PACIFIC JOURNAL OF MATHEMATICS
Vol. 49, No. 2, 1973

MULTIPLIERS AND THE GROUP L_p-ALGEBRAS

JOHN GRIFFIN AND KELLY MCKENNON

Let G be a locally compact group, p a number in $[1, \infty[$, and L_p the usual L_p-space with respect to left Haar measure on G. The space L_p^t consists of those functions f in L_p^t such that $g*f$ is well-defined and in L_p for each g in L_p. Since each function in L_p^t may be identified with a linear operator on L_p which, as it turns out, is bounded; the operator norm may be super-imposed on L_p^t and, under this norm $\| \|_p^t$, L_p^t is a normed algebra. The family of right multipliers (i.e., bounded linear operators which commute with left multiplication operators) on any normed algebra A will be written as $\mathfrak{m}_r(A)$ and the family of left multipliers as $\mathfrak{m}_l(A)$. The family of all bounded linear operators on L_p which commute with left translations will be written as $\mathfrak{M}_p$.

It was shown in a previous issue of this journal that the Banach algebra $\mathfrak{M}_p$ is linearly isomorphic to the normed algebra $\mathfrak{m}_r(L_p^t)$, whenever the group G is either Abelian or compact. This fact is shown, in the present paper, to hold for general locally compact G. The norm $\| \|_p^t$ is defective in that, unless $p = 1, (L_p^t, \| \|_p^t)$ is never complete.

An attempt will be made in the sequel to supply this deficiency by the introduction of a second norm $||| |||_p^t$ on L_p^t under which L_p^t is always a Banach algebra. It will be seen that, for $p=2$ (and of course for $p=1$), the Banach algebra $\mathfrak{m}_r(L_p^t, ||| |||_p^t)$ is linearly isometric to $\mathfrak{M}_p$.

Let G be a fixed, but arbitrary, locally compact topological group with left Haar measure λ. Write C_{00} for the family of continuous, complex-valued functions on G with compact support.

Let p be a fixed, but arbitrary, number in $[1, \infty[$ and write $\| \|_p$ for the norm on the Banach space $L_p = L_p(G, \lambda)$. The group L_p-algebra L_p^t is the set

$$\{f \in L_p: g*f \in L_p \quad \text{for all} \quad g \in L_p\} .$$

A function $f \in L_p$ is said to be p-tempered and, as shown in [3], the number

$$\|f\|_p^t = \sup\{\|g*f\|_p: g \in C_{00}\ \|g\|_p \leqq 1\} \tag{1}$$

is finite. Conversely, if $\|f\|_p^t$ is finite for some $f \in L_p$, then—as proved in [3]—f is p-tempered and there exists precisely one operator W_f in $\mathfrak{M}_p$ such that

$$\|W_f\| = \|f\|_p^t \quad \text{and} \quad W_f(g) = g*f$$

for all $g \in L_p$.

Let Δ be the modular function for G and let

$$L_{1,p'} = \{f\Delta^{1/p'}: f \in L_1\} \quad (p' = p/(p-1))$$

which is linearly isometric to L_1 when it bears the norm $\|\ \|_{1,p'}$ defined by

$$\|h\|_{1,p'} = \int_G |h| \Delta^{-1/p'} d\lambda \tag{2}$$

for each $h \in L_{1,p'}$. As in [1], 20.13 and [2], 32.45, we see that L_p may be viewed as a right Banach $L_{1,p'}$-module and

$$\|g*h\|_p \leqq \|h\|_{1,p'} \|g\|_p \tag{3}$$

for all $h \in L_{1,p'}$ and $g \in L_p$. Consequently, for each $f \in L_{1,p'}$, there exists precisely one bounded linear operator W_f on L_p such that, for all $g \in L_p$,

$$W_f(g) = g*f \quad \text{and} \quad \|W_f\| \leqq \|f\|_{1,p'} . \tag{4}$$

It is clear that C_{00} is a dense subset of $L_{1,p'}$ and so, since $\{W_f: f \in C_{00}\}$ is a subset of the Banach space $\mathfrak{M}_p$, we have

$$\{W_f: f \in L_{1,p'}\} \subset \mathfrak{M}_p . \tag{5}$$

We define the space of p-well tempered functions to be

$$L_p^{wt} = \{h*f: h \in L_p^t, f \in L_{1,p'}\} .$$

The closure $\mathfrak{A}_p$ of the set $\{W_f: f \in L_p^{wt}\}$ in $\mathfrak{M}_p$ was studied in [3]. Its Banach algebra of left multipliers can be identified with $\mathfrak{M}_p$ ([3], Th. 6) and it possesses a minimal left approximate identity $\{W_{h_\gamma}\}$ such that $\{h_\gamma\} \subset C_{00}*C_{00}$ and

$$\lim_\gamma \|W_{h_\gamma} \circ T \circ W_{h_\gamma}(g) - T(g)\|_p^t = 0 \tag{6}$$

for each $g \in L_p^{wt}$ and $T \in \mathfrak{M}_p$ (see [3], proofs to Theorem 3 and Lemma 1).

