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REPRESENTATION OF SUPERHARMONIC FUNCTIONS MEAN CONTINUOUS AT THE BOUNDARY OF THE UNIT BALL

J. R. DIEDERICH

In this paper it will be shown that superharmonic functions can be represented by a Green potential together with their boundary values if taken mean continuously at the boundary of the unit ball.

Introduction. It is well known that if $u(r, \theta, \phi)$ is harmonic inside the unit ball and has radial limit $\lim_{r\to 1} u(r, \theta, \phi) = 0$ everywhere on the surface, then u is not necessarily identically null inside and thus cannot be represented by its radial boundary values. Furthermore, there is an L_1 (Lebesgue class) harmonic function, see §2. *Remarks*, which satisfies $\lim_{r\to 1} u(r, \theta, \phi) = 0$ except for $(1, 0, 0)$. In [1] and [3], Shapiro established the representation of harmonic functions in the two dimensional unit disc by their radial limits when a certain radial growth condition is satisfied. However, the set of functions satisfying the radial growth condition does not contain the class L_1, and conversely. Also, the analogues of [1] and [3] have not been established in the N-dimensional unit ball, $3 \leqq N$.

Our intention is to establish a representation of superharmonic functions in L_1 on the N-dimensional unit ball by their boundary values if taken mean continuously. Definitions and the statement of the theorems follow in the next section.

1. Preliminaries. We shall work in N-dimensional Euclidean space R^N, $3 \leqq N$, and shall use the following notation: $x = (x_1, \cdots, x_N)$ and $B(x, r) =$ the open N-ball centered at x with radius r; $\tilde{B}(x, r) = B(x, r) \cap B(0, 1)$; $|E|$, the Lebesgue measure of E; ∂E, the boundary of E; $\bar{\partial} B(x, r) = \partial B(0, 1) \cap B(x, r)$; $d\omega_N$, the natural surface area on $\partial B(0, 1)$; and subscripted A's, positive absolute constances though possibly different from one occurrence to another. For a point $y_0 \in \partial B(0, 1)$, $u(x)$ a measurable function on some $\tilde{B}(y_0, r_0)$, and $f(y)$ a function on $\partial B(0, 1)$, we set for $\rho \leqq r_0$

$$u_f(y_0, \rho) = |\tilde{B}(y_0, \rho)|^{-1} \int_{\tilde{B}(y_0, \rho)} |u(x) - f(y_0)|\, dx \,.$$

We use the notation $u(y_0, \rho)$ when $f \equiv 0$.

THEOREM 1. *Let $u(x)$ be superharmonic in $\Omega = B(0, 1)$. If*

(1) $\quad u(y, \rho) = O(1) \quad as \quad \rho \longrightarrow 0 \quad for\ each \quad y \in \partial\Omega$

(2) $\quad u(y, \rho) = o(1) \quad as \quad \rho \longrightarrow 0 \quad a.e.\ [d\omega] \quad on \quad \partial\Omega$

then $0 \leqq u(x)$ *on* Ω.

Theorem 1 is the main step in establishing

THEOREM 2. *Let* $u(x)$ *be superharmonic in* $B(0, 1)$. *Let* $f(y)$ *be in* L_1 *on* $\partial\Omega$ *and satisfy*

$$\int_{\bar{\partial}B(y_0,\rho)} |f(y) - f(y_0)|\, d\omega_N(y) = O(\rho^{N-1}) \qquad (3)$$

$$as \quad \rho \longrightarrow 0 \quad for\ each \quad y_0 \in \partial\Omega\,.$$

If $u_f(y, \rho)$ *satisfies* (1) *and* (2), *then*

$$u(x) = \int_\Omega G(x, x')d\eta(x') + PI(f, x) \qquad (4)$$

where $G(x, x')$ *is the Green function for* Ω, η *is a nonnegative additive measure on* Ω, *and* $PI(f, x)$ *is the Poisson integral of* f.

2. REMARK. Theorem 1 is best possible in two respects. If (1) is required for all but one $y_0 \in \partial B(0, 1)$, then the conclusion fails as is demonstrated by $u(x) = (|x|^2 - 1)[\omega_N |x - y_0|^N]^{-1}$, with $y_0 = (1, 0, \cdots, 0)$. Secondly, if the modulus is eliminated in the definition of $u(y, \rho)$ and the integral is defined improperly, then the conclusion fails even if (2) is strengthened to "for each $y \in \partial\Omega$". Simply consider a non-radial partial of the above function. In Theorem 2 the necessity of (3) is not clear.

