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A PRODUCT INTEGRAL SOLUTION OF A RICCATI EQUATION

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In Memory of Professor H. S. Wall

Product integrals are used to show that, if dw, G, H and K are functions from number pairs to a normed complete ring N which are integrable and have bounded variation on $[a, b]$ and v^{-1} exists and is bounded on $[a, b]$, then the integral equation

$$\beta(x) = w(x) + (LRLR)\int_a^x (\beta H + G\beta + \beta K\beta)$$

has a solution $\beta(x) = v^{-1}(x)u(x)$ on $[a, b]$, where u and v are defined by the matrix equation

$$[u(x),\ v(x)] = [w(a),\ 1]\ {}_a\prod^x \left(I + \begin{bmatrix} H & -K \\ dw & -G \end{bmatrix}\right)$$

The above results are used to show that if p, q, h and r are quasicontinuous functions from the numbers to N such that h is left continuous and has bounded variation and p, q and h commute, then the solution on $[a, b]$ of the differential-type equation $f^{**} + f^*p + fq = r$ is

$$f(x) = f(a)\ {}_a\prod^x (1 - \beta dh) + (R)\int_a^x dz\ {}_t\prod^x (1 - \beta dh),$$

where $f(x) - f(a) = (L)\int_a^x f^* dh$, β is the solution of

$$\beta(x) = (L)\int_a^x q dh + (LL)\int_a^x \beta(-pdh) + (LR)\int_a^x \beta dh\beta,$$

and z is defined in terms of p, q, r, h and β.

1. Introduction. Adam [1] introduced the concept of continuous continued fractions and showed that the solution of $y' = g'y^2 - f'$ could be given as a continuous continued fraction, provided f' and g' are continuous and positive. Wall [11] [12] showed that, if F_{11}, F_{12}, F_{21} and F_{22} are continuous functions of bounded variation from the real numbers to the complex numbers and $|b - a|$ is sufficiently small, then the solution of

$$(1) \qquad w(x) = z + \int_b^x w^2 dF_{21} + \int_b^x w d(F_{22} - F_{11}) - \int_b^x dF_{12}$$

is $w(x) = [M_{11}(x, b)z + M_{12}(x, b)][M_{21}(x, b)z + M_{22}(x, b)]^{-1}$, where $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$ and $\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ is the function such that $M(x, y) = 1 + \int_x^y M(x, s) dF(s)$. MacNerney, using the Stieltjes integral in **[7]** and the subdivision-refinement-type mean integral in **[8]**, extended Wall's results to some types of quasicontinuous linear transformations and showed that the solution of Equation (1) can also be expressed as a continuous continued fraction [**8**, Theorem 5.3]. In this paper the product integral theory developed by MacNerney **[8]** **[9]** and the author **[3]** is used to find and express (in §3) the solution of

$$\beta(x) = w(x) + (LRLR) \int_a^x (\beta H + G\beta + \beta K \beta)$$

and to find and express (in §4) the solution of

$$f^{**} + f^*p + fq = r,$$

where w, p, q, r, G, H, K are quasicontinuous functions from numbers or pairs of numbers to a normed complete ring N.

2. Definitions and notations. The symbol R denotes the set of real numbers and N is a ring which has an identity element **1** and a norm $|\cdot|$ with respect to which N is complete and $|\mathbf{1}| = 1$ (henceforth, the symbol 1 will be used for this identity element). Functions from R to N and from $R \times R$ to N will be represented by lower case letters and upper case letters, respectively. All sum and product integrals are subdivision-refinement-type limits. If G is a function from $R \times R$ to N, the product integral of G exists on $[a, b]$ iff there exists $A \in N$ such that if ϵ is a positive number then there is a subdivision D of $[a, b]$ such that if $\{x_i\}_0^n$ is a refinement of D then $|A - G_1 G_2 \cdots G_n| < \epsilon$, where $G_i = G(x_{i-1}, x_i)$ for $i = 1, 2, \cdots, n$. The symbol ${}_a\Pi^b G$ will be used to represent the limit A. A similar definition holds for the sum integral. Upper case letters preceding an integral symbol show how the integrand is to be evaluated: i.e., $(LRLR)\int_a^b (fH + Gf + fKf) = \int_a^b M$, where for $x < y$

$$M(x, y) = f(x)H(x, y) + G(x, y)f(y) + f(x)G(x, y)f(y).$$

Also, $G \in OA^0$ on $[a, b]$ iff $\int_a^b G$ exists and $\int_a^b |G - \int G| = 0$; $G \in OM^0$ on $[a, b]$ iff ${}_x\Pi^y(1+G)$ exists for $a \leqq x \leqq y \leqq b$ and $\int_a^b |(1+G) - \Pi(1+G)| = 0$; $G \in OB^0$ on $[a, b]$ iff there is a number M and a subdivision D of $[a, b]$ such that, if $\{x_i\}_0^n$ is a refinement of D ,then $\Sigma_1^n |G(x_{i-1}, x_i)| \leqq M$; the function v^{-1} exists on $[a, b]$ means $v(x)v(x)^{-1} = v(x)^{-1}v(x) = 1$ for $x \in [a, b]$. The function G^{-1} exists on $[a, b]$ means there is a subdivision $\{x_i\}_0^n$ of $[a, b]$ such that if $0 < i \leqq n$ and $x_{i-1} \leqq x < y \leqq x_i$, then $G(x, y)^{-1}G(x, y) = G(x, y)G(x, y)^{-1} = 1$. If $\{x_i\}_0^n$ is a subdivision, the symbols f_{i-1}, f_i, and G_i will be used as shorthand notations for $f(x_{i-1})$, $f(x_i)$ and $G(x_{i-1}, x_i)$, respectively. For additional details pertaining to these definitions, see **[3]**, **[4]**, and **[9]**. The main results of this paper will be designated as theorems; the supporting theorems will be labeled as lemmas.