LEMMA 1. *Let $T \in \mathfrak{m}_r(L_p^t, \|\ \|_p^t)$ be such that $T(g) = 0$ for all $g \in L_p^{wt}$. Then $T = 0$.*

Proof. Assume that $T \neq 0$. Then there exists some $h \in L_p^t$ such that $T(h) \neq 0$ and some $g \in C_{00}$ such that $g*T(h) \neq 0$. Let $\{h_\gamma\}$ be the net in $C_{00}*C_{00}$ which appears in (6). It follows from (6) that

$$\begin{aligned} 0 &= \lim_\gamma \|W_{h_\gamma} \circ W_h \circ W_{h_\gamma}(g) - W_h(g)\|_p^t \\ &= \lim_\gamma \|g*h_\gamma*h*h_\gamma - g*h\|_p^t . \end{aligned}$$

Note that $g*h_\gamma*h*h_\gamma$ is in L_p^{wt} for each γ and so

$$\begin{aligned}||g*T(h)||_p^t &= ||T(g*h)||_p^t \\ &= \lim_\gamma ||T(g*h_\gamma*h*h_\gamma)||_p^t = 0 :\end{aligned}$$

an absurdity. Thus, $T = 0$.

THEOREM 1. *Define $\omega|\mathfrak{M}_p \rightarrow \mathfrak{m}_r(L_p^t, ||\ ||_p^t)$ by letting $\omega_T(f) = T(f)$ for each $T \in \mathfrak{M}_p$ and $f \in L_p^t$. Then ω is a surjective, isometric, algebra isomorphism.*

Proof. Assume false. By [4], Theorem 1, there exists some $T \in \mathfrak{m}_r(L_p^t, ||\ ||_p^t)$ such that $T \neq 0$ and

$$T(V(f)) = 0 \quad \text{for all} \quad V \in \mathfrak{A}_p \quad \text{and} \quad f \in L_p^t .$$

Since $\mathfrak{A}_p$ possesses a left minimal approximate identity, it is clear that the set $\{V(f): f \in L_p^t,\ V \in \mathfrak{A}_p\} \cap L_p^{wt}$ is dense in $(L_p^{wt}, ||\ ||_p^t)$. This implies that

$$T(g) = 0 \quad \text{for all} \quad g \in L_p^{wt} .$$

By Lemma 1, $T = 0$: an absurdity.

For each $f \in L_p^t$, let

$$|||f|||_p^t = ||f||_p^t + ||f||_p . \tag{7}$$

We have used the symbol $||\ ||$ to represent the operator norm on $\mathfrak{M}_p$. The map ω defined in Theorem 1 shows that $||\ ||$ also is the operator norm on $\mathfrak{M}_p$ when $\mathfrak{M}_p$ is regarded as a family of operators on $(L_p^t, ||\ ||_p^t)$. We may regard $\mathfrak{M}_p$ as a family of operators on the normed space $(L_p^t, |||\ |||_p^t)$ and, in this case, we shall write $|||\ |||$ for the operator norm.

LEMMA 2. *For each $T \in \mathfrak{M}_p$, we have*

$$|||T||| \leqq ||T|| .$$

Proof. For $g \in L_p^t$, we have

$$\begin{aligned}|||T(g)|||_p^t &= ||T(g)||_p^t + ||T(g)||_p \\ &\leqq ||T||\cdot||g||_p^t + ||T||\cdot||g||_p = ||T||\cdot|||g|||_p^t .\end{aligned}$$

THEOREM 2. *The algebra $(L_p^t, |||\ |||_p^t)$ is a Banach algebra. The set L_p^{wt} is a closed two-sided ideal in $(L_p^t, |||\ |||_p^t)$.*

Proof. From Lemma 2, we have

$$\begin{aligned}|||f*g|||_p^t &= |||W_g(f)|||_p^t \leqq |||W_g|||\cdot|||f|||_p^t \leqq ||W_g||\cdot|||f|||_p^t\\ &= ||g||_p^t\cdot|||f|||_p^t \leqq |||g|||_p^t\cdot|||f|||_p^t\end{aligned}$$

for all f and g in L_p^t. Hence $(L_p^t, |||\ |||_p^t)$ is a normed algebra.