Clearly, Theorem 1 offers a uniqueness theorem for harmonic functions which are mean continuous at the boundary of the unit ball. Also, contained in the proof of Theorem 1 is a generalization of the reflection principle for harmonic functions.

Finally, an open question regarding a converse to Theorem 1 will be considered in §5.

3. *Proof of Theorem* 1. Set $u^-(x) = \min(u(x), 0)$. Then $u^-(x)$ is superharmonic and clearly satisfies both (1) and (2). We intend, of course, to show that $u^-(x) \equiv 0$ which we shall do in the following steps.

Let Z be the set of points z on $\partial\Omega$ such that $u^-(x)$ is unbounded in every neighborhood $\tilde{B}(z, \rho)$. $\partial\Omega - Z$ clearly open so that Z is a closed set.

Step 1. If $y_0 \in \partial\Omega$ and $\bar{\partial}B(y_0, 2\rho_0) \cap Z = \phi$, then $\lim_{x \to y} u^-(x) = 0$ for

$x \in \tilde{B}(y_0, \rho_0)$ and $y \in \bar{\partial} B(y_0, \rho_0)$.

Proof. Let y be a point of $\bar{\partial} B(y_0, \rho_0)$ for which (2) is satisfied. Let x be a point on the line segment l_y through the center of Ω and y. Select $\rho_x = |x - y|$, then by the superharmonicity of $u^-(x)$

$$\begin{aligned} 0 &\geqq u^-(x) \geqq |B(x, \rho_x)|^{-1} \int_{B(x,\rho_x)} u^-(x')dx' \\ &\geqq |B(x, \rho_x)|^{-1} \int_{\tilde{B}(y,2\rho_x)} u^-(x')dx' \\ &\geqq 2^N |\tilde{B}(y, 2\rho_x)|^{-1} \int_{\tilde{B}(y,2\rho_x)} u^-(x')dx' \\ &= -2^N u^-(y, 2\rho_x) . \end{aligned}$$

As $x \longrightarrow y$, $2\rho_x \longrightarrow 0$, thus $u^-(y, 2\rho_x) \longrightarrow 0$, since y is selected to satisfy (2). So

$$\lim_{\substack{x \to y \\ x \in l_y}} u^-(x) = 0 \qquad \text{a.e. on } \bar{\partial} B(y_0, 2\rho_0) . \tag{5}$$

By the definition of Z and the superharmonicity of $u^-(x)$ it is clear that $u^-(x)$ is bounded in $\tilde{B}(y_0, \rho_0)$, and hence can be represented

$$u^-(x) = \int_{\tilde{B}(y_0,\rho_0)} G_0(x, x')d\eta_0(x') + h^-(x)$$

where $G_0(x, x')$ is the Green function for $\tilde{B}(y_0, \rho_0)$, η_0 is a nonnegative set function and $h^-(x)$ is the greatest harmonic minorant of $u^-(x)$. By Theorem 1 [4, p. 527], we have that

$$\lim_{\substack{x \to y \\ x \in l_y}} \int_{\tilde{B}(y_0,\rho_0)} G_0(x, x')d\eta_0(x') = 0 \quad \text{a.e. on } \bar{\partial} B(y_0, \rho_0) .$$

By this and (5)

$$\lim_{\substack{x \to y \\ x \in l_y}} h^-(x) = 0 \quad \text{a.e. on } \bar{\partial} B(y_0, \rho_0) . \tag{6}$$

Clearly $h^-(x)$ is bounded in $\tilde{B}(y_0, \rho_0)$ and therefore can be represented by its radial limits. Hence $\lim_{x \to y} h^-(x) = 0$ for $x \in \tilde{B}(y_0, \rho_0)$ and $y \in \bar{\partial} B(y_0, \rho_0)$. Since $0 \geqq u^-(x) \geqq h^-(x)$, the desired conclusion follows.