3. A Riccati integral equation. In this section we derive a solution for the integral equation

$$f(x) = w(x) + (LRLR)\int_a^x (fH + Gf + fKf).$$

Since the OA^0 property plays an important role in this paper, please note that the function $G \in OA^0$ if at least one of the following conditions is satisfied:

(1) there is a function g such that

$$G(x, y) = g(y) - g(x);$$

(2) if $G(x, y) = f(x)H(x, y)$, where f is quasicontinuous and $H \in OA^0$ and OB^0, **[4**, Theorem 2];

(3) if G is an integrable function from number pairs to a real Hilbert space which is finite dimensional, **[2**, Theorem 2].

Also note that, if H, K, W, G are functions from $R \times R$ to N which belong to OA^0 and OB^0, then $\begin{bmatrix} H & K \\ W & G \end{bmatrix}$ represents a matrix Q such that $Q \in OA^0$ and OB^0 and, by Lemma 3.1, $Q \in OM^0$.

LEMMA 3.1. *If G is a function from $R \times R$ to a normed complete ring and $G \in OB^0$, then the following statements are equivalent:*

(1) *$G \in OA^0$ on $[a, b]$ and*

(2) *$G \in OM^0$ on $[a, b]$.*

This is Theorem 3.4 *of* **[3]**.

THEOREM 3.2. *Given.* (1) $[a, b]$ *is a number interval.* (2) w *is a function from* R *to* N *and* H, G *and* K *are functions from* $R \times R$ *to* N *such that each of* dw, H, G *and* K *belongs to* OA^0 *and* OB^0.

(3) u *and* v *are functions from* R *to* N *such that if* $x \in [a, b]$ *then* $u(x)$ *and* $v(x)$ *are defined by the matrix equation*

$$[u(x),\, v(x)] = [w(a),\, 1]\ {}_a\prod^x \left(I + \begin{bmatrix} H & -K \\ dw & -G \end{bmatrix}\right);$$

and v^{-1} *exists and is bounded.*

(4) f *is a bounded function from* R *to* $N, f(a) = w(a)$ *and* $f(x) = v(x)^{-1}u(x)$ *for* $x \in [a, b]$.

Conclusion. If $x \in [a, b]$, then

$$f(x) = w(x) + (LRLR) \int_a^x (fH + Gf + fKf).$$

Furthermore, if w is a constant function, then

$$f(x) = \left[{}_a\prod^x (1 - G) - w(a)(LR) \int_a^x {}_a\prod^t (1 + H)K\ {}_t\prod^x (1 - G) \right]^{-1} \left[w(a)\ {}_a\prod^x (1 + H) \right].$$

Proof. Let Q be the function such that $Q = \begin{bmatrix} 1 + H & -K \\ dw & 1 - G \end{bmatrix}$; then $Q - I \in OA^0$ and OB^0 and, by Lemma 3.1, $Q - 1 \in OM^0$. Suppose $x \in (a, b]$ and $\{x_i\}_0^n$ is a subdivision of $[a, x]$. If $0 < i \leqq n$, then there exist a_i and $b_i \in N$ such that

$$\begin{aligned}
[v(x_i)f(x_i),\, v(x_i)] &= [u(x_i),\, v(x_i)] \\
&= [w(a),\, 1]\ {}_a\prod^{x_{i-1}} Q\ {}_{x_{i-1}}\prod^{x_i} Q \\
&= [u(x_{i-1}),\, v(x_{i-1})]\ {}_{x_{i-1}}\prod^{x_i} \begin{bmatrix} 1 + H & -K \\ dw & 1 - G \end{bmatrix} \\
&= [u_{i-1},\, v_{i-1}] \begin{bmatrix} 1 + H_i & -K_i \\ \Delta w_i & 1 - G_i \end{bmatrix} + [a_i,\, b_i] \\
&= v_{i-1}[f_{i-1},\, 1] \begin{bmatrix} 1 + H_i & -K_i \\ \Delta w_i & 1 - G_i \end{bmatrix} + [a_i,\, b_i]
\end{aligned}$$

$$= v_{i-1}[f_{i-1}(1+H_i)+\Delta w_i,\ -f_{i-1}K_i+(1-G)]+[a_i, b_i].$$

Therefore,

$$(v^{-1}{}_{i-1}v_i)f_i = f_{i-1}(1+H_i)+\Delta w_i + v^{-1}{}_{i-1}a_i$$

and

$$v^{-1}{}_{i-1}v_i = -f_{i-1}K_i + 1 - G_i + v^{-1}{}_{i-1}b_i;$$

hence,

$$(-f_{i-1}K_i + 1 - G_i + v^{-1}{}_{i-1}b_i)f_i = f_{i-1}(1+H_i)+\Delta w_i + v^{-1}{}_{i-1}a_i$$

and

$$f_i - f_{i-1} = \Delta w_i + f_{i-1}H_i + G_i f_i + f_{i-1}K_i f_i - v^{-1}{}_{i-1}b_i f_i + v^{-1}{}_{i-1}a_i.$$

