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A COUNTER EXAMPLE TO THE BLUM HANSON THEOREM IN GENERAL SPACES

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Let T and S be two bounded linear operators on a Banach space B. One studies the question whether weak convergence of the powers T^n to S implies convergence of the Cesaro averages $1/n \sum_{k=1}^n T^{i(k)}$ to S for all subsequences $0 \leqq i(1) < i(2) < \cdots$ of the integers. It is well known that this implication holds if B is the L^2 of a finite measure space and T is induced by a measure preserving transformation of that space (this is the Blum Hanson theorem) or, more generally, if B is a Hilbert space and T of norm at most 1, or if B is a L^1 space and T a positive operator of norm at most 1. In the present paper the conjecture that the above implication holds in general Banach spaces for all T with $\|T\| \leqq 1$ is disproved by constructing a counterexample in a Banach space of the type $B = \mathscr{C}(X)$, X a compact Hausdorff space.

Specifically, let B and B^* be a Banach space and its adjoint, respectively, and let $T: B \rightarrow B$ and $S: B \rightarrow B$ be two linear and bounded operators. Consider the following two statements:

(i) T^n converges weakly to S; i.e., for each $f \in B$ and $F \in B^*$, $\lim_{n\to\infty} F(T^n f) = F(Sf)$.

(ii) Let $i(n)$ be a sequence of integers so that $0 \leqq i(n) < i(n+1)$ for each $n \geqq 1$. Then, $1/n \sum_{k=1}^n T^{i(k)}$ converges strongly to S; i.e., if $f \in B$, then $\lim \| 1/n \sum_{k=1}^n T^{i(k)} f - Sf \| = 0$.

It is easy to see that (ii) always implies (i). The Blum Hanson theorem [2] states that if B is the L_2 space of a finite measure space and if T is induced by a measure preserving transformation of this measure space, then (i) also implies (ii). Later it was shown that the equivalence of (i) and (ii) is true if T is a contraction (i.e., if $\|T\| \leqq 1$) and B is a Hilbert space [1], [3] or the L_1 space of a σ-finite measure space [1]. It is then natural to ask if these two conditions (i) and (ii) are always equivalent. In this note, we give a counterexample to show that in general (i) does not imply (ii), even if T is a contraction.

2. Reducing the question to a topological one. Let X be a compact Hausdorff space and let $\mathscr{C} = \mathscr{C}(X)$ be the Banach space of all real valued continuous functions on X, with the usual, supremum norm. If $\tau: X \rightarrow X$ is a continuous transformation, then there is an induced linear contraction $T: \mathscr{C} \rightarrow \mathscr{C}$, defined as $(Tf)(x) = f(\tau x)$ for each $f \in \mathscr{C}$ and $x \in X$. Note that $T^n f$ converges weakly in $\mathscr{C}$ if and only if $f(\tau^n x)$ converges for each $x \in X$, as a sequence of real numbers.

Hence, if there is a point $x_0 \in X$ so that $\lim_{n\to\infty} \tau^n x = x_0$ for every $x \in X$, then T^n converges weakly to $S: \mathscr{C} \to \mathscr{C}$, defined as $(Sf)(x) = f(x_0)$ for each $x \in X$ and $f \in \mathscr{C}$.

Now assume that τ is such a transformation and also that there is a compact $K \subset X$, not containing the point x_0 and satisfying the following condition:

(A) For each integer $N \geqq 0$ there is a point $x = x(N)$ in X so that K contains more than N terms of the sequence $\tau^n x$, $n = 0, 1, 2, \cdots$.

Before we give an example for such an X and τ in the next section, here we note that in this case (i) does not imply (ii). Let $f \in \mathscr{C}$ be a nonnegative function so that $f(x_0) = 0$ and $f(x) \geqq 1$ for all $x \in K$. Hence, $T^n f$ converges weakly to zero. Now define a sequence $i(n)$ of integers as follows. Let $i(1) = 0$. For each $r \geqq 1$, if the first 2^{r-1} terms are determined then the next 2^{r-1} terms [i.e., the terms $i(2^{r-1}+1), \cdots, i(2^r)$] are chosen as follows. With the notations of Condition (A), let $x_r = x(i(2^{r-1}) + 2^{r-1})$ and let the following conditions be satisfied: $\tau^i(2^{r-1}+s)_{x_r} \in K$ for each $s = 1, 2, \cdots, 2^{r-1}$ and $i(2^{r-1}) < i(2^{r-1}+1) < i(2^r)$. Then,

$$\frac{1}{2^r}\sum_{k=1}^{2^r} f(\tau^{i(k)}x_r) \geqq \frac{1}{2}$$

for each $r \geqq 1$. Hence,

$$\frac{1}{n}\sum_{k=1}^{n} T^{i(k)}f$$

does not converge strongly to zero.