Let $\{f_n\}$ be a Cauchy sequence in $(L_p^t, |||\ |||_p^t)$. There exists a function $f \in L_p$ and a bounded linear operator W on L_p such that

$$\lim_n ||f_n - f|| = 0 = \lim_n ||W_{f_n} - W|| \,.$$

For all $g \in C_{00}$ such that $||g|| \leqq 1$, we have

$$||g*f||_p = \lim ||g*f_n||_p \leqq \overline{\lim}\, ||f_n||_p^t\, ||g||_p \leqq \overline{\lim}\, |||f_n|||_p^t \,.$$

This implies via (1) that f is in L_p^t For all $h \in C_{00}$, we have

$$W(h) = \lim_n W_{f_n}(h) = \lim_n h*f_n = h*f = W_f(h) \,,$$

all the limits being taken in L_p. Since C_{00} is dense in L_p, this yields that $W = W_f$. We have shown that

$$\lim_n |||f_n - f|||_p^t = 0 \,.$$

Thus, $(L_p^t, |||\ |||_p^t)$ is complete.

Evidently $(L_p^t, |||\ |||_p^t)$ is a right $L_{1,p'}$-module and so by [2], 32.22, $L_p^t * L_{1,p'}$ is a closed linear subspace. But this is just L_p^{wt}.

That L_p^{wt} is a left ideal of L_p^t is clear. Let g and h be in L_p^{wt} and L_p^t respectively. Choose the net $\{h_\gamma\}$ so that (6) holds. We have

$$\begin{aligned}0 &= \lim_n ||W_{h_\gamma}\circ W_h \circ W_{h_\gamma}(g) - W_h(g)||_p^t\\ &= \lim ||g*h_\gamma*h*h_\gamma - h*h||_p^t \,.\end{aligned}$$

From Lemma 2 of [3] we see that the nets $\{W_{h_\gamma}\}$ and $\{W_{h*h_\gamma}\}$ converge to the identity operator and to W_h, respectively, in the strong operator topology (as operaters on L_p). Consequently,

$$\begin{aligned}&\overline{\lim_\gamma}\, ||g*h_\gamma*h*h_\gamma - g*h||_p\\ &\quad\leqq \overline{\lim_\gamma}\, ||g*h_\gamma*h*h_\gamma - g*h*h_\gamma||_p + \overline{\lim_\gamma}\, ||g*h*h_\gamma - g*h||_p\\ &\quad\leqq \overline{\lim_\gamma}\, ||g*h_\gamma - g||\, ||h*h_\gamma||_p^t + \overline{\lim_\gamma}\, ||g*h*h_\gamma - g*h||_p\\ &\quad\leqq \overline{\lim_\gamma}\, ||W_{h_\gamma}(g) - g||_p\, ||h||_p^t + \overline{\lim_\gamma}\, ||W_{h*h_\gamma} - W_h(g)||_p = 0 \,.\end{aligned}$$

Thus, we have proved

$$\lim_{r} |||g*h_r*h*h_r - g*h|||_p^t = 0$$

and so, since each $g*h_r*h*h_r$ is in the closed set L_p^{wt}, it follows that $g*h$ is there as well. This shows that L_p^{wt} is a right ideal.

COROLLARY 1. *The subspace L_p^{wt} of L_p is $\mathfrak{M}_p$-invariant.*

Proof. Let T be in $\mathfrak{M}_p$ and $f \in L_p^{wt}$. It follows from Lemmas 1 and 2 of [3] that there exists a net $\{f_\alpha\}$ in L_p^{wt} such that

$$\lim_{\alpha} \| T(f) - W_{f_\alpha}(f)\| = 0 = \lim_{\alpha} \| T(f) - W_{f_\alpha}(f)\|_p \,.$$

But this just means

$$\lim_{\alpha} \| T(f) - f*f_\alpha\|_p^t = 0 = \lim_{\alpha} \| T(f) - f*f_\alpha\|_p$$

and so

$$\lim_{\alpha} |||T(f) - f*f_\alpha|||_p^t = 0 \,.$$

But, by Theorem 2, each $f*f_\alpha$ is in L_p^{wt} and so $T(f)$ is as well.