Since f, u, v and v^{-1} are bounded and since $\Sigma_1^n(|a_i|+|b_i|)$ can be made arbitrarily small with an appropriate choice of a subdivision (since $Q \in OM^0$), then the following integral exists and

$$f(x)-f(a) = w(x)-f(a)+(LRLR)\int_a^x (fH+Gf+fKf).$$

Since

$$\prod_1^n \begin{bmatrix} p_i & q_i \\ 0 & r_i \end{bmatrix} = \begin{bmatrix} p & q \\ 0 & r \end{bmatrix},$$

where $p = \Pi_1^n p_i$, $q = \Sigma_{j=1}^n(\Pi_{i=1}{}^{j-1}p_i)q_j(\Pi_{i=j+1}{}^n r_i)$ and $r = \Pi_{i=1}{}^n r_i$, and since all the following integrals and product integrals exist, then

$$[w(a), 1]\,{}_a\prod^x \begin{bmatrix} 1+H & -K \\ 0 & 1-G \end{bmatrix} = [w(a), 1]\begin{bmatrix} A & B \\ 0 & D \end{bmatrix},$$

where $A = {}_a\Pi^x(1+H)$, $B = (LR)\int_a^x [{}_a\Pi^t(+H)](1-K)[{}_t\Pi^x(1-G)]$ and $D = {}_a\Pi^x(1-G)$; hence, if w is a constant function, then

$$f(x) = [w(a)B+D]^{-1}[w(a)A].$$

THEOREM 3.3. *Given.* (1) $[a, b]$ *is a number interval;*
(2) w *is a function from* R *to* N *and* H, G *and* K *are functions from*

$R \times R$ to N such that each of dw, H, G and K belongs to OA^0 and OB^0;

(3) *u and v are functions from R to N such that, if $x \in [a, b]$, then $u(x)$ and $v(x)$ are defined by the matrix equation*

$$[u(x),\ v(x)] = [w(a),\ 1]\ {}_a\prod^x \left(I + \begin{bmatrix} H & -K \\ dw & -G \end{bmatrix}\right)$$

and $v(x)^{-1}$ exists;

(4) *f is a bounded function from R to N, $f(a) = w(a)$, $(1 - G_i - f_{i-1} K_i)^{-1}$ exists and*

$$f(x) = w(x) + (LRLR) \int_a^x (fH + Gf + fKf)$$

for $x \in [a, b]$.

Conclusion. If $x \in [a, b]$, then $f(x) = v(x)^{-1} u(x)$.

Proof. Suppose $x \in [a, b]$ and $\{x_i\}_0^n$ is a subdivision of $[a, b]$. If $0 < i \leqq n$, then there exists $\epsilon_i \in N$ such that

$$\begin{aligned} f(x_i) &= w(x_i) + (LRLR) \int_a^{x_i} (fH + Gf + fKf) \\ &= \Delta w_i + f_{i-1} + f_{i-1} H_i + G_i f_i + f_{i-1} K_i f_i + \epsilon_i \end{aligned}$$

and $f_i = b_i^{-1} a_i$, where $b_i = 1 - G_i - f_{i-1} K_i$ and $a_i = f_{i-1}(1 + H_i) + (\Delta w_i + \epsilon_i)$. For $i = 1, 2, 3, \cdots, n$, let R_i be the 2×2 matrix $R_i = \begin{bmatrix} 1 + H_i & -K_i \\ \Delta w_i + \epsilon_i & 1 - G_i \end{bmatrix}$; let $a_0 = w(a)$ and $b_0 = 1$; then $\{a_i\}_0^n$ and $\{b_i\}_0^n$ are elements of N such that, if $0 < i \leqq n$, then $f_i = b_i^{-1} a_i$ and

$$[a_i, b_i] = [f_{i-1},\ 1]R_i = [b_{i-1}^{-1} a_{i-1},\ 1]R_i = b_{i-1}^{-1}[a_{i-1}, b_{i-1}]R_i.$$

Therefore

$$[a_n, b_n] = \left(\prod_{i=n}^{1} b_{i-1}^{-1}\right)[f_0,\ 1] \prod_{i=1}^{n} R_i$$

and

$$\left(\prod_{i=1}^{n} b_{i-1}\right) b_n [f_n,\ 1] = \prod_{i=1}^{n} b_{i-1}[a_n, b_n] = [f_0,\ 1] \prod_{i=1}^{n} R_i. \tag{1}$$

Let Q be the function from $R \times R$ to the set of 2×2 matrices such that $Q = \begin{bmatrix} 1+H & -K \\ dw & 1-G \end{bmatrix}$. Since f is quasicontinuous and since each of dw, H, G and K belong to OA^0 and OB^0, then $Q - I$ and $-G - fK \in OA^0$ and OB^0 and it follows from Lemma 3.1 that $Q - I$ and $-G - fK$ belong to OM^0, the corresponding product integrals exist, $\int_a^b |Q - \Pi Q| = 0$ and $\int_a^b |(1 - G - fK) - \Pi(1 - G - fK)| = 0$. For each subdivision $\{x_i\}_0^n$ of $[a, x]$, there exist elements d_1, d_2, and d_3 such that Equation (1) can be rewritten

$$\left\{(L)\ {}_a\prod^x (1 - G - fK) + d_1\right\}[f_n, 1] = [f_0, 1]\left(\prod_a^x Q + d_2 + d_3\right),$$

where $1 - G_i - f_{i-1}K_i$ is playing the role of b_i and

$$d_1 = \prod_{i=1}^n (1 - G_i - f_{i-1}K_i) - (L)\ {}_a\prod^x (1 - G - fK),$$

$$d_2 = \prod_{i=1}^n Q_i - {}_a\prod^x Q$$

and

$$d_3 = \prod_{i=1}^n R_i - \prod_{i=1}^n Q_i = \sum_{i=1}^n \left(\prod_{j=1}^{i-1} Q_j\right)(R_i - Q_i) \prod_{j=i+1}^n R_j.$$

Since $R_i - Q_i = \begin{bmatrix} 0 & 0 \\ \epsilon_i & 0 \end{bmatrix}$, it follows from the OM^0 and OA^0 properties that each of $|d_1|$, $|d_2|$ and $|d_3|$ can be made arbitrarily small; hence $(L)\,{}_a\Pi^x(1 - G - fK)[f(x), 1] = [f_0, 1]\,{}_a\Pi^x Q = [u(x), v(x)]$. It follows from the meaning of equality for matrices that $(L)\,{}_a\Pi^x(1 - G - fK) = v(x)$, $v(x)f(x) = u(x)$ and $f(x) = v(x)^{-1}u(x)$.

