PACIFIC JOURNAL OF MATHEMATICS
Vol. 53, No. 1, 1974

A FIXED POINT THEOREM FOR k-SET-CONTRACTIONS DEFINED IN A CONE

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Let X be a Banach space and H a solid closed cone in X with interior H^0. Suppose B is a bounded open set in X containing the origin. For $G = B \cap H^0$, let $\partial_H \bar{G}$ denote the relative boundary of the closure $\bar{G}$ of G in H. In this paper mappings $T: \bar{G} \to H$ are considered where T is a k-set-contraction, $k < 1$. It is shown for such mappings that if $(I - tT)(G)$ is open, $t \in [0, 1]$, and if T satisfies (i) $Tx \neq \lambda x$ for all $x \in \partial_H \bar{G}$ and $\lambda > 1$, then T has a fixed point in $\bar{G}$. In the special case when T is a contraction mapping, $(I - tT)(G)$ is always open and boundedness of B can be dispensed with.

The Leray-Schauder boundary condition (i) is an assumption which in particular holds *for convex* G if $T: \partial_H \bar{G} \to \bar{G}$, or even more generally if T is 'inward' in the sense of Halpern and Bergman [7] (cf. also, Vidossich [15] (Theorem 5(ii)) for an equivalent condition on $f = I - T$). Conditions similar to (i) have been imposed by several authors recently in proving fixed point theorems in functional analysis, although, as we note in more detail below, it is usually assumed that the origin is an interior point of the domain of T, with the condition $Tx \neq \lambda x$, $\lambda > 1$, required of all x in the boundary of this domain.

We are concerned here with the "k-set-contractions", $k < 1$, a class of mappings which includes not only the usual "contraction mappings" (mappings $U: D \to X$ satisfying for some $\alpha < 1$, $\|Ux - Uy\| \leqq \alpha \|x - y\|$, $x, y \in D$), but also mappings of the form $T = U + C$ with U a contraction mapping and C compact. This class is defined by Kuratowski [9] as follows: For a bounded subset A of X define the measure of noncompactness, $\gamma(A)$, of A by $\gamma(A) = \text{g.l.b.}\ \{d > 0$: there exists a finite number of sets $S_1, \cdots, S_n$ such that $A \subset \bigcup_{i=1}^n S_i$ and diam $S_i \leqq d$, $i = 1, \cdots, n\}$. A continuous mapping $T: D \to X$, $D \subset X$, is called a k-set-contraction if there is a fixed constant $k \geqq 0$ such that $\gamma(T(A)) \leqq k\gamma(A)$ for all bounded $A \subset D$. There has been intensive study of these mappings recently including, notably, Nussbaum's development [10] of a theory of topological degree for them.

With H and G as above, we prove in this paper that if $T: \bar{G} \to H$ is a k-set-contraction, $k < 1$, satisfying (i) on $\partial_H \bar{G}$, and if $(I - tT)(G)$ is open, $t \in [0, 1]$, then T has a fixed point in $\bar{G}$. This theorem is specifically related to a number of recent results; for example, Nussbaum has proved [10] that if D is a bounded closed and convex subset of X with nonempty interior and if $T: D \to X$ is a k-set-contraction, $k < 1$, satisfying for some $x_0 \in D$, $Tx - x_0 \neq \lambda(x - x_0)$ for all $x \in \partial D$ and $\lambda > 1$, then T has a fixed point in D. (See also Nussbaum [11, 12].) This result reduces to a theorem of Browder [3] under the stronger assumption that T is semicontractive, and to one of Darbo [4] if $T: D \to D$. The boundary condition used by Nussbaum is similar to (i) (if x_0 is the origin), but it requires x_0 to be an interior point of the domain D of T. Another related fact, due to Petryshyn [13], is that if G is a bounded open subset of X with $0 \in G$ and if $T: \bar{G} \to X$ is a 'condensing mapping' (i.e., $\gamma(T(A)) < \gamma(A)$ for all $A \subset D$ such that $\gamma(A) > 0$), then the assumption $Tx \neq \lambda x$ for all $x \in \partial G$ and $\lambda > 1$ implies that the fixed point set of T in $\bar{G}$ is nonempty and compact. Thus this result holds for a more general class of mappings and convexity of the domain is no longer required.

