PACIFIC JOURNAL OF MATHEMATICS
Vol. 55, No. 1, 1974

ON THE INNER APERTURE AND INTERSECTIONS OF CONVEX SETS

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If $C_1, \cdots, C_n$ are n convex surfaces or sets in d-dimensional Euclidean space E^d, then it is of some interest to study the invariance properties of $\bigcap_{i=1}^n (C_i + \boldsymbol{a}_i)$ for all choices of vectors $\boldsymbol{a}_i$ in E^d. Such considerations occur naturally in identifying an object irrespective of the direction in which it approaches the observer.

For example, Melzak [2] and Lewis [1] have investigated the conditions under which the intersection $\bigcap_{i=1}^d (C_i + \boldsymbol{a}_i)$ of certain convex surfaces always is a single point. These surfaces arise from the work of Ratcliff and Hartline [3] concerning varying light intensities upon different visual elements of the eye.

In this article we study such intersections and in Theorem 1, we show that the result of Melzak [1] has an associated Helly number in E^2 but not in E^3. In Theorem 2 we give a necessary and sufficient condition for $\bigcap_{i=1}^n C_i + \boldsymbol{a}_i$ to be nonempty, whenever $C_1, \cdots, C_n$ are convex sets, in terms of the outward normals. This condition is not easy to apply in that it involves the outward normals to intersections of d-membered subsets. So in Theorem 3 we give a sufficient condition in terms of inner and outer apertures which is widely applicable. Finally, in Theorem 4, we give a characterization of the sets which can arise as inner apertures. I am indebted to Z. A. Melzak for suggesting these problems to me.

To define the inner and outer aperture, let D be a convex subset of E^d. If $l \equiv l(\boldsymbol{u}, \boldsymbol{v})$,

$$l = \{\boldsymbol{u} + \lambda \boldsymbol{v}, \lambda \geqq 0\}$$

is a typical ray in E^d, $\boldsymbol{u}, \boldsymbol{v} \in E^d$, $\boldsymbol{v} \neq \boldsymbol{o}$, define

$$\theta(\lambda, D) = \text{dist.} \{\boldsymbol{u} + \lambda \boldsymbol{v}, E^d \backslash D\}$$

and

$$\theta(D) = \sup_{\lambda \geqq 0} \theta(\lambda)$$

where

$$\text{dist.} \{A, B\} = \inf_{\substack{\boldsymbol{a} \in A \\ \boldsymbol{b} \in B}} ||\boldsymbol{a} - \boldsymbol{b}||$$

when A, B are nonempty subsets of E^d. The inner aperture $\mathscr{I}(D)$ of D is the union of those rays $l(\boldsymbol{u}, \boldsymbol{v}) - \boldsymbol{u}$ emanating from the origin

$\boldsymbol{o}$ such that $\theta(l(\boldsymbol{u}, \boldsymbol{v}), D) = +\infty$. So, if D contains $\boldsymbol{o}$, $\mathscr{I}(D)$ is the union of those rays $l \equiv l(\boldsymbol{o}, \boldsymbol{u})$ in D such that $\lambda\boldsymbol{u}$ can be made an arbitrarily large distance from the boundary of D for λ sufficiently large. The outer cone $O(D)$ of D is what is usually known as the characteristic cone namely the set of all rays $l(\boldsymbol{u}, \boldsymbol{v}) - \boldsymbol{u}$ emanating from $\boldsymbol{o}$ with $l(\boldsymbol{u}, \boldsymbol{v})$ contained in D. Both $O(D)$ and $\mathscr{I}(D)$ are convex cones and $O(D)$ is closed whenever D is closed. In general, of course, $O(D)$ can be any convex cone in E^d but this is not the case for $\mathscr{I}(D)$. It will follow from Theorem 4 that $\mathscr{I}(D)$ is a G_δ-convex cone with the property that whenever a ray $l \in \text{cl.}\{\mathscr{I}(D)\}\backslash\mathscr{I}(D)$ then the smallest exposed face $F(l)$ of $\text{cl.}\{\mathscr{I}(D)\}$ containing l also is contained in $\{\text{cl.}\,\mathscr{I}(D)\}\backslash\mathscr{I}(D)$.

THEOREM 1. *Let $C_1^*, \cdots, C_n^*$ be n convex sets in E^d whose d-dimensional interiors are nonempty and do not contain a line. Let $C_1, \cdots, C_n$ be the convex surfaces bounding $C_1^*, \cdots, C_n^*$ respectively. Then $\bigcap_{j=1}^n (C_j + \boldsymbol{a}_j)$ is at most a single point for all choices $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n$ of points in E^d if and only if there does not exist n parallel lines of support $l_1, \cdots, l_n$ to $C_1^*, \cdots, C_n^*$ respectively. In E^2 this is true if and only if some four membered subset $C_{j_1}^*, \cdots, C_{j_4}^*$ do not have parallel lines of support. However, in E^3 and for every $n \geqq 3$ there exist convex sets $C_1^*, \cdots, C_n^*$, whose relative interiors do not contain a line, such that every $n-1$ membered subset have parallel lines of support but this is not so for $C_1^*, \cdots, C_n^*$.*