LEMMA 3.4. *If $G \in OB^0$ on $[a, b]$ and $\epsilon > 0$, then there is a number $p \in (a, b]$ such that, if $\{x_i\}_0^n$ is a subdivision of $[a, p]$, then $\Sigma_2^n |G_i| < \epsilon$.*

THEOREM 3.5. *Given. H, W, K and G are functions from $R \times R$ to N such that each of H, W, K and G belongs to OA^0 and OB^0 on $[a, b]$ and u and v are functions from R to N and are defined by the matrix equation*

$$[u(x), v(x)] = [u(a), v(a)]\ {}_a\prod^x \left(I + \begin{bmatrix} H & W \\ K & G \end{bmatrix}\right)$$

for $x \in [a,b]$. *Conclusion.* (1) *If* $p \in (a,b]$ *and* $0<k<1$ *and* $|v(a)-1| + \Sigma_1^n |u_{i-1} W_i + v_{i-1} G_i| < k$ *for each subdivision* $\{x_i\}_0^n$ *of* $[a,p]$, *then* v^{-1} *exists and is bounded on* $[a,p]$. (2) *If* $|v(a)-1| + |u(a)W(a,a^+) + v(a)G(a,a^+)| < 1$, *then there exists* $p \in (a,b]$ *such that* v^{-1} *exists and is bounded on* $[a,p]$.

Proof. Since H, W, K and $G \in OA^0$ and OB^0 on $[a,b]$, then $\begin{bmatrix} H & W \\ K & G \end{bmatrix} \in OA^0$ and OB^0 on $[a,b]$ and, by Lemma 3.1, $\begin{bmatrix} H & W \\ K & G \end{bmatrix} \in OM^0$ on $[a,b]$; also, u and v are quasicontinuous and bounded on $[a,b]$.

We now prove Conclusion 1. Let $x \in [a,p]$ and let $\{x_i\}_1^n$ be a subdivision of $[a,x]$. For $i = 1,2,\cdots,n$, there exist a_i and $b_i \in N$ such that

$$
\begin{aligned}
[u(x_i),\ v(x_i)] &= [u(a), v(a)]\ {}_a\prod^{x_i} \left(I + \begin{bmatrix} H & W \\ K & G \end{bmatrix}\right) \\
&= [u_{i-1},\ v_{i-1}]\ {}_{x_{i-1}}\prod^{x_i} \left(I + \begin{bmatrix} H & W \\ K & G \end{bmatrix}\right) \\
&= [u_{i-1},\ v_{i-1}] \begin{bmatrix} 1+H_i & W_i \\ K_i & 1+G_i \end{bmatrix} + [a_i, b_i] \\
&= [u_{i-1}(1+H_i) + v_{i-1}K_i,\ u_{i-1}W_i + v_{i-1} + v_{i-1}G_i] + [a_i, b_i]
\end{aligned}
$$

and

$$
v_i - 1 = (v_{i-1} - 1) + u_{i-1}W_i + v_{i-1}G_i + b_i\,;
$$

hence, by iteration and the norm properties,

$$
\begin{aligned}
|v(x)-1| = |v_n - 1| &\leqq |v_0 - 1| + \sum_1^n |u_{i-1} W_i + v_{i-1} G_i| + \sum_1^n |b_i| \\
&< k + \sum_1^n |b_i|.
\end{aligned}
$$

Let $r = (k+1)/2$. Since $\begin{bmatrix} H & W \\ K & G \end{bmatrix} \in OM^0$ and u and v are bounded on $[a,b]$, then there is a subdivision $\{x_i\}_0^n$ of $[a,x]$ such that $\Sigma_1^n |b_i| < r-k$ and, hence, $|v(x)-1| < r < 1$. Let v denote $v(x)$; then $v = 1 + (v-1)$, v^{-1} exists, and

$$
v^{-1} = 1 - (v-1) + (v-1)^2 - (v-1)^3 + \cdots
$$

and

$$|v^{-1}| \leqq (1 - |v - 1|)^{-1} \leqq (1 - r)^{-1}.$$

Therefore, v^{-1} exists and is bounded by $[1 - (k+1)/2]^{-1}$ on $[a, p]$.

Since u and v are bounded and G and $W \in OB^0$ on $[a, b]$, then there exist numbers p and k satisfying Conclusion 1, provided $|v(a) - 1| + |u(a)W(a, a^+) + v(a)G(a, a^+)| < 1$; hence, Conclusion 2 follows as a corollary to Conclusion 1.

LEMMA 3.6. *If G is a function from $R \times R$ to N such that $G \in OA^0$ and OB^0, then $|G| \in OA^0$.*

A proof for this lemma is given in **[6]**.

LEMMA 3.7. *If G is a function from $R \times R$ to N, and $G \in OA^0$ and OB^0, then $\left|\int_a^b G\right| \leqq \int_a^b |G|$.*

Outline of proof.

$$\left|\int_a^b G\right| \leqq \sum_1^n \left|\int_{x_{i-1}}^{x_i} G - G_i\right| + \sum_1^n |G_i|.$$

LEMMA 3.8. *Given. H and G are functions from $R \times R$ to R and c is a number such that $H \geqq 0$, $G \geqq 0$, $1 - G \geqq c > 0$, and H and $G \in OA^0$ and OB^0 on $[a, b]$; f is a bounded function from R to R and k is a number such that $f(x) \leqq k + (LR)\int_a^x (fH + fG)$ for $x \in [a, b]$.*

Conclusion. If $x \in [a, b]$, then $f(x) \leqq k_a \Pi^x (1 + H)(1 - G)^{-1}$. This is Theorem 4 of **[4]**.