In attempting to weaken the assumption that the origin be an interior point of the domain of the mapping, Gatica and Kirk [6] have proved existence of fixed points for contraction mappings $T: \bar{G} \to H$ where H is any closed and convex set in X, with $G \subset X$ open relative to H and $0 \in G$. The boundary condition assumed for this result is: (i)′ $Tx \neq \lambda x$, $\lambda > 1$, for all nonzero $x \in \partial_{\mathscr{H}} G$, where $\partial_{\mathscr{H}} G$ denotes the relative boundary of G in the closed subspace $\mathscr{H}$ of X spanned by H. Subsequently, Gatica [7] has extended this result to the case where T is a k-set-contraction, $k < 1$, under the additional assumptions that G be bounded and $I - tT$ one-to-one, $t \in [0, 1]$. These results differ from the theorem of this paper in that by assuming H is a cone we are now able to replace the assumption that the boundary condition (i)′ hold on the relative boundary of G in $\mathscr{H}$ with the much weaker assumption that it hold only on $\partial_H \bar{G}$. This new result does not appear to follow directly from our preceding results and arguments.

THEOREM. *Let H be a solid closed cone with interior H^0 in the Banach space X, let B be a bounded open subset of X containing the origin, and let $G = B \cap H^0$. Suppose $T: \bar{G} \to H$ is a k-set-contraction, $k < 1$, with the property that $(I - tT)(G)$ is open, $t \in [0, 1]$. If T satisfies:*

(i) $Tx \neq \lambda x$ for all $x \in \partial_H \bar{G}$ and $\lambda > 1$,
then T has a fixed point in $\bar{G}$.

The assumption that $(I\text{-}tT)(G)$, $t \in [0, 1]$, is open always holds if $I - tT$ is one-to-one (see Nussbaum [12], Theorem 2), and in particular it always holds in the important case when T is a contraction mapping. Also in this case, the assumption that B is bounded may be dispensed with by the reasoning of Gatica-Kirk in the proof of Theorem 2.2 of [6]. Thus we have the following:

COROLLARY 1. *With H as in the theorem, let $B \subset X$ be an open set containing the origin, and let $G = B \cap H^0$. If $T: \bar{G} \to H$ is a contraction mapping satisfying*

(i) *$Tx \neq \lambda x$ for all $x \in \partial_H \bar{G}$ and $\lambda > 1$,*

then T has a fixed point in $\bar{G}$.

If $T: \bar{G} \to H$ is a nonexpansive mapping (i.e., if $\|Tx - Ty\| \leqq \|x - y\|$, $x, y \in \bar{G}$) then tT is a contraction mapping for $t \in [0, 1)$. Also, if (i) holds for T on $\partial_H \bar{G}$ then (i) also holds for tT, $t < 1$, so the above implies:

COROLLARY 2. *With H as in the theorem, let $B \subset X$ be an open set containing the origin and let $G = B \cap H^0$. If $T: \bar{G} \to H$ is a nonexpansive mapping satisfying:*

(i) *$Tx \neq \lambda x$ for all $x \in \partial_H \bar{G}$ and $\lambda > 0$.*

(ii) *$\{x_n\} \subset \bar{G}$ such that $x_n - Tx_n \to 0$ as $n \to \infty$ implies existence of $x' \in \bar{G}$ such that $x' - Tx' = 0$.*

Then T has a fixed point in $\bar{G}$.

We should mention that condition (ii) above was first used by Petryshyn [14], where he notes that it always holds if T is a generalized contraction in the sense of [2, 8]. Also (ii) always holds if X is uniformly convex and B is bounded and convex, because Browder has shown [3] that in such situations $I - T$ is demiclosed for nonexpansive T. Thus we have:

COROLLARY 3. *Suppose X is a uniformly convex Banach space, $H \subset X$ is a solid closed cone, and B a bounded open convex subset of X with $0 \in B$. Let $G = B \cap H^0$. If $T: \bar{G} \to H$ is a nonexpansive mapping satisfying* (i) *on $\partial_H \bar{G}$, then T has a fixed point in $\bar{G}$.*

In what follows we use ∂A to denote the boundary of a subset A of X. We also use γ to denote the measure of noncompactness, and in particular, if $\{y_n\}$ is a bounded sequence in X, $\gamma(\{y_n\}) = \gamma(\{y_n: n = 1, 2, \cdots\})$.

The following lemma, which is contained implicitly in [5], is used repeatedly and we include its proof for the sake of completeness.

