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BEST APPROXIMATION BY A SATURATION CLASS OF POLYNOMIAL OPERATORS

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The problem of determining a saturation class has been considered by Zamanski, Sunouchi and Watari and others. Zamanski has considered the Cesaro means of order 1 and Sunouchi and Watari have studied the Riesz means of type n. The object of the present paper is to extend these results by considering Nörlund means which include the above-mentioned results as particular cases.

1. Let $\{p_n\}$ be a sequence of positive constants such that

$$P_n = p_0 + \cdots + p_n \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty .$$

A given series $\sum_{n=0}^{\infty} d_n$ with the sequence of partial sums $\{S_n\}$ is said to summable (N, p_n) to d, provided that

$$(1.1)\qquad \begin{aligned} N_n\left[\sum_{l=0}^{\infty} d_l\right] &= \frac{1}{P_n}\sum_{k=0}^{n} P_{n-k} d_k \\ &= \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k} S_k \longrightarrow d , \quad \text{as} \quad n \longrightarrow \infty , \end{aligned}$$

and N_n are called the Nörlund operators.

Let

$$(1.2)\qquad \frac{1}{2}a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{\infty} A_k(x)$$

be the Fourier series associated with a continuous periodic function $f(x)$, with period 2π.

We define

$$(1.3)\qquad N_n(x) \equiv N_n(f; x) \equiv \frac{1}{P_n}\sum_{k=0}^{n} P_{n-k} A_k(x)$$

and the norm

$$\|f(x) - N_n(x)\| \equiv \max_{0 \leq x \leq 2\pi} |f(x) - N_n(x)| .$$

If there exists positive nonincreasing function $\phi(n)$ and a class of functions K, with the following properties:

(I) $\|f(x) - N_n(x)\| = o(\phi(n)) \Longrightarrow f(x)$ is constant,

(II) $\|f(x) - N_n(x)\| = O(\phi(n)) \Longrightarrow f(x) \in K$

and

(III) $f(x) \in K \Longrightarrow ||f(x) - N_n(x)|| = O(\phi(n))$,

then the Nörlund operators are saturated with the order $\phi(n)$ and the class K.

In this paper we prove that the above method of summations is saturated with the order p_n/P_n and that the class K consists of all continuous functions f such that $\tilde{f} \in Lip\ 1$, where $\tilde{f}$ is the conjugate function of f. By definition

$$\tilde{f}(x) = \frac{1}{2\pi} \int_0^\pi [f(x+t) - f(x-t)] \cot \frac{1}{2} t \, dt ,$$

if the integral converges absolutely for all x and if

$$\int_0^\pi | f(x+t) - f(x-t) | \cot \frac{t}{2} dt$$

is an integrable function.

The problem of determining a saturation class by considering $(C, 1)$ means of the Fourier series of $f(x)$ has been considered by Zamanski [6]. Sunouchi and Watari [4] have considered the problem by taking (R, λ, k) means of the Fourier series. Some of these results were later extended by Sunouchi [3] and others [2, 5].

2. We shall prove the following theorem.

THEOREM. *Let $\{p_n\}$ be a sequence of positive constants satisfying the following conditions,*

$$\text{(2.1)} \qquad \frac{p_{n-k}}{p_n} \longrightarrow 1 \quad as \quad n \longrightarrow \infty \quad for\ a\ fixed \quad k \leqq n ,$$

and

$$\text{(2.2)} \qquad \sum_{k=0}^{n} |p_{n-k} - p_{n-k-1}| = O(p_n) \quad where \quad [p_{-1} = 0] .$$

Then the operators N_n are saturated with order p_n/P_n and the class of all continuous functions f for which $\tilde{f} \in Lip\ 1$.

The following lemmas are required for the proof of the theorem.

LEMMA 2.1. *If*

$$||f(x) - N_n(x)|| = o\left[\frac{p_n}{P_n}\right]$$

then f is a constant.

Proof. From (1.3) we obtain

$$\frac{1}{\pi}\int_{-\pi}^{\pi} N_n(x)\cos rx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\sum_{k=0}^{n}\frac{P_{n-k}}{P_n}A_k(x)\cos rx\,dx$$
$$= \frac{1}{\pi}\sum_{k=0}^{n}\frac{P_{n-k}}{P_n}\int_{-\pi}^{\pi}A_k(x)\cos rx\,dx$$
$$= \frac{P_{n-r}}{P_n}a_r\,.$$

Thus,

$$a_r - \frac{P_{n-r}}{P_n}a_r = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos rx\,dx - \frac{1}{\pi}\int_{-\pi}^{\pi}N_n(x)\cos rx\,dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos rx\,[f(x) - N_n(x)]dx\,,$$

hence

$$\left|a_r - \frac{P_{n-r}}{P_n}a_r\right| \leqq \|f(x) - N_n(x)\|\frac{1}{\pi}\int_{-\pi}^{\pi}1\cdot dx = o\left[\frac{p_n}{P_n}\right].$$

Consequently

$$a_r\left\{\frac{p_n + \cdots + p_{n-r+1}}{p_n}\right\} = o(1)\,, \tag{2.3}$$

and since $p_r > 0$ for all r, we have $(p_n + \cdots + p_{n-r+1})/p_n \geqq 1$ for $r \geqq 1$.