LEMMA 3.9. *If $G \in OA^0$ and OB^0 and f is quasicontinuous on $[a, b]$, then fG and $Gf \in OA^0$ on $[a, b]$.*

This is a special case of [**4**, Theorem 2].

THEOREM 3.10. *Given.* (1) *$[a, b]$ is a number interval;*

(2) *w is a function from R to N and H, G and K are functions from $R \times R$ to N such that each of dw, H, G and K belongs to OA^0 and OB^0 on $[a, b]$;*

(3) *f and g are bounded functions from R to N and c is a number such that $1 - |B| \geqq c > 0$, where $B(x, y) = G(x, y) + g(x)K(x, y)$ and on $[a, b]$ each of f and g is a solution of the integral equation*

$$f(x) = w(x) + (LRLR)\int_a^x (fH + Gf + fKf).$$

Conclusion. If $x \in [a, b]$, then $f(x) = g(x)$.

Proof. Since f and g are bounded and since dw, H, G and $K \in OA^0$ and OB^0, then each of f, g and $|f - g|$ is a quasicontinuous function. Let A be the function $A(x, y) = H(x, y) + K(x, y)f(y)$ for $a \leqq x < y \leqq b$; then it follows from Lemmas 3.6 and 3.9 that $A, B, |A|$ and $|B| \in OA^0$ and OB^0 and that $(LR)\int_a^b [|f - g|\ |A| + |B|\ |f - g|]$ exists. If $x \in [a, b]$, then

$$|f(x) - g(x)| = \left|(LR)\int_a^x [(f - g)A + B(f - g)]\right|$$

$$\leqq 0 + (LR)\int_a^x [|f - g||A| + |B||f - g|] \quad \text{(Lemma 3.7)}.$$

It follows from Lemma 3.8 that

$$|f(x) - g(x)| \leqq 0 \cdot {}_a\prod^x (1 + |A|)(1 - |B|)^{-1} = 0.$$

Therefore, if $x \in [a, b]$, then $f(x) = g(x)$.

The restrictions $1 - |B| \geqq c > 0$ and $(1 - G_i - f_{i-1}K_i)^{-1}$ cannot be deleted from the hypothesis of Theorem 3.10 and Theorem 3.3, respectively. Consider the following example. Let u, v, and g be functions from R to R such that $u(x) = 0$ for $x \in [0, 2]$, $v(x) = g(x) = 0$ for $x \in [0, 1]$ and $v(x) = g(x) = 1$ for $x \in (1, 2]$. Each of u and v is a solution on $[0, 2]$ for the equation $f(x) = (R)\int_0^x f dg$. See **[5]** for solutions of equations in which the restriction $1 - |B| \geqq c > 0$ does not hold.

Theorems similar to Theorems 3.2, 3.3 and 3.10 can be proved for $f(x) = u(x)v(x)^{-1}$,

$$f(x) = w(x) + (RLRL)\int_a^x (fG + Hf + fKf),$$

and

$$\begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = {}_a\prod^x Q \begin{bmatrix} w(a) \\ 1 \end{bmatrix},$$

where $Q = \begin{bmatrix} 1 + H & dw \\ -K & 1 - G \end{bmatrix}$ and

$${}_a\prod^x Q = \lim Q(x_{n-1}, x_n) \cdots Q(x_1, x_2)Q(x_0, x_1).$$

We will now compare the Riccati equation for Riemann-Stieltjes integrals with the Riccati equation for the $(LRLR)$-integral. In this and the next paragraph, G is continuous at p means $G(p^-,p)=0=G(p,p^+)$; also, the symbol $(RS)\int_a^b E(f)$ is used to denote a Riemann-Stieltjes-type integral: i.e., for each subdivision $\{x_i\}_0^n$ of $[a,b]$, the approximating sum has the form $\Sigma_1^n E[f(c_i)]$, where $x_{i-1} \leqq c_i \leqq x_i$ for $i = 1,2,\cdots,n$. Suppose that w, H, G and K satisfy the hypothesis of Theorem 3.2. If f is the solution of the Riccati equation

$$f(x) = w(x) + (RS)\int_a^x fH + (RS)\int_a^x Gf + (RS)\int_a^x fKf$$

on $[a,b]$, then f is the solution of

$$f(x) = w(x) + (LRLR)\int_a^x (fH + Gf + fKf) \tag{1}$$

on $[a,b]$. If f is a solution of

$$f(x) = w(x) + (RS)\int_a^x (fH + Gf + fKf) \tag{2}$$

on $[a,b]$ and either f is continuous on $[a,b]$ or each of H, G and K is continuous on $[a,b]$, then f is the solution of Equation 1 on $[a,b]$. Equation 2 can have a solution f on $[a,b]$ even though each of f, w, H, G and K has a discontinuity.

Example. Suppose that N is a field, $a < p \leqq b$, and g is a function of bounded variation which is continuous on $[a,p)$ and on $[p,b]$; f is the function such that

$$f(x) = 1 + (LRLR)\int_a^x (fdg + dgf + fdgf)$$

for $x \in [a,p)$ and

$$f(x) = -2 - f(p^-) + (LRLR)\int_p^x (fdg + dgf + fdgf)$$

for $x \in [p,b]$; also,

$$g(p) - g(p^-) = -2[1 + f(p^-)]/f(p^-)[f(p^-) + 2].$$

The function f is the solution on $[a, b]$ of Equation (2) with $dg = H = G = K$; however, f is not the solution of Equation (1) unless $f(p^-) = -1$. Furthermore, if $g(p)$ is defined differently, then Equation (2) has no solution on $[a, p]$.