LEMMA. *Let X be a Banach space, D a bounded subset of X, and $T: D \to Y$ a k-set-contraction, $k < 1$. Suppose $\{\alpha_n\}$ is a sequence of numbers converging to $\alpha \in [-1, 1]$, and suppose for $\{y_n\} \subset D$, $y_n + \alpha_n Ty_n = z_n$ where $\{z_n\}$ converges. Then $\gamma(\{y_n\}) = 0$ and thus $\{y_n\}$ has a convergent subsequence.*

Proof. Since D is bounded it follows that $M=\sup\{\|Tx\|: x \in D\} < \infty$, and clearly we may suppose $M > 0$. Let $\varepsilon > 0$. Since $\alpha_n \to \alpha$ as $n \to \infty$ there exists a positive integer N such that if $m, n \geqq N$, $|\alpha_n - \alpha_m| < \varepsilon/M$, $|\alpha_m| \leqq 1 + \varepsilon$, and $\|z_m - z_n\| < \varepsilon$. If $\gamma(\{y_n\}) = d$, then clearly $\gamma(\{y_n\}_{n=N}^{\infty}) = d$ and $\gamma(\{Ty_n\}_{n=N}^{\infty}) \leqq kd$. There exists a finite cover $\{S_1, \cdots, S_r\}$ of $\{Ty_n\}_{n=N}^{\infty}$ such that if $Ty_k, Ty_s \in S_i$ for $i = 1, \cdots, r$, then $\|Ty_k - Ty_s\| \leqq kd + \varepsilon$. Now let

$$F_i = T^{-1}(S_i) \cap \{y_n: n = N, N+1, \cdots\}\,, \qquad i = 1, \cdots, r\,,$$

and observe that $\{y_n\}_{n=N}^{\infty} \subset \bigcup_{i=1}^{r} F_i$. Also if $y_n, y_m \in F_i$, then

$$\begin{aligned} \|y_m - y_n\| &= \|\alpha_n Ty_n - \alpha_m Ty_n + \alpha_m Ty_n - \alpha_m Ty_m + z_m - z_n\| \\ &\leqq |\alpha_n - \alpha_m|\,\|Ty_n\| + |\alpha_m|\,\|Ty_n - Ty_m\| + \|z_m - z_n\| \\ &\leqq |\alpha_n - \alpha_m| M + |\alpha_m|(kd + \varepsilon) + \varepsilon \\ &< 2\varepsilon + (1 + \varepsilon)(kd + \varepsilon)\,. \end{aligned}$$

Since ε is arbitrary this implies $\gamma(\{y_n\}) \leqq kd$ which in turn yields $d \leqq kd$. Since $k < 1$ we conclude $d = 0$. But this implies that the closure of the set $\{y_n: n = 1, 2, \cdots\}$ is compact.

Proof of the theorem. We may assume without loss of generality that G is the interior of its closure. For $t \in [0, 1]$, let $G_t = (I - tT)(G)$. By assumption, G_t is open and since tT is also a tk-set-contraction, $tk < 1$, $I - tT$ is proper (Nussbaum [11], Corollary 2). Thus $(I - tT)(\bar{G})$ is closed and hence $\partial G_t \subset (I - tT)(\partial G)$.

Fix $x \in G$ and let $\alpha = \alpha(x) = \sup\{t \in [0, 1]: x \in G_t\}$. We first show that either $\alpha = 1$, or there exists $w \in \partial_H \bar{G}$ such that $w - \alpha Tw = x$. (Note that α is well-defined since $x \in G_0 = G$.)

Let $\mu_n \to \alpha$ as $n \to \infty$ where $x \in G_{\mu_n}$. Then for each n there exists $y_n \in G$ such that $y_n - \mu_n Ty_n = x$. By the lemma there exists $y \in \bar{G}$ and a convergent subsequence $\{y_{n_i}\}_{i=1}^{\infty}$ of $\{y_n\}$ such that $y_{n_i} \to y$ as $i \to \infty$. Since $\mu_{n_i} T \to \alpha T$ uniformly on $\bar{G}$ as $i \to \infty$ it follows that $y - \alpha Ty = x$, i.e., $x \in \bar{G}_\alpha$.