As an immediate consequence of Step 1, we have

Step 2. If $\bar{\partial} B(y_0, 2\rho_0) \cap Z = \phi$, then the function $u_0^-(x) = u^-(x)$ for $x \in \tilde{B}(y_0, \rho_0)$, $u_0^-(x) \equiv 0$ for $x \in B(y_0, \rho_0) - \tilde{B}(y_0, \rho_0)$ is superharmonic in $B(y_0, \rho_0)$.

Proof. $u^-(x)$ is continuously 0 at $\bar{\partial} B(y_0, \rho_0)$ and nonpositive in

$\tilde{B}(y_0, \rho_0)$.

Step 3. If $Z \neq \phi$, then there is a $z_0 \in Z$, an $r_0 > 0$, and a constant A_1, such that

$$(7) \qquad u^-(z, \rho) \leqq A_1 \quad \text{for} \quad z \in \bar{\partial} B(z_0, 2r_0) \cap Z \quad (0 < \rho < 1) .$$

Proof. Since $u^-(x)$ is superharmonic and satisfies (2), it is in L_1 on Ω. Consequently by continuity of the integral $u(y, \rho)$ is jointly continuous for $0 < \rho < 1$ and $y \in \partial\Omega$. Proceeding as in [2, p. 69] and again employing (2) the conclusion (7) follows.

By Step 1, the conclusion of Theorem 1 follows immediately if $Z = \phi$. Assuming $Z \neq \phi$, select z_0 as in Step 3. Let x_1 be an arbitrary point in $\tilde{B}(z_0, r_0)$, and let ρ_{x_1} be the largest value for which $B(x_1, 2\rho_{x_1}) \cap Z = \phi$. Clearly there is a point z^* which lies in $\bar{\partial} B(z_0, 2r_0)$ and is on the boundary of $B(x_1, 2\rho_{x_1})$. By Step 2, we can extend $u^-(x)$ by $u_0^-(x)$ in the part of $B(x_1, \rho_{x_1})$ lying outside Ω. So

$$\begin{aligned} u^-(x_1) = u_0^-(x_1) &\geqq |B(x_1, \rho_{x_1})|^{-1} \int_{B(x_1, \rho_{x_1})} u_0^-(x')dx' \\ &= |B(x_1, \rho_{x_1})|^{-1} \int_{\tilde{B}(x_1, \rho_{x_1})} u^-(x')dx' \\ &\geqq A_0 |\tilde{B}(x_1, \rho_{x_1})|^{-1} \int_{\tilde{B}(x_1, \rho_{x_1})} u^-(x')dx' \\ &\geqq 4^N A_0 |\tilde{B}(z^*, 4\rho_{x_1})|^{-1} \int_{\tilde{B}(z^*, 4\rho_{x_1})} u^-(x')dx' \\ &= -4^N A_0 u^-(z^*, 4\rho_{x_1}) \geqq -4^N A_0 A_1 \end{aligned}$$

by (7). Thus $u^-(x)$ is bounded in $\tilde{B}(z_0, r_0)$. Thus $z_0 \notin Z$, a contradiction based on the assumption that $Z \neq \phi$; thus $Z = \phi$ and Theorem 1 is established.

4. *Proof of Theorem* 2. The theorem will follow directly from

Step 4. Let $f(y)$ satisfy (3) and set $h(x) = PI(f, x)$. Then $h_f(x, \rho)$ satisfies (1) and (2).

To see this, set $v(x) = u(x) - h(x)$; then

$$v(x, \rho) = [u - h](x, \rho) \leqq u_f(x, \rho) + h_f(x, \rho)$$

so $v(x, \rho)$ satisfies (1) and (2) since both $u_f(x, \rho)$ and $h_f(x, \rho)$ do. So by Theorem 1, $0 \leqq v(x)$ and thus

$$v(x) = \int_\Omega G(x, x')d\nu(x') + g(x)$$

with all the terms nonnegative. So $g(x, \rho)$ satisfies (1) and (2) and thus $0 \leqq g(x)$; clearly then $0 \leqq -g(x)$ and $g(x) \equiv 0$, whereby (4) follows.