In order for the Riemann-Stieltjes equation to have a solution which is not a solution of the $(LRLR)$-equation, there must be an interdependence between the functions w, H, G and K. The following discussion illustrates this. Suppose that N is a field and that w, H, G and K are functions that satisfy the hypothesis of Theorem 3.2 and that on $[a, b]$ the function f is a solution of Equation (2) but is not a solution of Equation (1); then there is a number $p \in [a, b]$ such that f is not continuous at p. For convenience suppose that $f(p^-) \neq f(p)$ and, in the following manipulations, let $f_1, f_2, \Delta w, H, G$ and K denote $f(p^-)$, $f(p)$, $w(p) - w(p^-)$, $H(p^-, p)$, $G(p^-, p)$ and $K(p^-, p)$, respectively. Then

$$f(p) = f(p^-) + \Delta w + (RS) \int_{p^-}^{p} (fH + Gf + fKf),$$

$$f_2 = f_1 + \Delta w + f_1 H + G f_1 + f_1 K f_1,$$

$$= f_1 + \Delta w + f_2 H + G f_2 + f_2 K f_2,$$

$$f_2 H + G f_2 + f_2 K f_2 = f_1 H + G f_1 + f_1 K f_1$$

and

$$(f_2 - f_1)(H + K f_2) + (G + f_1 K)(f_2 - f_1) = 0.$$

Since $f_2 - f_1 \neq 0$ and N is a field, then

$$H + G + K f_2 + f_1 K = 0.$$

Substituting for f_2 and simplifying, we obtain

(3) $$K^2 f_1^2 + (2 + H + G) K f_1 + (H + G + \Delta w K) = 0.$$

Since $f_1 = f(p^-) = w(p^-) + (RS) \int_a^{p^-} (fH + Gf + fKf)$, then the value of $f(p^-)$ depends only on the values of w, H, G and K on the half open interval $[a, p)$; however, Equation (3) depends on the values of w, H, G and K on the closed interval $[a, p]$. Hence, these functions cannot be defined independently. For example, if $K \neq 0$ and a different value is assigned to $w(p)$, then Equation (3) is no longer true and the Riemann-Stieltjes equation has no solution on $[a, p]$ unless compensating values are assigned to $H(p^-, p)$, $G(p^-, p)$ and $K(p^-, p)$. However, the new $(LRLR)$-Riccati equation will have a solution on $[a, p]$.

4. A differential-type equation. In this section we find the solution of $f^{**}+f^{*}p+fq=r$, where f^{*} and f^{**} are defined as follows. If $[a,b]$ is a number interval and h is a left continuous function from R to N such that $dh \in OB^{0}$, then $D(h,a,b)$ denotes the set of ordered pairs of functions such that $(f,g)\in D(h,a,b)$ iff g is a quasicontinuous function from R to N such that $f(x)-f(a)=(L)\int_a^x gdh$ for $x\in[a,b]$. If $(f,g)\in D(h,a,b)$, then g is denoted by f^{*}. Also,

$f^{**}=(f^{*})^{*}$ and $f\cong w$ iff $(L)\int_a^x fdh=(L)\int_a^x wdh$ for $x\in[a,b]$. In this section all integrals and product integrals are Cauchy-left-type integrals unless indicated otherwise.

LEMMA 4.1. *If* (f,f^{*}) *and* $(g,g^{*})\in D(h,a,b)$, *then* $(f+g,\, f^{*}+g^{*})\in D(h,a,b)$.

LEMMA 4.2. *If* (f,f^{*}) *and* $(g,g^{*})\in D(h,a,b)$, g^{*},h *and* g *commute and* z *is the function such that* $z(x)=g(x^{+})-g(x)$ *for* $x\in[a,b]$, *then* $(fg, f^{*}g+fg^{*}+f^{*}z)\in D(h,a,b)$.

Indication of proof. Since (g,g^{*}) and $(f,f^{*})\in D(h,a,b)$, then g is left continuous and $df\in OB^{0}$; hence,

$$\int_a^x dfdg=(L)\int_a^x (df)z,$$

$$(L)\int_a^x (df)g=(R)\int_a^x [(df)g-(df)(dg)]$$

and

$$\begin{aligned}(L)\int_a^x (f^{*}g+fg^{*}+f^{*}z)dh &= (LLL)\int_a^x [(df)g+fdg+(df)z]\\ &= (RLL)\int_a^x [(df)g+fdg-(df)dg+(df)z]\\ &= (RL)\int_a^x [(df)g+fdg]\\ &\qquad = f(x)g(x)-f(a)g(a)\end{aligned}$$

LEMMA 4.3. *Given.* $[a,b]$ *is a number interval*; f *and* h *are functions from* R *to* N *such that* $f(a)=h(a)$ *and* $dh\in OB^{0}$; G *is a function from* $R\times R$ *to* N *such that* $G\in OB^{0}$ *and* OA^{0}

Conclusion. The following statements are equivalent:

(1) if $x \in [a, b]$, then $f(x) = h(x) + (L)\int_a^x fG$; and

(2) if $x \in [a, b]$, then

$$f(x) = f(a)\ {}_a\prod^x (1 + G) + (R)\int_a^x dh\ {}_t\prod^x (1 + G).$$

This lemma is a special case of Theorem 5.1 of **[3]**.