LEMMA 1. *Let $A_1, \cdots, A_n$ be spherically convex subsets (possibly open, half-open or closed semicircles) of the unit circle S^1 such that*

$$\bigcap_{\nu=1}^{4} (A_{i_\nu} \cup -A_{i_\nu}) \neq \emptyset,\ 1 \leqq i_\nu \leqq n,\ \nu = 1, \cdots, 4\,.$$

Then

$$\bigcap_{i=1}^{n} (A_i \cup -A_i) \neq \emptyset\,.$$

Proof. We parametrise S^1 in terms of the angle θ made with some fixed line through the origin and consider the semicircular interval $[0, \pi]$. The intersection $A_i \cup -A_i$ with $[0, \pi]$ is either

(i) an interval $\langle c_i, d_i \rangle$ not containing either 0 or π,

or (ii) $[0, \pi]$,

or (iii) two intervals $[0, a_i >, < b_i, \pi]$, the first containing 0 and the second containing π.

The classification yields a corresponding subdivision I_1, I_2, I_3 of $\{1, \cdots, n\}$. Let

$$[0, a_{i_1}\rangle = \bigcap_{i \in I_3} [0, a_i\rangle$$
$$\langle b_{i_2}, \pi] = \bigcap_{i \in I_3} \langle b_i, \pi] \,.$$

If $\langle c_i, d_i\rangle$ and $\langle c_j, d_j\rangle$, $i, j \in I_1$ both meet $[0, a_{i_1}\rangle$ and

$$\langle c_i, d_i\rangle \cap \langle c_j, d_j\rangle \cap [0, a_{i_1}\rangle = \varnothing \tag{1}$$

then at least one of these intervals is contained in $[0, a_{i_1}\rangle$. But then

$$(A_i \cup - A_i) \cap (A_j \cup - A_j) \cap (A_{i_1} \cup - A_{i_1}) \cap (A_{i_2} \cup - A_{i_2})$$

is contained in $[0, a_{i_1}\rangle \cup - [0, a_{i_1}\rangle$ and consequently, by (1), is empty, which is contradiction. So, if

$$I_1^1 = \{i \in I_1 \colon \langle c_i, d_i\rangle \cap [0, a_{i_1}\rangle \neq \varnothing\}$$

we have, from Helly's theorem, that

$$[0, a_{i_1}\rangle \cap \bigcap_{i \in I_1^1} \langle c_i, d_i\rangle \neq \varnothing \,. \tag{2}$$

Similarly, if

$$\begin{gathered} I_1^2 = \{i \in I_1 \colon \langle c_i, d_i\rangle \cap \langle b_{i_2}, \pi] \neq \varnothing\} \\ \langle b_{i_2}, \pi] \cap \bigcap_{i \in I_1^2} \langle c_i, d_i\rangle \neq \varnothing \,. \end{gathered} \tag{3}$$

If there exists $i_3 \in I_1 \backslash I_1^1$ and $i_4 \in I_1 \backslash I_1^2$ then

$$\bigcap_{\nu=1}^{4} A_{i_\nu} \cup - A_{i_\nu} = \varnothing \,,$$

so either $I_1^1 = I_1$ or $I_1^2 = I_1$ and, using (2) and (3),

$$\bigcap_{i=1}^{n} A_i \cup - A_i \neq \varnothing \,.$$

REMARK. This is the best possible result for if $A_1 = [0, \pi/2]$, $A_2 = [\pi/4, 3\pi/4]$, $A_3 = [\pi/2, \pi]$, $A_4 = [3\pi4, 5\pi/4]$ then

$$\bigcap_{\nu=1}^{3} A_{i_\nu} \cup - A_{i_\nu} \neq \varnothing, 1 \leqq i_1 < i_2 < i_3 \leqq 4$$

but

$$\bigcap_{i=1}^{4} A_i \cup - A_i = \varnothing \,.$$

LEMMA 2. *There exist n closed spherically convex two dimensional subsets $D_1, \cdots, D_n$ on S^2, none of which contain antipodal points, such that for every $n-1$ membered subset $D_{i_1}, \cdots, D_{i_{n-1}}$ there exists* *a great circle of* S^2 *which meets each* D_{i_ν}, *but there does not exist a great circle meeting each of* $D_1, \cdots, D_n$.

Proof. In [4], Santalo constructs, for each $n \geqq 3$, a family of n compact convex two dimensional sets $F_1, \cdots, F_n$ in E^2 so that each $n-1$ members of the family admit a common transversal but the entire family does not have a common transversal. We mention that such an example is the family of n circular discs whose centers have polar coordinates $\rho = 1$ and $\theta = 2k\pi/n$, $k = 1, \cdots, n$ and whose radii are all equal to $\cos^2 \pi/n$ or $\cos^2 \pi/n + \cos^2 \pi/2n - 1$ according as whether n is even or odd.

Now, if we place the configuration $F_1, \cdots, F_n$ into a plane tangent to S^2, let $D_1, \cdots, D_n$ be the corresponding closed spherically convex subsets of S^2 obtained by the projection of $F_1, \cdots, F_n$ into S^2 from the origin. Clearly $D_1, \cdots, D_n$ satisfy the requirements of the lemma.