Proof of Step 4. For $y_0 \in \partial\Omega$, there is a γ and a $0 < \rho_0$ such that

$$\rho^{1-N} \int_{\tilde{\partial} B(y_0, \rho)} |f(y) - f(y_0)| dy < \gamma \quad \text{for} \quad \rho < \rho_0 .$$

Clearly we can assume that $f(y_0) = 0$. Consider

$$\begin{aligned} &|\tilde{B}(y_0, \rho)|^{-1} \int_{\tilde{B}(y_0, \rho)} \int_{\partial\Omega} \{(1 - |x|^2)/\omega_N |x - y|^N\} |f(y)| d\omega_N(y) dx \\ &= \int_{\tilde{\partial} B(y_0, 2\rho)} + \int_{\partial\Omega - \tilde{\partial} B(y_0, 2\rho)} |\tilde{B}(y_0, \rho)|^{-1} \int_{\tilde{B}(y_0, \rho)} (1 - |x|^2)/\omega_N |x \\ &\quad - y|^N dx |f(y)| d\omega_N(y) \\ &= I_1 + I_2 . \end{aligned}$$

In the second integral we have $1/2|y_0 - y| \leqq |x - y| \leqq 2|y_0 - y|$, which gives

$$\begin{aligned} I_2 &\leqq A_1\rho \int_{\partial\Omega - \tilde{\partial} B(y_0, 2\rho)} |f(y)| \, |y - y_0|^{-N} d\omega_N(y) \\ &\leqq A_2\rho \int_{2\rho}^{1} r^{-N} \int_{s(y_0, r)} |f(y)| ds_r(y) dr \end{aligned}$$

where $s(y_0, r) = \partial B(y_0, r) \cap \partial\Omega$

$$\begin{aligned} &= A_2\rho r^{-N} \int_0^r \int_{s(y_0, r')} |f(y)| ds_{r'}(y) dr' \Big|_{2\rho}^{1} \\ &\quad + A_2 N\rho \int_{2\rho}^{\rho_0} + \int_{\rho_0}^{1} \Big\{ r^{-N-1} \int_0^r \int_{s(y_0, r')} |f(y)| ds_{r'}(y) \Big\} dr' \\ &\leqq A_3\gamma + o(\rho) \qquad \text{as } \rho \longrightarrow 0 . \end{aligned}$$

For I_1 we use the inequality

$$\int_{\tilde{B}(y_0, \rho)} (1 - |x|^2)/\omega_N |x - y|^N dx \leqq \int_{\tilde{B}(y_0, 2\rho)} (1 - |x|^2)/\omega_N |x - y_0|^N dx$$

to obtain

$$\begin{aligned} I_1 &\leqq A_1 |\tilde{B}(y_0, \rho)|^{-1} \int_{\tilde{B}(y_0, 2\rho)} (1 - |x|^2)/|x - y_0|^N dx \cdot \int_{\tilde{\partial} B(y_0, 2\rho)} |f(y)| dy \\ &\leqq A_2\rho^{1-N} \int_{\tilde{\partial} B(y_0, 2\rho)} |f(y)| dy \\ &\leqq A_3\gamma , \end{aligned}$$

which shows that $h_f(x, \rho)$ satisfies (2). Since γ can be taken arbitrarily small for almost every $y_0 \in \partial\Omega$, $h_f(x, \rho)$ also satisfies (1).

5. Converse to Theorem 1. Let $u(x) = \int_\Omega G(x, x')d\eta(x')$, with $u(x)$ in L_1 on Ω. Zygmund constructed, see [5, p. 644], such a $u(x)$ which fails to have a finite nontangential limit at every point of the boundary of unit disc. Even so, Tolsted and Solomentseff have established in R^2 and R^N respectively that u must have radial limit zero a.e. along any nontangential ray. However, Zygmund's example as well as the other examples in [5], have a zero mean continuous boundary limit a.e., i.e., they satisfy (2).

Open Question: Is there an L_1, Green potential which does not satisfy (2)?[1]

It is interesting to note that continuity at a boundary point y_0 implies mean continuity at y_0 which implies nontangential limit at y_0 for harmonic functions. From the above examples, we see that this hierarchy fails for superharmonic functions. Furthermore it is not clear that mean continuity at y_0 implies a radial limit at y_0 for superharmonic functions.

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UNIVERSITY OF CALIFORNIA, DAVIS

[1] The answer is negative, i.e., every L_1 Green potential satisfies (2). See the Notices, Jaw. 1975.