PACIFIC JOURNAL OF MATHEMATICS
Vol. 51, No. 2, 1974

SUBALGEBRAS OF FINITE CODIMENSION IN THE ALGEBRA OF ANALYTIC FUNCTIONS ON A RIEMANN SURFACE

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Let R be a finite open Riemann surface with boundary Γ. We set $\bar{R} = R \cup \Gamma$ and let $A(R)$ denote the algebra of functions which are continuous on $\bar{R}$ and analytic on R. Suppose A is a uniform algebra contained in $A(R)$. The main result of this paper shows that if A contains a function F which is analytic in a neighborhood of $\bar{R}$ and which maps $\bar{R}$ in a n-to-one manner (counting multiplicity) onto $\{z: |z| \leqq 1\}$, then A has finite codimension in $A(R)$.

We say that A is a uniform algebra on $\bar{R}$ if A is a uniformly closed subalgebra of the complex-valued continuous functions on $\bar{R}$ which separates points of $\bar{R}$ and contains the constant functions. If A is contained in $A(R)$, then we say A has finite codimension in $A(R)$ if $A(R)/A$ is a finite dimensional vector space over $\boldsymbol{C}$. A reference for uniform algebras is Gamelin [2].

Let U be the open unit disk in $\boldsymbol{C}$. We call F an unimodular function if F is analytic in a neighborhood of $\bar{R}$ and maps $\bar{R}$ onto $\bar{U}$ so that F is n-to-one if we count the multiplicity of F where dF vanishes. If T is the unit circle, then F maps Γ onto T. The existence of such a function was first proved by Ahlfors [1]. Later, Royden [4] gave another proof of this result.

1. Main results. Let A be a uniform algebra on $\bar{R}$ which is contained in $A(R)$. If $J = \{f \in A(R): fA(R) \subset A\}$, then J is a closed ideal in $A(R)$ and J is contained in A.

LEMMA. *Let $F \in A$ be an unimodular function of order n. If $\zeta_1 \in \bar{R}$ is such that $F^{-1}(F(\zeta_1))$ consists of n distinct points, then there is $G \in J$ such that $G(\zeta_1) \neq 0$.*

Proof. Since A separates points on $\bar{R}$, there is $g \in A$ such that g separates $F^{-1}(F(\zeta_1))$. If $z_1 \in \bar{R}$, let $F^{-}(F(z_1)) = \{z_1, z_2, \cdots, z_n\}$ (perhaps with repetitions) and let $f \in A(R)$.

Define $Q(u) = f(z_1)\{u - g(z_2)\}\{u - g(z_3)\} \cdots \{u - g(z_n)\} + f(z_2)\{u - g(z_1)\}\{u - g(z_3)\} \cdots \{u - g(z_n)\} + \cdots + f(z_n)\{u - g(z_1)\}\{u - g(z_2)\} \cdots \{u - g(z_{n-1})\}$ (cf. [5], p. 290). Then $Q(u)$ is a polynomial in u of the form $Q(u) = \alpha_{n-1}(z_1, \cdots, z_n)u^{n-1} + \alpha_{n-2}(z_1, \cdots, z_n)u^{n-2} + \cdots + \alpha_0(z_1, \cdots, z_n)$. The coefficients α_j are symmetric functions in $z_1, \cdots, z_n$. Hence, if

$w = F(z_1)$, then $a_j(w) = \alpha_j(z_1, \cdots, z_n)$ for $j = 0, \cdots, n-1$ is well-defined on $\bar{U}$. Using Riemann's removable singularity theorem, it follows that $a_j(w) \in A(U)$ for $j = 0, \cdots, n-1$.

Since $a_j(w) \in A(U)$ for each j, there are polynomials $\{p_k^j(w)\}_{k=1}^{\infty}$ such that the p_k^j's converge uniformly to a_j on $\bar{U}$. Then $p_k^j(F(z)) \in A$ for each k, and we conclude that $a_j(F(z)) \in A$. Letting $z = z_1$ and setting $u = g(z)$, we obtain $Q(g(z)) = a_{n-1}(F(z))g(z)^{n-1} + a_{n-2}(F(z))g(z)^{n-2} + \cdots + a_0(F(z)) = f(z) \prod_{i=2}^{n} \{g(z) - g(z_i)\} \in A$. Let $G(z) = \prod_{i=2}^{n} \{g(z) - g(z_i)\}$. Then $G(\zeta_1) \neq 0$ and we have shown that $fG \in A$ for any $f \in A(R)$. Therefore, $G \in J$.

THEOREM. *Let A be a uniform algebra on $\bar{R}$ which is contained in $A(R)$. If A contains an unimodular function, then A has finite codimension in $A(R)$.*

Proof. Suppose $F \in A$ is an unimodular function of order n. Let hull $J = \{z \in \bar{R}: f(z) = 0$ for all $f \in J\}$. If $\zeta \in \Gamma$, then $dF(\zeta) \neq 0$ ([7], p. 367) and consequently $F^{-1}(F(\zeta))$ consists of n distinct points. By the lemma, hull $J \subset R$. It follows that hull J is a finite set. By applying [6], Theorem 1 and [3], Lemma 2.5, we conclude that $A(R)/J$ is finite dimensional. Hence, A has finite codimension in $A(R)$.

Let $R = \{z \in C: 1 < |z| < 2\}$. Again let $J = \{f \in A(R): fA(R) \subset A\}$ where A is a uniform algebra on $\bar{R}$. Using the same technique we prove the proposition below.

PROPOSITION. *Let A be a uniform algebra on $\bar{R}$ which is contained in $A(R)$. If A contains z^n and z^{-m} for some positive integers n and m, then $A = A(R)$.*

Proof. Let N be the least common multiple of n and m. Then z^N and $z^{-N} \in A$. Also, z^N is an N-to-one map of $\bar{R}$ onto $\bar{R}$ without branch points. For any $\zeta_1 \in \bar{R}$ there are N distinct points $\{\zeta_1, \zeta_2, \cdots, \zeta_N\}$ which satisfy $\zeta_i^N = \zeta_1^N$. Fix $\zeta_1 \in \bar{R}$ and let $g \in A$ separate $\{\zeta_1, \zeta_2, \cdots, \zeta_N\}$. Let $f \in A(R)$.

Letting z^N take the role of F and using g and f, we form $Q(u)$ just as in the proof of the lemma. The coefficients $a_j(w)$ of $Q(u)$ belong to $A(R)$. Hence there are polynomials in w and w^{-1} which converge uniformly to $a_j(w)$ on $\bar{R}$. Since z^N and z^{-N} belong to A, it follows that $a_j(z^N)$ is in A.

Consequently, $Q(g(z)) = f(z) \prod_{i=2}^{N} \{g(z) - g(z_i)\} \in A$ for all $f \in A(R)$. Let $G(z) = \prod_{i=2}^{N} \{g(z) - g(z_i)\}$. Then $G \in J$ and $G(\zeta_1) \neq 0$. Therefore, hull $J = \phi$. This implies $A = A(R)$.

2. Question.

The theorem of this paper gives an affirmative answer to a special case of the following question. Suppose A is a uniform algebra on $\bar{R}$ and A is contained in $A(R)$. If A contains a nonconstant function which is analytic in a neighborhood of $\bar{R}$, does it follow that A has finite codimension in $A(R)$?

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Received February 22, 1973 and in revised form June 25, 1973.

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