Proof of Theorem 1. The proof of the first part is essentially due to Melzak [1] but as he makes the restriction that $d = n$ we repeat the details.

If there exist n parallel lines of support $l_1, \cdots, l_n$ to $C_1^*, \cdots, C_n^*$ respectively then by translating the line l_j into the relative interior of C_j if necessary, $j = 1, \cdots, n$ we obtain n nondegenerate similarly orientated chords $[\boldsymbol{p}_j, \boldsymbol{q}_j]$ of C_j^* parallel to l_j such that

$$||\boldsymbol{p}_1 - \boldsymbol{q}_1|| = \cdots = ||\boldsymbol{p}_n = \boldsymbol{q}_n|| .$$

Hence, if $\boldsymbol{a}_j = \boldsymbol{p}_i - \boldsymbol{p}_j$, $j = 1, \cdots, n$

$$\bigcap_{j=1}^{n} C_j^* + \boldsymbol{a}_j \supset \{\boldsymbol{p}_1, \boldsymbol{q}_1\}$$

and so contains at least two points.

On the other hand, if there exist vectors $\boldsymbol{a}_j$, $j = 1, \cdots, n$ such that $\bigcap_{j=1}^{n} C_j^* + \boldsymbol{a}_j$ contains at least two points say $\boldsymbol{p}, \boldsymbol{q}$ then, by considering two dimensional sections of C_j, C_j has a line of support l_j parallel to $[\boldsymbol{p}, \boldsymbol{q}]$ and hence $l_1, \cdots, l_n$ are parallel lines of support to $C_1, \cdots, C_n$ respectively which completes the proof of the first part.

In E^2 we may select a set A_i of unit tangent vectors $\boldsymbol{u}$ to C_i^* by ensuring that the outward normal lies on the left hand side of $\boldsymbol{u}$ when viewed from the point of contact on C_i in a clockwise direction. Then A_i is a spherically convex subset of S^1 which is either S^1 or is contained in semicircle according to whether or not C_i is bounded. Now $C_1^*, \cdots, C_n^*$ do not have parallel lines of support if and only if

$$\bigcap_{i=1}^{n} (A_i \cup - A_i) = \varnothing .$$

This, by Lemma 1, is true if and only if there exists some four membered subset of $C_1^*, \cdots, C_n^*$ which do not possess parallel lines of support which completes the proof of the second part of the theorem.

In E^3 and for each $n \geqq 2$ consider the n closed spherically convex subsets $D_1, \cdots, D_n$ of S^2 afforded by Lemma 2. If $\langle , \rangle$ denotes scalar product consider the set of closed half-spaces $\mathscr{H}_i$ such that $H^- \in \mathscr{H}_i$ if

$$H^- = \{x: \langle x, u \rangle \leqq 1\} \quad \text{for some} \quad u \in D_i \,.$$

Let

$$C_i^* = \bigcap_{\mathscr{H}_i} H^- \,, \qquad\qquad i = 1, \cdots, n \,.$$

Then D_i is the set of outward normals to C_i^* and so as D_i is two dimensional, C_i^* does not contain a line, $i = 1, \cdots, n$. Also for every $n-1$ membered subset $C_{i_1}^*, \cdots, C_{i_{n-1}}^*$ of $C_1, \cdots, C_n$ the corresponding set of outward normals $D_{i_1}, \cdots, D_{i_{n-1}}$ all meet some great sphere $S \equiv S(i_1, \cdots, i_{n-1})$. Consequently, if l is a line perpendicular to aff. S, $C_{i_1}, \cdots, C_{i_{n-1}}$ each possess lines of support parallel to l.

On the other hand, if $C_1, \cdots, C_n$ possess parallel lines of support then there would exist a great sphers S^1 of S^2 which meets each of $D_1, \cdots, D_n$ which, by Lemma 2, is not so. Hence $C_1, \cdots, C_n$ do not possess parallel lines of support, which completes the proof of Theorem 1.

We observe the following lemma which is easily established by separating two disjoint convex sets by a hyperplane.

LEMMA 3. *Two convex sets C_1, C_2 in E^d cannot be separated by translation if and only if $N(C_1) \cap (-N(C_2)) = o$, where $N(C_i)$ is the convex cone of outward normals to C_i, $i = 1, 2$.*

Using Helly's theorem we readily verify the following lemma.

LEMMA 4. *If $C_1, \cdots, C_n$ are convex sets in E^d, then $\bigcap_{i=1}^n (C_i + a_i) \neq \emptyset$ for all points $a_1, \cdots, a_n$ in E^d if and only if $\bigcap_{\nu=1}^{d+1} (C_{i_\nu} + a_{i_\nu}) \neq \emptyset$ for all points $a_1, \cdots, a_n$ in E^d and for every $d+1$ membered subset $\{C_{i_\nu}\}_{\nu=1}^{d+1}$ of $\{C_i\}_{i=1}^n$.*

Using Lemmas 3 and 4 we obtain

THEOREM 2. *If $C_1, \cdots, C_n$ are convex sets in E^d then $\bigcap_{i=1}^n (C_i + a_i) \neq \emptyset$ for all points $a_1, \cdots a_n$ in E^d if and only if*

$$\{-N(C_{i_1})\} \cap N\Big(\bigcup_{\nu=2}^{d+1} C_{i_\nu}\Big) = \emptyset$$

for all $d+1$ membered subcollections $\{C_{i_\nu}\}_{\nu=1}^{d+1}$ of $\{C_i\}_{i=1}^{n}$.