3. The topological example.

We are now going to give an example of a compact Hausdorff space X and a continuous transformation $\tau: X \to X$ so that all the assumptions of the second section are satisfied.

Let R be the real line with the usual topology and let $C = [0, 1) = \{x \mid 0 \leqq x < 1\}$ be the unit interval with its circle topology. Let $\varphi: C \to C$ be a homeomorphism that is linear in $[0, 1/2)$ and in $[1/2, 1)$ and satisfies $\varphi 0 = 0$, $\varphi 1/2 = 3/4$. Note that if $0 < x < 1$ then $\varphi^n x \to 1$ in R. Hence, $\varphi^n x \to 0$ in C, for each $x \in C$. Also, let $\alpha: C \to C$ be a continuous function that is linear in $[1/4, 1/2)$, vanishes identically on $[1/2, 1)$ and is equal to $-A/\log x$ at every $x \in (0, 1/4)$. Here, A is a positive constant so that $\max_{x\in C} \alpha x = +A/\log 4$ is less than $1/4$.

Now let $X = C^2$ be the two-dimensional torus with its usual topology. The points of X are denoted as (x, y), where $x, y \in C$. Let a mapping $\tau: X \to X$ be defined as $\tau(x, y) = ([\varphi x + \alpha y] \bmod 1, \varphi y)$. It is then clear that τ is continuous.

LEMMA. *If* $(x, y) \in X$, *then* $\lim_{n\to\infty} \tau^n(x, y) = (0, 0)$.

Proof. Let $\tau^n(x, y) = (x_n, y_n)$. Hence, $x_0 = x$, $y_0 = y$ and if $n \geqq 1$, then $y_n = \varphi^n y$, $x_n = \xi_n \bmod 1$, where $\xi_n = \varphi x_{n-1} + \alpha y_{n-1}$. If $y_0 = 0$, then $y_n = 0$ for all $n \geqq 0$ and $x_n = \varphi x_{n-1} = \varphi^n x_0 \to 0$ in C. If $y_0 > 0$, then there is an integer $m \geqq 0$ so that $y_n = \varphi^n y_0 > 1/2$ for all $n \geqq m$. This means that $\alpha y_n = 0$ and $x_n = \varphi^{n-m} x_m$ for all $n \geqq m$. Hence, $(x_n, y_n) = (\varphi^{n-m} x_m, \varphi^n y_0)$ converges to $(0, 0)$ for all $(x_0, y_0) \in X$.

LEMMA. *If* $K = \{(x, y) \mid (x, y) \in X, 1/8 \leqq x \leqq 7/8\}$, *then* K *satisfies Condition* A *of* § 2.

Proof. With the notations of the previous proof, let

$$\delta_n = \delta_n(x, y) = \xi_n - x_{n-1} = \varphi x_{n-1} - x_{n-1} + \alpha y_{n-1} .$$

Then,

$$x_n = \left[x_0 + \sum_{k=1}^{n} \delta_k\right] \bmod 1 .$$

Now note that if

$$\sum_{k=n_1}^{n_2} \delta_k \geqq 1$$

then there is an integer n, so that $n_1 \leqq n \leqq n_2$ and that $(x_n, y_n) \in K$. In fact, for each $k \geqq 1$, $0 \leqq \delta_k \leqq \max_{x\in C}(\varphi x - x) + \max_{y\in C} \alpha y \leqq 1/2$, and hence, if

$$\sum_{k=n_1}^{n_2} \delta_k \geqq 1$$

then,

$$\left[x_0 + \sum_{k=1}^{n} \delta_k\right] \bmod 1$$

is between 1/8 and 7/8 for some n, $n_1 \leqq n \leqq n_2$. Therefore, to prove the present lemma, it is enough to show that given any number N, there is a point $(x, y) \in X$ so that

$$\sum_{n=1}^{\infty} \delta_n(x, y) \geqq N .$$

Let $0 < y < 1/4$ be given and let $M = M(y)$ be the largest integer in the set $\{n \mid n \geqq 0, \varphi^n y < 1/4\}$. Let $z = \varphi^M y$. Hence, $z < 1/4$, but $\varphi z = (3/2)z \geqq 1/4$. Therefore, if $0 \leqq n \leqq M$, then $y_n = \varphi^n y = (3/2)^n y = (3/2)^{n-M} z \geqq (3/2)^{n-M} 1/6$, and $\alpha y_n = -A/\log y_n \geqq A/((M-n)\log 3/2 + \log 6)$. This means that

$$\sum_{k=1}^{M} \delta_n \geqq \sum_{n=1}^{M} \alpha y_n \geqq A \sum_{n=0}^{M-1} \frac{1}{n \log 3/2 + \log 6} .$$

But it is clear that there are points $y \in (0, 1/4)$ for which $M = M(y)$ is arbitrarily large, hence, for which $\sum_{n=1}^{\infty} \delta_n$ is also arbitrarily large.

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