Now suppose $\alpha < 1$ and let $t_n \downarrow \alpha$ as $n \to \infty$ where $t_n \in (\alpha, 1)$, $n = 1, 2, \cdots$. Since $x \notin G_{t_n}$ and $y - t_n Ty \in \bar{G}_{t_n}$ the segment joining $y - t_n Ty$ and x must contain a point $z_n \in \partial G_{t_n}$. But $\partial G_{t_n} \subset (I - tT)(\partial G)$, so for each n there exists $w_n \in \partial G$ such that

$$(*) \qquad\qquad w_n - t_n Tw_n = z_n\,.$$

As $n \to \infty$, $y - t_n Ty \to x$ so it follows that $z_n \to x$. Thus by the lemma some subsequence $\{w_{n_i}\}_{i=1}^{\infty}$ of $\{w_n\}$ converges to a point $w \in \partial G$, and $(*)$ yields: $w - \alpha Tw = x$.

Now $x \in H^0$ and $\alpha Tw \in H$. Since H is a cone and $w = x + \alpha Tw$ we have $w \in H^0$. This, with the fact that $w \in \partial G$, implies $w \in \partial_H \bar{G}$.

(Notice that since, by assumption, G is the interior of its closure, $\partial_H G = \partial_H \bar{G}$.) Therefore, we have established the following:

If $x \in G$ *and if* $\alpha = \sup\{t \in [0, 1]: x \in G_t\}$, *then either* (a) $\alpha = 1$, *or* (b) *there exists* $w \in \partial_H \bar{G}$ *such that* $w - \alpha Tw = x$. *Moreover, if* (a) *holds for* $x \in G$, *there exists* $y \in \bar{G}$ *such that* $y = x + Ty$.

Now to complete the proof, let $\{h_i\}$ be a sequence of points of G such that $h_i \to 0$ as $i \to \infty$. For each i, let $\alpha_i = \alpha(h_i)$. From the above for each i either (a) $\alpha_i = 1$, or there exists $w_i \in \partial_H \bar{G}$ such that $w_i - \alpha_i Tw_i = h_i$. But if (a) holds for infinitely many i, then we may suppose (by passing to a subsequence) that for each i there exists $y_i \in \bar{G}$ satisfying $y_i = h_i + Ty_i$. Since $h_i \to 0$ it follows from the lemma that there exists a subsequence $\{y_{i_j}\}_{j=1}^{\infty}$ of $\{y_i\}$ which converges to a point $y_0 \in \bar{G}$. Since

$$y_{i_j} = h_{i_j} + Ty_{i_j}, \qquad j = 1, 2, \cdots,$$

we must have $y_0 = Ty_0$ and T has a fixed point in $\bar{G}$.

On the other hand, if (a) does not hold for infinitely many i, there exists an integer N such that for $w_i \in \partial_H \bar{G}$,

$$(**) \qquad w_i - \alpha_i Tw_i = h_i, \qquad i = N, N+1, \cdots.$$

Moreover, by passing to a subsequence we may suppose $\alpha_i \to \alpha \leqq 1$ as $i \to \infty$. Then since $h_i \to 0$, the lemma again applies and there exists a subsequence $\{w_{i_j}\}_{j=1}^{\infty}$ of $\{w_i\}$ which converges to a point $w \in \partial_H \bar{G}$. This, with $(**)$, yields $w = \alpha Tw$. Since $\alpha \in [0, 1]$ and $w \in \partial_H \bar{G}$, (i) implies either $\alpha = 0$ or $\alpha = 1$. But $\alpha = 0$ is not possible because this implies $0 \in \partial_H \bar{G}$, a contradiction in view of the fact $G = B \cap H^0$ with B open, and $0 \in B$. And if $\alpha = 1$ then $w = \alpha Tw = Tw$, completing the proof.

As a final comment we note that in relation to Corollary 1, if the stronger assumption $T: \partial_H \bar{G} \to \bar{G}$ is made then H need not be a solid cone, and in fact may be taken to be any closed and convex set with $G = B \cap H \neq \varnothing$. This fact is a consequence of results of [1].

We wish to thank the referee for suggestions which improved our exposition.

Added in proof. Recent generalizations of the theorem of this paper based on degree theory results of Nussbaum and Petryshyn and not requiring $(I - tT)(G)$ to be open for $0 < t \leqq 1$ will appear in [W. A. Kirk, "A remark on condensing mappings," J. Math. Anal. Appl.] and in [W. V. Petryshyn and P. M. Fitzpatrick, "Fixed point theorems and the fixed point index for multivalued mapping defined in cones"]. The latter paper extends the theorem to multi-valued mappings in Frechet spaces.

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Received June 18, 1973 and in revised form September 12, 1973. The second author's research was supported by National Science Foundation Grant GP 18045.

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