Thus from (2.3) it follows that $a_r = 0$, for each $r \geqq 1$. Similarly we can show that $b_r = 0$ for each $r \geqq 1$. Hence $f(x) = 1/2a_0$, a constant.

LEMMA 2.2. *If*

$$\|f(x) - N_n(x)\| = O\left[\frac{p_n}{P_n}\right]$$

and condition (2.1) *is satisfied, then* $\tilde{f}(x) \in Lip$ 1.

Proof. It can be shown without much difficulty that if

$$\|f(x) - N_n(x)\| = O\left[\frac{p_n}{P_n}\right],$$

then

$$\left\|\sum_{k=1}^{N}\frac{p_n + \cdots + p_{n-k+1}}{p_n}A_k(x)\left[1 - \frac{k}{N+1}\right]\right\| = O(1),\ N \leqq n\,.$$

Taking the limit as $n \longrightarrow \infty$, and using condition (2.1), we obtain

$$\left\| \sum_{k=1}^{N} kA_k(x)\left[1 - \frac{k}{N+1}\right]\right\| = O(1) . \tag{2.4}$$

The left hand side of the above equation represents the $(C, 1)$ mean of the series

$$\sum_{k=1}^{\infty} - kA_k(x) .$$

Since $-kA_k(x) = B'_k(x)$, where $\sum_{k=1}^{\infty} B_k(x) \equiv \sum_{k=1}^{\infty} (b_k \cos kx - a_k \sin kx)$ is the conjugate series of (1.2), then (2.4) is equivalent to

$$\| \tilde{\sigma}'_N(f) \| < M$$

which implies that $\tilde{f}(x) \in Lip\ 1$, [1].
($\tilde{\sigma}_N(f)$ represents the $(C, 1)$ mean of the conjugate series.)

LEMMA 2.3. *Assume* $\tilde{f} \in Lip\ 1$. *If the sequence* $\{p_n\}$ *satisfies condition* (2.2), *then*

$$\| f(x) - N_n(x) \| = O\left[\frac{p_n}{P_n}\right] .$$

Proof. Since, by definition

$$\tilde{S}_n(\tilde{f}, x) = \frac{1}{\pi} \int_0^{\pi} [\tilde{f}(x, t) - \tilde{f}(x - t)] \frac{\cos \frac{t}{2} - \cos\left[n + \frac{1}{2}\right]t}{2 \sin \frac{t}{2}} dt$$

where $\tilde{S}_n(\tilde{f}, x)$ denotes the partial sums of the conjugate series associated with $\tilde{f}(x)$, we have

$$\begin{aligned} N_n(\tilde{S}_n(\tilde{f}, x)) &= \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} \tilde{S}_k(\tilde{f}, x) \\ &= \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} \frac{1}{2\pi} \int_0^{\pi} [\tilde{f}(x + t) - \tilde{f}(x - t)] \cot \frac{1}{2} t\, dt \\ &\quad - \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} \frac{1}{2\pi} \int_0^{\pi} [\tilde{f}(x + t) - \tilde{f}(x - t)] \frac{\cos\left[k + \frac{1}{2}\right]t}{\sin \frac{1}{2} t} dt . \end{aligned}$$

Since the function $\tilde{f}(x) \in Lip\ 1$, $-f + (1/2)a_0$ is identical to $\tilde{\tilde{f}}$, therefore

$$f(x) - N_n(f, x) = \frac{1}{2\pi} \int_0^{\pi} [\tilde{f}(x + t) - \tilde{f}(x - t)] K_n(t) dt , \tag{2.5}$$

where

$$K_n(t) = \frac{1}{P_n \sin\frac{1}{2}t} \sum_{k=0}^{n} p_{n-k} \cos\left[k + \frac{1}{2}\right]t\,.$$