However, this condition is not completely satisfactory in that $N(\bigcup_{\nu=2}^{d+1} C_{i_\nu})$ is a function of $\bigcup_{\nu=2}^{d+1} C_{i_\nu}$ rather than a combination of functions of each C_{i_ν}. We shall resolve this problem to a certain extent in Theorem 3 by giving a widely applicable sufficient condition.

THEOREM 3. *Let $C_1, \cdots, C_n$ be n convex sets in E^d. Then*

$$\bigcap_{i=1}^{n} (C_i + \boldsymbol{a}_i) \neq \varnothing \tag{4}$$

for all choices of $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n$ if there exists j such that

$$O(\text{cl. } C_j) \cap \bigcap_{\nu=1}^{d+1} \mathscr{I}(C_{i_\nu}) \neq \varnothing$$

for all $d+1$ membered subcollections $\{C_{i_\nu}\}_{\nu=1}^{d+1}$ of $\{C_i\}_{i=1, i\neq j}^{n}$. Further, if at least of cl. $C_1, \cdots,$ cl. C_n *does not contain a line, each is unbounded and $C_1, \cdots, C_n$ cannot be separated by translation,* i.e., (4) *holds for all $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n$ then*

$$\bigcap_{j=1}^{n} O(\text{cl. } C_j) \neq \varnothing \; .$$

Proof. Let l be a ray of $O(\text{cl. } C_j) \cap \bigcap_{i=1}^{n} \mathscr{I}(C_i)$ which, by Helly's theorem, is nonempty. We may suppose, without loss of generality, that $\boldsymbol{o} \in C_1 \cap \cdots \cap C_n$. Then, if $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n$ are points of E^d,

$$l + \boldsymbol{a}_i \subset C_i + \boldsymbol{a}_i \; , \qquad i = 1, \cdots, n \; .$$

If $l = \{\lambda \boldsymbol{u}, \lambda \geqq 0\}$, then, as $l \subset \mathscr{I}(C_i)$, $i \neq j$, there exists λ_i such that $\lambda \boldsymbol{u} + \boldsymbol{a}_j$ is in C_i, $\lambda \geqq \lambda_i$.

So, if $\lambda^* = \max_{1 \leq i \leq n} \lambda_i$,

$$\lambda^* u + a_j \in \bigcap_{i=1}^{n} C_i \quad \text{as required} \; .$$

To prove the second part, let C_i^* denote the closure of C_i, $i = 1, \cdots, n$. We may assume that C_1 and C_1^* do not contain a line and that for some n, $\bigcap_{i=1}^{n-1} C_i^*$ is unbounded, which is certainly true for $n = 2$. As $\bigcap_{i=1}^{n-1} C_i^*$ is convex closed and unbounded it follows that $O(\bigcap_{i=1}^{n-1} C_i^*)$ is nonempty. Further, as $\bigcap_{i=1}^{n-1} C_i^*$ is contained in C_1^*, $\bigcap_{i=1}^{n-1} C_i^*$ and $O(\bigcap_{i=1}^{n-1} C_i^*)$ do not contain a line. Let l be a ray of $O(\bigcap_{i=1}^{n-1} C_i^*)$, say $l = \{\lambda \boldsymbol{u}, \lambda \geqq 0\}$. If $O(\bigcap_{i=1}^{n} C_i^*)$ is empty then, in particular, $\bigcap_{i=1}^{n} C_i^*$ must be a compact convex set.

If $\lambda \geqq 0$,

$$\lambda \boldsymbol{u} + \bigcap_{i=1}^{m-1} C_i \subset \bigcap_{i=1}^{m-1} C_i \; ,$$

and consequently,

$$(5)\qquad \left(\lambda \boldsymbol{u} + \bigcap_{i=1}^{m-1} C_i\right) \cap C_m = \left(\lambda \boldsymbol{u} + \bigcap_{i=1}^{m-1} C_i\right) \cap \left(\bigcap_{i=1}^{m} C_i\right).$$

If no matter how large λ is taken, $(\lambda \boldsymbol{u} + \bigcap_{i=1}^{m-1} C_i) \cap C_m$ contains a point $z(\lambda)$ say then, by (5), $z(\lambda)$ is confined to a compact set $\bigcap_{i=1}^{m} C_i$ and $z(\lambda) - \lambda \boldsymbol{u} \in \bigcap_{i=1}^{m-1} C_i$, $\lambda \geqq 0$. It follows that $-l$ is a ray of $O(\bigcap_{i=1}^{m-1} C_i^*)$ which is a contradiction to C_1^* not containing a line. So $\bigcap_{i=1}^{m} C_i^*$ is an unbounded closed convex set and hence $O(\bigcap_{i=1}^{m} C_i^*)$ is nonempty. So repeating this process for $m = 1, 2, \cdots, n$ we conclude that $O(\bigcap_{i=1}^{n} C_i^*)$ is nonempty as required.