THEOREM 4.4. *Given.* (1) $[a, b]$ *is a number interval*; (2) h, p, q, u, v, β *and* s *are functions from* R *to* N *such that* h *is left continuous,* $dh \in OB^0$, p *and* q *are quasicontinuous on* $[a, b]$ *and, if* $x \in [a, b]$, *then* $u(x)$ *and* $v(x)$ *are defined by the matrix equation*

$$[u(x),\ v(x)] = [0,\ 1](\dot{L})\ {}_a\prod^x \left(I + \begin{bmatrix} -p & -1 \\ q & 0 \end{bmatrix} dh\right),$$

$v(x)^{-1}$ *exists,* $\beta(x) = v(x)^{-1}u(x)$ *and* $s(x) = \beta(x^+) - \beta(x)$; *also,* v^{-1} *is bounded on* $[a, b]$; (3) *if* $a \leqq x \leqq y \leqq b$, *then* $p(x)$, $p(y)$, $q(x)$, $q(y)$, $h(x)$ *and* $h(y)$ *commute*; (4) f *and* r *are functions from* R *to* N *and* r *is quasicontinuous.*

Conclusion. The following statements are equivalent.

(1) There exist functions f^* and f^{**} such that (f, f^*) and $(f^*, f^{**}) \in D(h, a, b)$ and such that on $[a, b]$

$$f^{**} + f^*p + fq = r.$$

(2) If $x \in [a, b]$, then

$$f(x) = f(a)(L)\ {}_a\prod^x (1 - \beta dh) + (R)\int_a^x dz(L)\ {}_t\prod^x (1 - \beta dh),$$

where $\alpha = p - \beta - s$, $z(x) = f(a) + (L)\int_a^x wdh$, $g(x) = f^*(a) + (L)\int_a^x rdh$ and

$$w(x) = f^*(a)(L)\ {}_a\prod^x (1 - \alpha dh) + (R)\int_a^x dg(L)\ {}_t\prod^x (1 - \alpha dh).$$

Proof. Since $dh \in OB^0$ and h is left continuous and since p and q are quasicontinuous, then u and v are left continuous and quasicontinuous. Since v^{-1} is bounded and $\beta = v^{-1}u$, then β is left continuous, quasicontinuous and commutes with h. If $x \in [a, b]$, it follows from Theorem 3.2 that

$$\beta(x) = (L)\int_a^x qdh + (LL)\int_a^x \beta(-pdh) + (LR)\int_a^x \beta dh\beta.$$

Let α, s and k be the functions such that $s(t) = \beta(t^+) - \beta(t)$, $\alpha = p - \beta - s$, $k(a) = 0$, and $k = q + \beta^2 - \beta p + \beta s$; then, for $x \in [a, b]$,

$$\begin{aligned}(L)\int_a^x kdh &= (L)\int_a^x (q + \beta^2 - \beta p + \beta s)dh \\ &= (L)\int_a^x qdh + \left[(LR)\int_a^x \beta dh\beta - (L)\int_a^x \beta dh\, d\beta\right] \\ &\quad + (LL)\int_a^x \beta(-pdh) + (LL)\int_a^x \beta sdh.\end{aligned}$$

Since β is left continuous, then

$$(L)\int_a^x \beta dh\, d\beta = (LL)\int_a^x \beta sdh,$$

$\int_a^x kdh = \beta(x) - \beta(a)$ and $(\beta, k) \in D(h, a, b)$; k will be denoted by β^*. Note that $\beta, \alpha, \beta^*, p, q$ and h commute on $[a, b]$ and that $q = \beta^* + \beta\alpha$.

Proof of $1 \to 2$. Since the triple (f, f^*), (β, β^*), s satisfies the hypothesis of Lemma 4.2, then $(f\beta, f^*\beta + f\beta^* + f^*s) \in D(h, a, b)$. Hence,

$$\begin{aligned}&(f^* + f\beta)^* + (f^* + f\beta)\alpha \\ &\quad \cong f^{**} + f^*\beta + f\beta^* + f^*s + f^*\alpha + f\beta\alpha \\ &\quad = f^{**} + f^*(\beta + s + \alpha) + f(\beta^* + \beta\alpha) \\ &\quad = f^{**} + f^*p + fq = r\end{aligned}$$

and

$$(f^* + f\beta)^* \cong r - (f^* + f\beta)\alpha.$$

If we integrate each member of the preceding equation with respect to h and recall that $\beta(a) = 0$, we obtain

$$(f^*+f\beta)(x)=g(x)+(L)\int_a^x (f^*+f\beta)(-\alpha dh),$$

where $g(x)=f^*(a)+(L)\int_a^x rdh$. It follows from Lemma 4.3, $1\to 2$, that

$$(f^*+f\beta)(x)=f^*(a)\ {}_a\prod^x (1-\alpha dh)+(R)\int_a^x dg\ {}_t\prod^x (1-\alpha dh)$$

for $x\in[a,b]$. Let $w(x)$ respresent the right member in the preceding equation. If $x\in[a,b]$, then $f^*(x)=w(x)-f(x)\beta(x)$ and by integrating both members we obtain

$$f(x)=z(x)+(L)\int_a^x f(-\beta dh),$$

where $z(x)=f(a)+(L)\int_a^x wdh$ and $z(a)=f(a)$. It follows from Lemma 4.3, $1\to 2$, that

$$f(x)=f(a)\ {}_a\prod^x (1-\beta dh)+(R)\int_a^x dz\ {}_t\prod^x (1-\beta dh).$$

Proof of $2\to 1$. Functions f^{**} and f^* will be defined such that (f,f^*) and $(f^*,f^{**})\in D(h,a,b)$ and such that on $[a,b]$ $f^{**}+f^*p+fq=r$.