Now by partial summation

$$\begin{aligned} K_n(t) &= \frac{1}{2P_n \sin^2\frac{1}{2}t} \sum_{k=0}^{n} (p_{n-k} - p_{n-k-1}) \sin(k+1)t \\ &= \frac{1}{P_n}\left\{\frac{2}{t^2} + O(1)\right\} \sum_{k=0}^{n} (p_{n-k} - p_{n-k-1}) \sin(k+1)t \\ &= \frac{2}{P_n t^2} \sum_{k=0}^{n} (p_{n-k} - p_{n-k-1}) \sin(k+1)t + O\left[\frac{p_n}{P_n}\right], \end{aligned}$$

by hypothesis. Since $\tilde{f}(x)$ is certainly bounded, the right hand side of (2.5) becomes

$$\text{(2.6)} \qquad \begin{aligned} &\frac{1}{\pi P_n} \int_0^{\pi} [\tilde{f}(x+t) - \tilde{f}(x-t)] \frac{1}{t^2}\left\{\sum_{k=0}^{n} (p_{n-k} - p_{n-k-1}) \sin(k+1)t\right\} dt \\ &\quad + O\left[\frac{p_n}{P_n}\right]. \end{aligned}$$

Let us write

$$F_n(t) = \frac{1}{P_n} \int_t^{\pi} \frac{1}{u^2}\left\{\sum_{k=0}^{n} (p_{n-k} - p_{n-k-1}) \sin(k+1)u\right\} du\,.$$

Since $\tilde{f}(u) \in Lip\ 1$, it is an indefinite integral of a bounded function, say $\tilde{f}'(u)$. Further, since $\tilde{f}(x+t) - \tilde{f}(x-t) = O(t)$, as $t \to 0$, while for fixed n, $F_n(t) = O(\log(1/t))$, we can integrate (2.6) by parts to obtain

$$\frac{1}{\pi} \int_0^{\pi} [\tilde{f}'(x+t) + \tilde{f}'(x-t)] F_n(t) dt + O\left[\frac{p_n}{P_n}\right],$$

noting that the integrated term vanishes at both limits. The absolute value of this above expression is now,

$$\text{(2.1)} \qquad O\left\{\int_0^{\pi} |F_n(t)|\, dt\right\} + O\left[\frac{p_n}{P_n}\right] \text{ since } \tilde{f}' \text{ is bounded}\,.$$

Now

$$\begin{aligned} F_n(t) &= \frac{1}{P_n} \sum_{k=0}^{n} (p_{n-k} - p_{n-k-1}) \int_t^{\pi} \frac{\sin(k+1)u}{u^2} du \\ &= \frac{1}{P_n} \sum_{k=0}^{n} (p_{n-k} - p_{n-k-1})(k+1) \int_{(k+1)t}^{(k+1)\pi} \frac{\sin\nu}{\nu^2} d\nu\,. \end{aligned}$$

However,

$$\int_{(k+1)t}^{(k+1)\pi} \frac{\sin \nu}{\nu^2} d\nu = \begin{cases} O(\log 1/(k+1)t) & \text{if} \quad (k+1)t < 1 \\ O(1/(k+1)^2 t^2) & \text{if} \quad (1(k+1)t \geqq 1 \,. \end{cases}$$

Hence

$$\int_0^{\pi} |F_n(t)|\, dt = O\left\{\frac{1}{P_n} \int_0^{\pi}\left[\sum_{\substack{(k+1)<1/t \\ k\geqq 0}} |p_{n-k} - p_{n-k-1}|(k+1)\log(1/(k+1)t)\right.\right.$$

$$\left.+ \sum_{\substack{(k+1)\geqq 1/t \\ k\leqq n}} |p_{n-k} - p_{n-k-1}|\, 1/(k+1)t^2\right] dt$$

$$= O\left\{\frac{1}{P_n}\sum_{k=0}^{n} |p_{n-k} - p_{n-k-1}|\left[\int_0^{1/(k+1)} (k+1)\log(1/(k+1)t)dt\right.\right.$$

$$\left.\left. + \int_{1/(k+1)}^{\pi} \frac{1}{(k+1)t^2} dt\right]\right\} .$$

Further,

$$\int_0^{1/(k+1)} \log(1/(k+1)t)dt = \int_0^1 \log\left(\frac{1}{u}\right)du = \text{constant}$$

and

$$\int_{1/(k+1)}^{\pi} \frac{1}{(k+1)t^2} dt < M \text{ (constant)} ,$$

therefore

$$\int_0^{\pi} |F_n(t)|\, dt = O\left\{\frac{1}{P_n}\sum_{k=0}^{n} |p_{n-k} - p_{n-k-1}|\right\} = O\left[\frac{p_n}{P_n}\right]$$

from (2.2).

Thus (2.7) and hence (2.6) is $O[p_n/P_n]$.
Consequently from (2.5), we have that

$$||f(x) - N_n(f, x)|| = O\left[\frac{p_n}{P_n}\right]$$

which proves the lemma.

The proof of the theorem now follows from Lemmas 2.1, 2.2, and 2.3.

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