DEFINITION. We say that a collection $\mathscr{H}$ of closed half-spaces in E^d is *closed* if whenever $\{H_i^-\}_{i=1}^{\infty}$ is a sequence of closed half-spaces in $\mathscr{H}$, where

$$H_i^- = \{\boldsymbol{x}: \langle \boldsymbol{x}, \boldsymbol{u}_i \rangle \leqq \alpha_i\},\ \boldsymbol{u}_i \text{ a unit vector},$$

and $\boldsymbol{u}_i \to \boldsymbol{u}$, $\alpha_i \to \alpha$ as $i \to \infty$ then the closed half-space

$$H^- = \{\boldsymbol{x}: \langle \boldsymbol{x}, \boldsymbol{u} \rangle \leqq \alpha\}$$

is in $\mathscr{H}$. We say that a collection $\mathscr{H}$ of closed half-spaces is F_σ if it is the countable union of closed collections.

If $\mathscr{H}$ is a closed collection of closed half-spaces notice that the set $\bigcup_{H^- \in \mathscr{H}} H$, where H is the bounding hyperplane of H^-, is a closed set and consequently $\bigcap_{H^- \in \mathscr{H}} \operatorname{int} H^-$ is a relatively open subset of $\bigcap_{H^- \in \mathscr{H}} H^-$.

THEOREM 4. *A set C in E^d is the inner aperture of some convex subset of E^d if and only if*

$$C = \boldsymbol{o} \cup \bigcap_{\mathscr{H}} \operatorname{int.} H^-$$

where $\mathscr{H}$ is an F_σ-collection of closed half-spaces and $\boldsymbol{o} \in H$, the bounding hyperplane of H^-, for all $H^- \in \mathscr{H}$.

REMARK. So, in particular, C has to be a G_δ-convex cone with apex the origin such that if $\boldsymbol{x} \in \{\text{cl.}\, C\}\backslash C$ then the smallest exposed face $F(\boldsymbol{x})$ of cl. C that contains $\boldsymbol{x}$ is also contained in $\{\text{cl.}\, C\}\backslash C$. In E^3 the converse is also true.

Proof. We shall assume that the theorem is true in $d-1$ dimensions, the theorem being trivial for $d = 1$.

(i) *Necessity.* Let C be the inner aperture of some convex set D in E^d where, since $\mathscr{I}(D) = \mathscr{I}(\text{cl.}\, D)$ we may suppose that D is closed. If $D = E^d$ then $C = E^d$ and, by convention,

$$C = \bigcap_{\mathscr{H}} \text{int. } H^- = E^d$$

where $\mathscr{H}$ is the empty set of closed half-spaces.

Otherwise $D \neq E^d$ and so possesses at least one hyperplane of support M say with D contained in the closed half-space M^-. We may suppose, without loss of generality, that $\boldsymbol{o} \in M$. If D contains a (maximal) linear subspace L of dimension at least one then $L \subset M$ and

$$D = F + L$$

where F is a closed convex subset of $L^\perp$. By the inductive assumption the inner aperture $\mathscr{I}(F)$ of F can be written

$$\mathscr{I}(F) = \boldsymbol{o} \cup \bigcap_{\mathscr{H}^*} \text{int. } H^{*-}$$

where $\mathscr{H}^*$ is a closed subset of the closed half-spaces in $L^\perp$. Then

$$C = \boldsymbol{o} \cup \bigcap_{\mathscr{H}} \text{int. } H^-$$

where $\mathscr{H}$ is the closed collection of closed half-spaces in E^d formed by taking H^- in $\mathscr{H}$ if

$$H^- = L + H^{*-}$$

where $H^{*-} \in \mathscr{H}^*$.

If D does not contain a line then the set of rays in D is a closed convex cone K which has a hyperplane of support say $\{x_d = 0\}$ with

$$K \cap \{x_d = 0\} = \boldsymbol{o} .$$

Let π_ν denote the hyperplane $x_d = \nu$, $\nu \geqq 0$. Let l be a typical ray of K,

$$\alpha_\nu(l) = \text{dist. } \{(l\pi_\nu), \pi_\nu(E^d \backslash D)\} ,$$

and

$$\alpha(l) = \sup_{\nu \geqq 0} \alpha_\nu(l) .$$

By considering two dimensional sections through l it is easily verified that $\alpha_\nu(l)$ increases with ν. Also

$$l \subset C \quad \text{if and only if} \quad \alpha(l) = +\infty .$$

So, if

$$C_i = \{l : l \text{ is a ray in } K, \alpha(l) > i\} ,$$

then

$$C = \bigcap_{i=1}^{\infty} C_i \,. \tag{6}$$

Now $C_i K$, $i = 1, 2, \cdots$ and

$$K = \boldsymbol{o} \cup \bigcap_{\mathscr{H}} \operatorname{int.} H^- \tag{7}$$

where $\mathscr{H}$ is the collection of closed half-spaces, whose bounding hyperplanes contain $\boldsymbol{o}$, such that $K\backslash\boldsymbol{o} \subset \operatorname{int.} H^-$. If $\hat{K} = K \cap S^{d-1}$, let $\mathscr{H}_j^*$ denote the closed set of the closed half-spaces H^-,

$$H^- = \{\boldsymbol{x} \colon \langle \boldsymbol{x}, \boldsymbol{u} \rangle \leqq 0\}$$

where

$$\langle -\boldsymbol{u}, \boldsymbol{k} \rangle \leqq -2^{-j} \,, \qquad \text{for all} \quad k \in \hat{K} \,.$$