Let $f^*=w-f\beta$. Since f satisfies the second statement of the conclusion, it follows from Lemma 4.3, $2\to 1$, that for $x\in[a,b]$

$$\begin{aligned}f(x)&=z(x)+(L)\int_a^x f(-\beta dh)\\&=f(a)+(L)\int_a^x wdh+(L)\int_a^x f(-\beta dh)\\&=f(a)+(L)\int_a^x f^*dh\end{aligned}$$

and $(f,f^*)\in D(h,a,b)$.

Let f^{**} be the function such that

$$f^{**}=r-(f^*+f\beta)\alpha-(f^*\beta+f\beta^*+f^*s).$$

Since $\beta(a)=0$ and

$$
\begin{aligned}
(f^*+f\beta)(x) &= w(x)\\
&= f^*(a)\ {}_a\prod^x (1-\alpha dh)+(R)\int_a^x dg\ {}_t\prod^x (1-\alpha dh)
\end{aligned}
$$

for $x\in[a,b]$, it follows from Lemma 4.3, $2\to 1$, that

$$(f^*+f\beta)(x)=g(x)+(L)\int_a^x (f^*+f\beta)(-\alpha dh)$$

and, hence,

$$f^*(x)=g(x)+(L)\int_a^x (f^*+f\beta)(-\alpha dh)-f(x)\beta(x).$$

Since $(f\beta, f^*\beta+f\beta^*+f^*s)\in D(h,a,b)$ and $\beta(a)=0$, it follows from the definition of f^{**} that

$$
\begin{aligned}
(L)\int_a^x f^{**}dh &= (L)\int_a^x [r-(f^*+f\beta)\alpha-(f^*\beta+f\beta^*+f^*s)]dh\\
&= -f^*(a)+\Big[g(x)+(L)\int_a^x (f^*+f\beta)(-\alpha dh)\\
&\qquad\qquad -f(x)g(x)\Big]\\
&= f^*(x)-f^*(a)
\end{aligned}
$$

for $x\in[a,b]$; hence, $(f^*,f^{**})\in D(h,a,b)$.

Since

$$
\begin{aligned}
f^{**}+f^*p+fq &= [r-(f^*+f\beta)\alpha-(f^*\beta+f\beta^*+f^*s)]\\
&\qquad +f^*(\alpha+\beta+s)+f(\beta^*+\alpha\beta)=r,
\end{aligned}
$$

then the triple f, f^*, f^{**} satisfies the given equation.

Suppose that on $[a,b]$ the functions h, p and q are defined as in Theorem 4.4 except for the restrictions pertaining to v^{-1}. If $h\in C^0$, it follows from Theorem 3.5 that there is a subdivision $\{x_i\}_0^n$ of $[a,b]$ and functions $\{\beta_i\}_1^n$, $\{u_i\}_1^n$ and $\{v_i\}_1^n$ such that for $i=1,2,\cdots,n$ and $x\in[x_{i-1},x_i]$

$$[u_i(x),\ v_i(x)]=[0,\ 1]\ {}_{x_{i-1}}\prod^x \left(I+\begin{bmatrix}-p & -1\\ q & 0\end{bmatrix}dh\right),$$

$\beta_i(x) = v_i(x)^{-1} u_i(x)$, and v_i^{-1} exists and is bounded on $[x_{i-1}, x_i]$. Hence, for $i = 1, 2, \cdots, n$, Theorem 4.4 gives the solution of $f^{**} + f^*p + fq = r$ on $[x_{i-1}, x_i]$ which is unique for a given pair $f^*(x_{i-1})$ and $f(x_{i-1})$. Therefore, Theorem 4.4 can be used to find a unique solution on $[a, b]$ for given values of $f(a)$ and $f^*(a)$.

A theorem similar to Theorem 4.4 can be stated and proved for the equation $f^{**} + pf^* + qf = r$; however, Theorem 5.2 of **[3]** would be used in the proof instead of Lemma 4.3. If (f, f^*) means $f(x) - f(a) = (R)\int_a^x f^* dh$ and h is right continuous, a theorem similar to Theorem 4.4 can be stated and proved.

References

1. P. Puig Adam, *Los fracciones continuas de cocientes incompletos differenciales y sus aplicaciones*, Revista Mat. Hisp. Amer. II, (1951), 180–190.
2. W. D. L. Appling, *Interval functions and real Hilbert spaces*, Rend.-Circl. Mat. Palermo, Ser. 2, **11** (1962), 154–156.
3. B. W. Helton, *Integral equations and product integrals*, Pacific J. Math. **16** (1966), 297–322.
4. ———, *A product integral representation for a Gronwall inequality*, Proc. Amer. Math. Soc., **23** (1969), 493–500.
5. ———, *Solutions of* $f(x) = f(a) + (RL)\int_a^x (fH + fG)$ *for rings*, Proc. Amer. Math. Soc., **25** (1970), 735–742.
6. Jon C. Helton, *An existence theorem for sum and product integrals*, Proc. Amer. Math. Soc., **39** (1973), 149–154.
7. J. S. MacNerney, *Stieltjes integrals in linear spaces*, Ann. of Math., (2) **61** (1955), 354–367.
8. ———, *Continuous products in linear spaces*, J. Elisha Mitchell Sci. Soc., **71** (1955), 185–200.
9. ———, *Integral equations and semigroups*, Illinois J. Math., **7** (1963), 148–173.
10. Alfred Stohr, *Über einen integralartigen Grenzubergang bei Kettenbruchen*, Math. Nachr., **6** (1951), 103–107.
11. H. S. Wall, *Concerning continuous continued fractions and certain systems of Stieltjes integral equations*, Rend. Circ. Mat. Palermo, (2), **2** (1953), 73–84.
12. ———, *Concerning harmonic matrices*, Archiv. der Math., **V** (1954), 160–167.

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