p. 178]).

3. *Proof of Theorem* 1. Let us write

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} s_k \quad \text{and} \quad u_n = \frac{1}{Q_n} \sum_{k=0}^{n} q_k s_k .$$

By Abel's transformation, we have

$$\begin{aligned} t_n &= \frac{1}{P_n} \sum_{k=0}^{n-1} Q_k u_k \Delta_k\left(\frac{p_{n-k}}{q_k}\right) + \frac{p_0 Q_n u_n}{P_n q_n} \\ &= \sum_{k=0}^{n} a_{nk} u_k , \end{aligned}$$

where

$$a_{nk} = \frac{Q_k}{P_n} \Delta_k\left(\frac{p_{n-k}}{q_k}\right) .$$

To prove theorem, we must verify that the conditions of Lemma 1 with $\{c_{nk}\}$ replaced by $\{a_{nk}\}$ are satisfied. Since $\{p_n\}$ and $\{q_n\}$ are positive and nondecreasing, $a_{nk} \geqq 0$. And if $s_k = 1$ for all k, then $t_n = 1$, $u_n = 1$ for all n.
Hence,

$$\sum_{j=0}^{n}|a_{nj}| = \sum_{j=0}^{n} a_{nj} = 1 .$$

Therefore, it is sufficient to show that

$$P \equiv \sum_{j=k}^{n}(a_{nj} - a_{n-1,j}) \geqq 0 \qquad \text{for } k = 0, 1, 2, \cdots, n .$$

By Lemma 2, we have

$$\begin{aligned} P &= \frac{1}{P_n}\Bigl(\sum_{j=0}^{n} - \sum_{j=0}^{k-1}\Bigr)Q_j\Delta_j\Bigl(\frac{p_{n-j}}{q_j}\Bigr) - \frac{1}{P_{n-1}}\Bigl(\sum_{j=0}^{n-1} - \sum_{j=0}^{k-1}\Bigr)Q_j\Delta_j\Bigl(\frac{p_{n-j-1}}{q_j}\Bigr) \\ &= \frac{1}{P_n}\Bigl(P_{n-k} + \frac{Q_{k-1}}{q_k}p_{n-k}\Bigr) - \frac{1}{P_{n-1}}\Bigl(P_{n-k-1} + \frac{Q_{k-1}}{q_k}p_{n-k-1}\Bigr) \\ &= \frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} + \frac{Q_{k-1}}{q_k}\Bigl(\frac{p_{n-k}}{P_n} - \frac{p_{n-k-1}}{P_{n-1}}\Bigr) . \end{aligned}$$

Since $\{p_{n+1}/p_n\}$ is nonincreasing, it is easily deducible that

$$\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \geqq 0 .$$

Thus, if $p_{n-k}/P_n - p_{n-k-1}/P_{n-1} \geqq 0$, P is nonnegative. Suppose on the other hand that

$$\frac{p_{n-k}}{P_n} - \frac{p_{n-k-1}}{P_{n-1}} < 0 .$$

Since $\{q_n\}$ is nondecreasing,

$$Q_{k-1} \leqq kq_{k-1} \leqq kq_k .$$

Hence, $Q_{k-1}/q_k \leqq k$.
Thus, we have, by Lemma 3,

$$\begin{aligned} P &\geqq \frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} + k\Bigl(\frac{p_{n-k}}{P_n} - \frac{p_{n-k-1}}{P_{n-1}}\Bigr) \\ &\geqq \frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} + \sum_{m=0}^{k-1}\Bigl(\frac{p_{n-m}}{P_n} - \frac{p_{n-m-1}}{P_{n-1}}\Bigr) \\ &= \frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} + \frac{P_n - P_{n-k}}{P_n} - \frac{P_{n-1} - P_{n-k-1}}{P_{n-1}} \\ &= 0 . \end{aligned}$$

This completes the proof of Theorem 1.

4. *Proof of Theorem* 2. Under the conditions of $\{p_n\}$, using McFadden's theorem [4, Theorem (2.28)], we see that $|N, p_n| \subseteqq |C, 1|$. Hence we need only verify, under the conditions of theorem, that

$|C, 1| \subseteqq |\bar{N}, q_n|$.

Let us write

$$\sigma_n = \frac{1}{n+1} \sum_{k=0}^{n} s_k \quad \text{and} \quad u_n = \frac{1}{Q_n} \sum_{k=0}^{n} q_k s_k \ .$$

By Abel's transformation, we have

$$\begin{aligned} u_n &= \frac{1}{Q_n} \sum_{k=0}^{n-1} (k+1)\sigma_k \Delta q_k + \frac{(n+1)q_n \sigma_n}{Q_n} \\ &= \sum_{k=0}^{n} b_{nk}\sigma_k \ , \end{aligned}$$

where

$$\begin{aligned} b_{nk} &= \frac{(k+1)\Delta q_k}{Q_n} \qquad \text{for } 0 \leqq k < n \ , \\ &= \frac{(n+1)q_n}{Q_n} \qquad \text{for } k = n \ . \end{aligned}$$

For our purposes, it is sufficient to show that the conditions of Lemma 1 with $\{c_{nk}\}$ replaced by $\{b_{nk}\}$ are satisfied.

Since $\{q_n\}$ is positive and nonincreasing, $b_{nk} \geqq 0$. And if $s_k = 1$ for all k, then $\sigma_n = 1$, $u_n = 1$ for all n.
Hence,

$$\sum_{j=0}^{n} |b_{nj}| = \sum_{j=0}^{n} b_{nj} = 1 \ .$$

Therefore, it is sufficient to show that

$$Q \equiv \sum_{j=k}^{n} (b_{nj} - b_{n-1,j}) \geqq 0 \qquad \text{for } k = 0, 1, 2, \cdots, n \ .$$

For $0 \leqq k \leqq n-1$, we have

$$\begin{aligned} Q &= \frac{1}{Q_n} \sum_{j=k}^{n-1} (j+1)\Delta q_j + \frac{(n+1)q_n}{Q_n} \\ &\quad - \frac{1}{Q_{n-1}} \sum_{j=k}^{n-2} (j+1)\Delta q_j - \frac{nq_{n-1}}{Q_{n-1}} \\ &= \frac{1}{Q_n}\{Q_n - (Q_{k-1} - kq_k)\} - \frac{1}{Q_{n-1}}\{Q_{n-1} - (Q_{k-1} - kq_k)\} \\ &= (kq_k - Q_{k-1})\left(\frac{1}{Q_n} - \frac{1}{Q_{n-1}}\right) . \end{aligned}$$

Since $\{q_n\}$ is positive and nonincreasing,

$$Q_{k-1} \geqq kq_{k-1} \geqq kq_k \quad \text{and} \quad \frac{1}{Q_n} \leqq \frac{1}{Q_{n-1}} \ .$$

Therefore, we have $Q \geqq 0$. For $k = n$, since $b_{nn} \geqq 0$, we have $Q = b_{nn} \geqq 0$. Hence, we have $Q \geqq 0$ for $k = 0, 1, 2, \cdots, n$.

This completes the proof of Theorem 2.

5. *Proof of Theorem* 3. Consider Theorem A for $p_n = q_n$. Then, by our assumption,

$$0 \leqq \frac{P_n}{P_{n+1} p_{n+1}} \leqq \frac{1}{p_{n+1}} \leqq \frac{1}{K} .$$

Therefore, we have $P_n/p_{n+1} = O(P_{n+1})$.
Thus, using our assumption, we see that the conditions of Theorem A are satisfied for $p_n = q_n$.

Hence, by Theorem A, we have $|\bar{N}, p_n| \subseteqq |N, p_n|$.

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