Then $\mathscr{H} = \bigcup_{j=1}^{\infty} \mathscr{H}_j^*$ and so, using (6), (7) it is enough to show that

$$C_i = K \cap \bigcap_{\mathscr{H}_i} \operatorname{int.} H^-$$

where $\mathscr{H}_i$ is a closed collection of closed half-spaces of E^d whose bounding hyperplanes goes through $\boldsymbol{o}$.

Suppose now that l is a ray of $K\backslash C_i$. Then

$$\alpha(l) \leqq i \,.$$

For $j = 1, 2, \cdots$, there exist points $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots$, with $\boldsymbol{a}_j \in \pi_j \cap \operatorname{bdy.} D$ such that

$$\|\boldsymbol{a}_j - \{\pi_j \cap l\}\| \leqq i \,. \tag{8}$$

Let H_j denote a hyperplane of support to D at $\boldsymbol{a}_j$, with $D \subset H_j^-$. As we may suppose that $K \neq \boldsymbol{o}$, H_j is not parallel to the hyperplane π_1. So $H_j \cap \pi_1$ is a line in π_1. If we consider the two plane σ_j through l and $\boldsymbol{a}_j$ then H_j meets σ_j in a line l_j. As l_j supports $\sigma_j \cap D$, it follows, using (8), that

$$\|l_j \cap \pi_1 - l \cap \pi_1\| \leqq i \,. \tag{9}$$

Consequently the $(d-2)$ affine space $\pi_1 \cap H_j$ lies within a distance i of $l \cap \pi_1$. So we may suppose, by picking subsequences if necessary, that $\pi_1 \cap H_j \to \pi_1 \cap H_0$ as $j \to \infty$ and $l_j \cap \pi_1$ tends to a point which, with a view to later developments, we denote by $l_0 \cap \pi_1$. Let the line through the points $\boldsymbol{a}_j$ and $l_j \cap \pi_1$ be l_j^*, $j = 1, 2, \cdots$. As (8), (9) hold, l_j^* converges to a line l_0 through $l_0 \cap \pi_1$ and parallel to l. Consequently $H_j \to H_0$ as $j \to \infty$. So $D \subset H_0^-$ and

(10) $$\|\pi_\nu \cap l_0 - \pi_\nu \cap l\| = \beta \leqq i\,, \qquad \text{if} \quad \nu \geqq 0\,,$$

β a constant. We claim that

$$H_0^- + \{\pi_1 l - \pi_1 l_0\} = H_0'^- \text{ say },$$

contains K and H_0' supports K and passes through $\boldsymbol{o}$. Certainly

(11) $$l \subset H_0'$$

and so H_0' passes through $\boldsymbol{o}$. If there exists a ray l^* in $K \backslash H_0'^-$, then l^* meets H_0 which contradicts $D \subset H_0^-$.

Now let $\mathscr{H}_i$ denote those closed half-spaces H^- such that the bounding hyperplane H supports K and there exists a closed half-space H^{*-} containing H^- such that H^* supports D; H^* is parallel to H and a distance, in the hyperplane π_1, at most i from H.

By (11),

(12) $$C_i \supset K \cap \bigcap_{\mathscr{H}_i} \text{int. } H^-\,,$$

where $\mathscr{H}_i$ is a closed set of closed half-spaces.

Conversely, if l is a ray of

$$K \backslash \{K \cap \bigcap_{\mathscr{H}_i} \text{int. } H^-\}$$

then there exists H^- in $\mathscr{H}_i$ such that $l \subset H$. Then there exists a closed half-space H^{*-} which contains D such that H^* is parallel to H and the distance between H and H^* is at most i. Consequently

$$\alpha_\nu(l) \leqq i,\ \nu \geqq 0$$

and so $l \not\subset C_i$. Hence

(13) $$C_i \subset K \cap \bigcap_{\mathscr{H}_i} \text{int. } H^-\,.$$

Combining (12) and (3),

$$C_i = K \cap \bigcap_{\mathscr{H}_i} \text{int. } H^-$$

which completes the proof of the necessity of the conditions.

(ii) *Sufficiency*. Suppose now that

$$C = \boldsymbol{o} \cup \bigcap_{\mathscr{H}} \text{int. } H^-$$

where $\mathscr{H}$ is an F_σ-collection of closed half-spaces and $\boldsymbol{o} \in H$ for all $H^- \in \mathscr{H}$. So we may write $\mathscr{H} = \bigcup_{i=1}^\infty \mathscr{H}_i$ where the $\mathscr{H}_i$ form an increasing sequence of closed collections.

