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# A NOTE ON THE ATIYAH-BOTT FIXED POINT FORMULA

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**Let $f$ be a holomorphic self map of a compact complex analytic manifold $X$. The differential of $f$ commutes with $\bar{\partial}$ and, hence, induces an endomorphism of the $\bar{\partial}$-complex of $X$. If $f$ has isolated simple fixed points, the Lefschetz formula of Atiyah-Bott expresses the Lefschetz number of this endomorphism in terms of local data involving only the map $f$ near the fixed points. For example, if $X$ is a curve, this Lefschetz number is the sum of the residues of $(z-f(z))^{-1}$ at the fixed points.**

**Using a well-known technique of Atiyah-Bott for computing trace formulas, we shall, in this note, give a direct analytic derivation of the Lefschetz number as a residue formula. The formula is valid for holomorphic maps having isolated, but not necessarily simple fixed points.**

1. Let $E$ be the $\bar{\partial}$-complex of a compact complex analytic manifold $X$ of dimension $n$.

$$E: 0 \longrightarrow \Gamma(\Lambda^{0,0}) \xrightarrow{\bar{\partial}} \Gamma(\Lambda^{0,1}) \longrightarrow \cdots \xrightarrow{\partial} \Gamma(\Lambda^{0,n}) \longrightarrow 0 .$$

Since $E$ is elliptic, $H^i(X) = \ker \bar{\partial}_i / im \bar{\partial}_{i-1}$ is finite dimensional. Denote by $T = \{T_i\}$ the endomorphism induced on $E$ by the holomorphic map $f$, and by $H^j T$ the resulting endomorphism on $H^i(X)$.

The Lefschetz number of $f$ is then defined by

$$L(f) = \sum_{i=0}^{n} (-1)^i tr H^i T$$

and the finite dimensionality of the spaces $H^i(X)$ insures that this number is finite.

The Atiyah-Bott method of computing trace formulas reduces the problem of calculating $L(f)$ to that of finding a good parametrix for the $\bar{\partial}$-operator. In fact, let us suppose we can find operators $P_i: \Gamma(\Lambda^{0,i}) \rightarrow \Gamma(\Lambda^{0,i-1})$, $i = 1, \cdots, n$, having the property that

$$(1) \qquad P_{i+1}\bar{\partial}_i + \bar{\partial}_{i-1}P_i = I - S_i$$

where $S_i: \Gamma(\Lambda^{0,i}) \rightarrow \Gamma(\Lambda^{0,i})$ are integral operators with sufficiently smooth kernels. Observe that if $\omega \in \Gamma(\Lambda^{0,i})$ is in the kernel of $\bar{\partial}_i$, then the left-hand side of (1) is a co-boundary. Hence, $H^i I - H^i S$ is the zero-endomorphism on homology. Similarly, since $T$ commutes with $\bar{\partial}$

$$T_i(P_{i+1}\bar{\partial}_i + \bar{\partial}_{i-1}P_i) = T_iP_{i+1}\bar{\partial}_i + \bar{\partial}_{i-1}T_{i-1}P_i = T_i - T_iS_i$$

so that $H^iT = H^iTS$. Therefore,

$$L(f) = \sum_{i=0}^{n}(-1)_i trH^i(TS)\ . \tag{2}$$

The generalized alternating sum formula of Atiyah-Bott says that the alternating sum of traces is the same on the chain level as on the homology level; that is,

$$L(f) = \sum_{i=0}^{n}(-1)^i trH^iTS = \sum_{i=0}^{n}(-1)^i trT_iS_i \tag{3}$$

provided the right-hand side is finite. This will be the case if the kernels of the operators $S_i$ are sufficiently smooth along the graph of $f$.

To carry out the above procedure and evaluate $L(f)$ we make an explicit choice of the operators $P_i$.

2. The most natural way to choose a parametrix on $X$ is to glue together the local fundamental solutions of the $\bar{\partial}$-operator using partitions of unity. Given any finite open covering $\{U_\alpha\}$ of $X$, there are, in each $U_\alpha$, integral operators $Q_{\alpha,i}\colon \Gamma(\Lambda^{0,i}(U_\alpha)) \to \Gamma(\Lambda^{0,i-1}(U_\alpha))$ $i = 1, \cdots, n$ such that for $\omega \in C_0^\infty(U_\alpha)$

$$\bar{\partial}Q_{\alpha,i}(\omega) = \omega - Q_{\alpha,i+1}(\bar{\partial}\omega) \tag{4a}$$

$$(Q_{\alpha,i}\omega)(z^\alpha) = \int_{U_\alpha}\omega(\zeta^\alpha) \wedge \Omega_i(z^\alpha, \zeta^\alpha) \tag{4b}$$

where $\Omega_i(z^\alpha, \zeta^\alpha) \in \Gamma(\Lambda^{0,i-1}(U_\alpha) \otimes \Lambda^{n,n-i}(U_\alpha))$ is a $C^\infty$-section off the diaganal and has an absolutely integrable singularity.