COROLLARY 2. *The Banach algebra $\mathfrak{M}_p$ is linearly isometric to $\mathfrak{m}_r(L_p^{wt}, \| \,\|_p^t)$.*

Proof. It is known that $\mathfrak{M}_p$ is linearly isometric to $\mathfrak{m}_1(\mathfrak{A}_p, \| \,\|)$. Each element of $\mathfrak{m}_r(L_p^{wt}, \| \,\|_p^t)$ clearly may be identified with an element of $\mathfrak{m}_r(\mathfrak{A}_p, \| \,\|)$. Thus, to prove this corollary, it will suffice to show that each element of $\mathfrak{m}_1(\mathfrak{A}_p, \| \,\|)$ can be identified with an element of $\mathfrak{m}_r(L_p^{wt}, \| \,\|_p^t)$. But this follows from Corollary 1.

LEMMA 3. *Let $T \in \mathfrak{m}_r(L_p^t, |||\,|||_p^t)$ be such that $T(g) = 0$ for all $g \in L_q^{wt}$. Then $T = 0$.*

Proof. Repeat the proof for Lemma 1, noticing that, as in the proof to Theorem 2,

$$\lim_{r} |||g*h_r*h*h_r - g*h|||_p^t = 0 \,.$$

It follows form Lemma 2 that the natural restriction mapping of $\mathfrak{M}_p$ into $\mathfrak{m}_r(L_p^t, |||\,|||_p^t)$ is a norm non-increasing algebra isomorphism. There arise natural questions:

(i) when is the mapping onto?

(ii) when is the mapping a homeomorphism?

(iii) when is the mapping an isometry?

Question (iii) clearly implies (ii).

PROPOSITION 1. *The restriction mapping of* $\mathfrak{M}_p$ *into* $\mathfrak{m}_r(L_p^t, ||| \ |||_p^t)$ *is surjective if and only if it is a homeomorphism.*

Proof. Let Ψ denote the restriction mapping. If Ψ is onto, the open mapping theorem implies that it is a homeomorphism.

Now suppose that Ψ is a homeomorphism. Let T be an element of $\mathfrak{m}_r(L_p^t, ||| \ |||_p^t)$. In view of Lemma 3, T is completely determined by its restriction to L_p^{wt}. Thus, T may be identified with a multiplier on $\{\Psi(W_f): f \in L_p^{wt}\}$, and so with a multiplier on its closure $\Psi(\mathfrak{A}_p)$ in $\Psi(\mathfrak{M}_p)$ as well. It follows that T may be identified with a multiplier on $\mathfrak{A}_p$, which, in view of [3], Theorem 6, may be identified with some $V \in \mathfrak{M}_p$. It follows that $\Psi(V) = T$. Hence, Ψ is surjective.

When $p = 1$, then $L_p^t = L_p^{wt} = L_p$ and $|| \ ||_1 = || \ ||_1^t = 1/2 \, ||| \ |||_1^t$. When $p = 2$, we have the following:

THEOREM 3. *The algebra* $\mathfrak{m}_r(L_2^t, ||| \ |||_2^t)$ *is linearly isometric and isomorphic with* $\mathfrak{M}_2$.

Proof. In view of the fact that $\mathfrak{M}_2$ is a C^*-algebra, it follows from [5], 4.8.4 that $||T||^2 \leqq |||T^*||| \cdot |||T|||$ for all $T \in \mathfrak{M}_2$. But Lemma 2 implies

$$|||T^*||| \leqq ||T^*|| = ||T|| \quad \text{and} \quad |||T||| \leqq ||T||$$

for $T \in \mathfrak{M}_2$ and so $|||T||| = ||T||$. Thus, Ψ is an isometry and Theorem 3 now follows from Proposition 1.

REFERENCES

1. Edwin Hewitt and Kenneth A. Ross, *Abstract Harmonic Analysis*, Vol. I. Berlin, Springer-Verlag, 1963.
2. ———, *Abstract Harmonic Analysis*, Vol. II. Berlin, Springer-Verlag, 1963.
3. Kelly McKennon, *Multipliers of type* (p, p), To be published in Pacific J. Math.
4. ———, *Multipliers of type* (p, p) *and multipliers of the group* L_p*-algebras*, Pacific J. Math., **45** (1973), 297-302.
5. Charles Rickert, *General Theory of Banach Algebras*, Princeton, N. J., Van Nostrand Company, Inc., 1960.

Received July 24, 1972.

WASHINGTON STATE UNIVERSITY