Consider the closed convex cone

$$C_0 = \text{cl.}\, C = \bigcap_{\mathscr{H}} H^- .$$

If $C_0 = E^d$ then $C = E^d$ and C is its own inner aperture. Otherwise C_0 possesses one hyperplane of support M through $\boldsymbol{o}$ with C_0 contained in the closed half-space M^-. If $M \cap C_0$ contains a maximal linear subspace L of dimension at least 1 then we may write $C_0 = F + L$ where F is a proper closed convex cone in L. Notice that $L \subset H$ for each $H^- \in \mathscr{H}$ and consequently we may write

$$H^- = L + H^{*-} \quad \text{for each} \quad H^- \in \mathscr{H} ,$$

where H^{*-} is a closed half-space in L whose bounding hyperplane H^* passes through $\boldsymbol{o}$. Consequently

$$C = \boldsymbol{o} \cup \{\{\bigcap_{\mathscr{H}} \text{int.}\, H^{*-}\} + L\} .$$

By the inductive assumption, there exists a closed convex set D^* in L such that

$$\boldsymbol{o} \cup \bigcap_{\mathscr{H}} \text{int.}\, H^{*-}$$

is the inner aperture of D^* in L. Let

$$D = D^* + L$$

and then C is the inner aperture of D.

Henceforth therefore we may suppose that C_0 is a proper closed convex cone in E^d i.e., C_0 does not contain a line and we can also suppose that the ray

$$X_d^+ = \{(0, \cdots, 0, x_d), x_d \geqq 0\}$$

is in C_0 and that the hyperplane $\pi_0 = \{x_d = 0\}$ supports C_0 with $\pi_0 \cap C_0 = \boldsymbol{o}$. Then, as for K in the proof of necessity,

$$C_0 = \boldsymbol{o} \cup \bigcap_{\mathscr{H}_0} \text{int.}\, H^-$$

where $\mathscr{H}_0$ is a closed set of closed half-spaces whose bounding hyperplanes pass through $\boldsymbol{o}$. We may suppose that

$$\mathscr{H}_0 \subset \mathscr{H}_1 \subset \mathscr{H}_2 \subset \cdots$$

and let

$$C_i = \boldsymbol{o} \cup \bigcap_{\mathscr{H}_i} \text{int.}\, H^- , \qquad i = 0, 1, 2, \cdots .$$

We shall produce inductively a nested sequence of closed convex sets $\{C_i^*\}_{i=0}^{*}$ such that C_i is the inner aperture of C_i^* and indeed

(14) $$C_{i+1}^* = C_i^* \cap \bigcap_{\mathscr{H}_i} H^{*-}, \ i \geqq 0$$

where, if $H^- \in \mathscr{H}_i$ then H^{*-} is that closed half-space containing H^- such that H^* and H are parallel and at a distance i apart in the hyperplane π_1.

We begin the induction by taking

$$C_0^* = \{\boldsymbol{x} = (x_1, \cdots, x_d),\ x_d \geqq 0 \quad \text{and} \quad \text{dist.}\,(\boldsymbol{x}, C_0 \cap \pi_{x_d}) \leqq x_d^{1/2}\}\ .$$

Clearly C_0^* is closed and it is convex since, from above, $C_0^* \cap \pi_\nu$ is convex, $\nu \geqq 0$ and so C_0^* cannot possess a point of concavity. We shall show that

(15) $$\mathscr{I}(C_0^*) = C_0\ .$$

First notice that if $\boldsymbol{u} = (u_1, \cdots, u_d)$ is a unit vector in C_0 then $u_d > 0$. So, if $l = \{\lambda u: \lambda \geqq 0\}$ is the corresponding ray in C_0

$$\theta_\lambda = \alpha_{\lambda u_d}(l) \geqq \sqrt{\lambda u_d} > 0\ .$$

So, if m is a positive number

(16) $$\theta_\lambda \geqq m$$

provided $m^2/u_d \leqq \lambda$. It is an almost immediate consequence of (16) that $l \subset \mathscr{I}(C_0^*)$ and hence $C_0 \subset \mathscr{I}(C_0^*)$.

Suppose next that the ray

$$l' = \{\lambda \boldsymbol{v},\ \lambda \geqq 0\}$$

is not in C_0. If $v_d \leqq 0$ then $\lambda \boldsymbol{v} \notin C_0^*$ for all $\lambda > 0$ and then certainly $l' \not\subset \mathscr{I}(C_0^*)$. If $v_d > 0$ then $l' \cap \pi_\nu$ is a single point for each $\nu \geqq 0$ and there exists $\eta > 0$ such that

$$\text{dist.}\,(\boldsymbol{v}, C_0 \cap \pi_{v_d}) > \eta\ .$$

So

(17) $$\text{dist.}\,(\lambda \boldsymbol{v}, C_0 \pi_{\lambda v_d}) > \lambda \eta\ .$$

But, if $l' \subset \mathscr{I}(C_0^*)$ then, in particular, $\lambda \boldsymbol{v} \in C_0^*$ for each $\lambda \geqq 0$. So

(18) $$\text{dist.}\,(\lambda \boldsymbol{v}, C_0 \pi_{\lambda v_d}) \leqq (\lambda v_d)^{1/2},\ \lambda \geqq 0\ .$$

However, provided $\lambda > v_d/\eta^2$ it follows from (17) that (18) is false. Consequently $l' \not\subset \mathscr{I}(C_0^*)$ which establishes (15).