Let $\Omega(z^\alpha, \zeta^\alpha) = \sum_{i=1}^{n}(-1)^i\Omega_i(z^\alpha, \zeta^\alpha)$. This is an $(n, n-1)$ form on $U_\alpha \times U_\alpha$ satisfying

$$\bar{\partial}\Omega = 0\ . \tag{4c}$$

For a detailed study of Cauchy-Fantappié forms see Koppelman [2], Lieb [3], Øvrelid [4]. An explicit expression for $\Omega$ appears near the end of § 3.

Suppose $f$ has $m$ isolated fixed points, $P_1, \cdots, P_m$. Let $U_k$ be a coordinate neighborhood containing $P_k$, chosen so that the sets $U_k$ are mutually disjoint. Let $N_k$ be a neighborhood of $P_k$, sufficiently small so that $f^{-1}(N_k) \subset U_k$ ($f$ is continuous and $f(P_k) = P_k$). The collection $U_1, \cdots, U_m$ can be extended to a covering $\{U_\alpha\}$ and a partition of unity $\{\lambda_\alpha\}$ subordinate to this covering can be chosen such that (for $k=1, \cdots, m$)

(i) $\operatorname{supp} \lambda_k \subset N_k$

(ii) $\lambda_k = 1$ in a neighborhood of $P_k$.

Then $\operatorname{supp} \lambda_k \circ f \subset f^{-1}(N_k) \subset U_k$ and $\lambda_k \circ f = 1$ in some (other) neighborhood of $P_k$.

Now choose nonnegative functions $\sigma_\alpha \in C_0^\infty(U_\alpha)$ such that

(iii) $\sigma_\alpha = 1$ on $\operatorname{supp} \lambda_\alpha$ $\alpha \neq 1, \cdots, m$

(iv) $\sigma_\alpha = 1$ on $\{\operatorname{supp} \lambda_\alpha\} \cup \{\operatorname{supp} \lambda_\alpha \circ f\}$ $\alpha = 1, \cdots, m$.

Define $P_i: \Gamma(\Lambda^{0,i}) \to \Gamma(\Lambda^{0,i-1})$ by

$$P_i\omega = \sum_\alpha \lambda_\alpha Q_{\alpha,i}(\alpha_\alpha \omega) \qquad i = 1, \cdots, n \tag{5}$$
$$P_0\omega = 0 .$$

From (4a) we obtain

$$\begin{aligned} \bar\partial P_i\omega + P_{i+1}\bar\partial\omega &= \omega + \sum_\alpha \bar\partial\lambda_\alpha Q_{\alpha,i}(\sigma_\alpha\omega) - \sum_\alpha \lambda_\alpha Q_{\alpha,i+1}(\bar\partial\sigma_\alpha \wedge \omega) \\ &= \omega - S_i\omega \qquad i = 0, \cdots, n \end{aligned} \tag{6}$$

where

$$\begin{aligned} S_i\omega(z) = &-\sum_\alpha \bar\partial\lambda_\alpha(z)\int_{U_\alpha} \sigma_\alpha(\zeta)\omega(\zeta) \wedge \Omega_i(z, \zeta) \\ &+ \sum_\alpha \lambda_\alpha(z)\int_{U_\alpha} \bar\partial\sigma_\alpha(\zeta) \wedge \omega(\zeta) \wedge \Omega_{i+1}(z, \zeta) . \end{aligned}$$

(We consistently suppress the coordinate superscript when possible: writing, for example, $\sigma_\alpha(\zeta)$ for $\sigma_\alpha(\zeta^\alpha)$.)

3. Because of the construction of the covering and the patching functions, the kernel of $S_i$ is smooth in a neighborhood of the graph of $f$. In fact, if $\alpha > m$, then $f$ has no fixed points in $U_\alpha$ and therefore, $\zeta - f(\zeta)$ is bounded away from zero so that $\Omega_i(f(\zeta), \zeta)$ is a $C^\infty$-function in $U_\alpha$. Furthermore, in $U_k$, $k \leqq m$, we have chosen $\lambda_k$ so that $\lambda_k(f(\zeta)) \equiv 1$ in a neighborhood of $P_k$. Then, $\bar\partial\lambda_k(f(\zeta)) = 0$ near $\zeta = f(\zeta)$. Also, since $\sigma_k(\zeta) \equiv 1$ on the support of $\lambda_k(f(\zeta))$, we have $\bar\partial\sigma_\alpha(\zeta) = 0$ near $\zeta = f(\zeta)$. Thus, the kernel of $S_i$ may be evaluated along the graph of $f$ to obtain:

$$\begin{aligned} \sum_0^n (-1)^i tr(T_iS_i) &= \sum_\alpha \Big\{ \sum_1^n (-1)^{i+1} \int_{U_\alpha} \bar\partial\lambda_\alpha(f(\zeta)) \wedge \sigma_\alpha(\zeta)\Omega_i(f(\zeta), \zeta) \Big\} \\ &\quad + \sum_\alpha \Big\{ \sum_0^{n-1} (-1)^i \int_{U_\alpha} \lambda_\alpha(f(\zeta))\bar\partial\sigma_\alpha(\zeta) \wedge \Omega_{i+1}(f(\zeta), \zeta) \Big\} \\ &= -\sum_\alpha \int_{U_\alpha} \bar\partial\{\lambda_\alpha(f(\zeta))\sigma_\alpha(\zeta)\} \wedge \sum_1^n (-1)^i \Omega_i(f(\zeta^\alpha), \zeta^\alpha) \end{aligned}$$

from which

$$L(f) = -\sum_{\alpha} \int_{U_\alpha} \bar{\partial}\{\lambda_\alpha(f(\zeta))\sigma_\alpha(\zeta)\} \wedge \Omega(f(\zeta), \zeta) . \tag{7}$$

In $U_\alpha$, for $\alpha > m$, $f$ has no fixed points. Using (4c), integrating by parts, and making use of the fact that $\sigma_\alpha$ has compact support in $U_\alpha$, we have

$$\begin{aligned}\int_{U_\alpha} \bar{\partial}\{\lambda_\alpha(f(\zeta))\sigma_\alpha(\zeta)\} \wedge \Omega(f(\zeta), \zeta) &= \int_{U_\alpha} \bar{\partial}\{\lambda_\alpha(f(\zeta))\sigma_\alpha(\zeta)\Omega(f(\zeta), \zeta)\} \\ &= \int_{\partial U_\alpha} \lambda_\alpha(f(\zeta))\sigma_\alpha(\zeta)\Omega(f(\zeta), \zeta) \equiv 0 .\end{aligned}$$

For $\alpha = k \leqq m$, let $B_k$ be a ball around $P_k$ on which $\lambda_k(f(\zeta)) \equiv 1$. Since $\sigma_k(\zeta) \equiv 1$ on the support of $\lambda_k(f(\zeta))$,

$$\begin{aligned}L(f) = -\sum_{k=1}^{m} \int_{U_k - B_k} \bar{\partial}\{\lambda_k(f(\zeta))\Omega(f(\zeta), \zeta)\} &= \sum_{k=1}^{m} \int_{\partial B_k} \lambda_k(f(\zeta))\Omega(f(\zeta), \zeta) \\ &= \sum_{k=1}^{m} \int_{\partial B_k} \Omega(f(\zeta), \zeta) .\end{aligned} \tag{8}$$

Using local coordinates in $B_k$, let $g_i(\zeta^k) = \zeta_i^k - f_i(\zeta^k)$, $i = 1, \cdots, n$. Then, for $n > 1$,

$$\Omega(z^k, \zeta^k) = \frac{(n-1)!}{(2\pi i)^n} |\zeta^k - z^k|^{-2n} \sum_{i=1}^{n} (-1)^{i+1} \overline{\zeta_i^k - z_i^k} \bigwedge_{\substack{j=1 \\ j \neq i}}^{n} \overline{d\zeta_j^k} - \overline{dz_j^k}) \bigwedge_{l=1}^{n} d\zeta_l^k$$

and

$$L(f) = \frac{(n-1)!}{(2\pi i)^n} \sum_{k=1}^{m} \int_{\partial B_k} (\Sigma |g_i^k|^2)^{-n} \sum_{i=1}^{n} (-1)^{i+1} \overline{g_i^k} \bigwedge_{\substack{j=1 \\ j \neq i}}^{n} \overline{dg_j^k} \bigwedge_{l=1}^{n} d\zeta_l^k \tag{9}$$

which is the desired formula.

For $n = 1$, $\Omega(z^k, \zeta^k) = (1/2\pi i)(d\zeta^k/\zeta^k - z^k)$ and

$$L(f) = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{\partial B_k} \frac{d\zeta^k}{\zeta^k - f(\zeta^k)} = \sum_{f(\zeta)=\zeta} \operatorname{Res}(\zeta - f(\zeta))^{-1} .$$

NOTE. Other proofs of this result have recently been given by Toledo [5] and Tong [6] using different techniques.

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