Suppose inductively that for some $m \geqq 1$ we have constructed m closed convex sets $C_0^*, \cdots, C_{m-1}^*$ in E^d with C_i being the inner aperture of C_i^*, $i = 0, \cdots, m-1$. Indeed,

(19) $$C_{i+1}^* = C_i^* \cap \bigcap_{\mathscr{H}_{i+1}} H^{*-}\,, \qquad i = 0, 1, \cdots, m-2\,,$$

where, if $H^- \in \mathscr{H}_{i+1}$ then H^{*-} is that closed half-space containing H^- such that H^* and H are parallel and at a distance $i+1$ apart in the plane π_1.

For each $H^- \in \mathscr{H}_m$, let H^{*-} be that closed half-space containing H^- such that H^* and H are parallel and at a distance m apart in the plane π_1. Define

(20) $$C_m^* = C_{m-1}^* \cap \bigcap_{\mathscr{H}_m} H^{*-}\,.$$

We claim that the inner aperture of C_m^* is C_m i.e.,

(21) $$\mathscr{I}(C_m^*) = C_m\,.$$

If l is a ray of C_0 not in C_m then l is in some hyperplane H where $H^- \in \mathscr{H}_m$. Consequently, by considering the corresponding closed half-space H^{*-}, we deduce that $\alpha(l) \leqq m$, and so $l \not\subset \mathscr{I}(C_m^*)$. Hence $\mathscr{I}(C_m^*) \subset C_m$.

On the other hand, suppose that $l \in C_m$. That the set

$$\bigcup_{\mathscr{H}_m} H^* = H_m \text{ say}$$

is a closed set and does not meet the ray $l \backslash \boldsymbol{o}$. As each hyperplane H, with $H^- \in \mathscr{H}_m$, passes through $\boldsymbol{o}$, it follows that

(22) $$\text{dist.}\,(l \cap \pi_\nu, H_m) \longrightarrow +\infty \quad \text{as} \quad \nu \longrightarrow +\infty\,.$$

Also $l \in \mathscr{I}(C_{m-1}^*)$ and so

(23) $$\text{dist.}\,(l \cap \pi_\nu, E^d \backslash C_{m-1}^*) \longrightarrow +\infty \quad \text{as} \quad \nu \longrightarrow +\infty\,.$$

Consequently using (20), (22), (23),

$$\text{dist.}\,(l \cap \pi_\nu, E^d \backslash C_m^*) \longrightarrow +\infty \quad \text{as} \quad \nu \longrightarrow +\infty\,.$$

Therefore, $l \subset \mathscr{I}(C_m^*)$ and so $C_m \subset \mathscr{I}(C_m^*)$ which completes the verification of (21).

The results (20), (21) verify (19) for m and we can now suppose that the C_m^* have been defined so that (20), (21) hold for $m = 0, 1, 2, \cdots$. Define

$$C^* = \bigcap_{m=0}^{\infty} C_m^*$$

and we shall show that $\mathscr{I}(C^*) = C$.

Suppose that l is a ray of C_0 not in $\mathscr{I}(C^*)$. Then there exists m such that $\alpha_\nu(l) \leqq m$, $\nu \geqq 0$. So l is not in $\mathscr{I}(C_{m+1}^*) = C_{m+1}$. Consequently l is not in C. So $C \subset \mathscr{I}(C^*)$.

On the other hand, suppose that l is a ray of C_0 which is not in C. Then l is not in C_m for some $m \geqq 0$. So

$$l \not\subset \mathscr{I}(C_m^*) \supset \mathscr{I}(C^*) .$$

Hence $\mathscr{I}(C^*) \subset C$ and this finally establishes that

$$\mathscr{I}(C^*) = C$$

which completes the proof of Theorem 4.

REFERENCES

1. J. E. Lewis, *On a problem of uniqueness arising in connection with a neurophysical control mechanism*, Submitted to Information and Control.
2. Z. A. Melzak, *On a uniqueness theorem and its application to a neurophysical control mechanism*, Information and Control, **5** (1962), 163-172.
3. F. Ratcliff and H. K. Hartline, *The response of Limus optic nerve fibers to patterns of illumination on the receptor mosaic*, J. Genl. Physiol., **42** (1959), 1241-1255.
4. L. A. Santaló, *Un teorema sobre conjuntos de paralelepipedos de aristas paralelas*, Publ. Inst. Mat. Univ. Nac. Litoral, **2** (1940), 49-60 and **3** (1942), 202-210.

Received April 30, 1973 and in revised form August 21, 1973. Research supported by the National Research Council of Canada, while visiting the University of British Columbia.

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