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MUTUAL EXISTENCE OF SUM AND PRODUCT INTEGRALS

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Functions are from $R \times R$ to N, where R denotes the set of real numbers and N denotes a normed complete ring. If G has bounded variation on $[a, b]$, then $\int_a^b G$ exists if and only if ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$. If each of $\lim_{x\to p^+} H(p,x)$, $\lim_{x\to p^-} H(x,p)$, $\lim_{x,y\to p^+} H(x,y)$ and $\lim_{x,y\to p^-} H(x,y)$ exists, G has bounded variation on $[a,b]$ and either $\int_a^b G$ exists or ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$, then $\int_a^b HG$ and $\int_a^b GH$ exist and ${}_x\Pi^y(1+HG)$ and ${}_x\Pi^y(1+GH)$ exist for $a \leqq x < y \leqq b$. If G has bounded variation on $[a,b]$ and ν is a nonnegative number, then $\int_a^b G$ exists and $\int_a^b \left| G - \int G \right| = \nu$ if and only if ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$ and

$$\int_a^b |1 + G - \Pi(1+G)| = \nu.$$

J. S. MacNerney **[4]** defines classes OA and OM of functions such that the integral-like formulas

$$V(a,b) = \int_a^b (W-1) \quad \text{and} \quad W(a,b) = {}_a\Pi^b(1+V)$$

are mutually reciprocal and establishes a one-to-one correspondence between the classes OA and OM. B. W. Helton **[1]** defines classes OA° and OM° of functions and shows that if G has bounded variation on $[a,b]$, then $G \in OA^\circ$ on $[a,b]$ if and only if $G \in OM^\circ$ on $[a,b]$, where $G \in OA^\circ$ on $[a,b]$ only if $\int_a^b G$ exists and $\int_a^b \left| G - \int G \right| = 0$, and $G \in OM^\circ$ on $[a,b]$ only if ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$ and

$$\int_a^b |1 + G - \Pi(1+G)| = 0.$$

The class OA is a proper subclass of OA° and OM is closely related to the class OM°. In the following, we establish a related result and show that if G has bounded variation on $[a, b]$, then $\int_a^b G$ exists if and only if ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$. This is not the same as the result of B. W. Helton since it is possible to construct a function G such that G has bounded variation on $[a, b]$, $\int_a^b G$ exists, ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$, $G \notin OA^\circ$ on $[a, b]$ and $G \notin OM^\circ$ on $[a, b]$ **[3]**. We then use this result and ideas from another theorem of B. W. Helton [**2**, Theorem 2, p. 494] to establish that if each of $\lim_{x \to p^+} H(p, x)$, $\lim_{x \to p^-} H(x, p)$, $\lim_{x,y \to p^+} H(x, y)$ and $\lim_{x,y \to p^-} H(x, y)$ exists, G has bounded variation on $[a, b]$ and either $\int_a^b G$ exists or ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$, then $\int_a^b HG$ and $\int_a^b GH$ exist and ${}_x\Pi^y(1+HG)$ and ${}_x\Pi^y(1+GH)$ exist for $a \leqq x < y \leqq b$. Further, we show that if G has bounded variation on $[a, b]$ and ν is a nonnegative number, then $G \in OA^\nu$ on $[a, b]$ if and only if $G \in OM^\nu$ on $[a, b]$, where $G \in OA^\nu$ on $[a, b]$ only if $\int_a^b G$ exists and

$$\int_a^b \left| G - \int G \right| = \nu,$$

and $G \in OM^\nu$ on $[a, b]$ only if ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$ and

$$\int_a^b |1 + G - \Pi(1+G)| = \nu.$$

Finally, we show that if the norm used has the property that $|AB| = |A||B|$ and if each of $\lim_{x \to p^+} H(p, x)$, $\lim_{x \to p^-} H(x, p)$, $\lim_{x,y \to p^+} H(x, y)$ and $\lim_{x,y \to p^-} H(x, y)$ exists, G has bounded variation on $[a, b]$ and either $G \in OA^\nu$ on $[a, b]$ or $G \in OM^\nu$ on $[a, b]$, then there exist nonnegative numbers α and β such that HG is in OA^α and OM^α on $[a, b]$ and GH is in OA^β and OM^β on $[a, b]$.

All integrals and definitions are of the subdivision-refinement type, and functions are from $R \times R$ to N, where R denotes the set of real numbers and N denotes a ring which has a multiplicative identity element represented by 1 and has a norm $|\cdot|$ with respect to which N is complete and $|1| = 1$. Unless noted otherwise, functions are assumed to be defined only for $\{x, y\} \in R \times R$ such that $x < y$. The statement that $G \in OB^\circ$ on $[a, b]$ means that there exist a subdivision D of $[a, b]$ and a number B such that if $\{x_i\}_{i=0}^n$ is a refinement of D, then $\Sigma_{i=1}^n |G_i| < B$, where G_i denotes $G(x_{i-1}, x_i)$. When convenient, we use

$$\sum_{J(I)} G \quad \text{and} \quad \prod_{J(I)} (1+G)$$

to denote

$$\sum_{i=1}^{n} G_i \quad \text{and} \quad \prod_{i=1}^{n} (1+G_i),$$

respectively, where $J = \{x_i\}_{i=0}^{n}$ represents a subdivision of some interval. The sets OA°, OM°, OA^ν and OM^ν have been defined previously, and $G \in OA^+$ only if G is an additive function from $R \times R$ to the nonnegative numbers. Also, $G \in OM^*$ on $[a,b]$ only if ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$ and if $\epsilon > 0$ then there exists a subdivision D of $[a,b]$ such that if $\{x_i\}_{i=0}^{n}$ is a refinement of D and $0 \leqq p < q \leqq n$, then

$$\left| {}_{x_p}\Pi^{x_q}(1+G) - \prod_{i=p+1}^{q} (1+G_i) \right| < \epsilon.$$

The symbols $G(p,p^+), G(p^-,p), G(p^+,p^+)$ and $G(p^-,p^-)$ denote $\lim_{x \to p^+} G(p,x)$, $\lim_{x \to p^-} G(x,p)$, $\lim_{x,y \to p^+} G(x,y)$ and $\lim_{x,y \to p^-} G(x,y)$, respectively, and $G \in OL^\circ$ on $[a,b]$ only if $G(p,p^+), G(p^-,p), G(p^+,p^+)$ and $G(p^-,p^-)$ exist for $p \in [a,b]$. Further, $G \in S_2$ on $[a,b]$ only if $G(p,p^+)$ and $G(p^-,p)$ exist for $p \in [a,b]$. Finally, statements of the form $G > \beta$ should be interpreted in terms of subdivisions and refinements. See B. W. Helton **[1]** and J. S. MacNerney **[4]** for additional background.

We now establish an approximation theorem for product integrals. To do this, we initially develop a sequence of lemmas.

LEMMA 1.1. *If $\beta > 0$, G is a function from $R \times R$ to N, $|G| < 1 - \beta$ on $[a,b]$, $G \in OB^\circ$ on $[a,b]$ and ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$, then $G \in OM^*$ on $[a,b]$.*

Proof. Let $\epsilon > 0$. There exist a subdivision D of $[a,b]$ and a number B such that if $\{x_i\}_{i=0}^{n}$ is a refinement of D, then

(1) $|G_i| < 1 - \beta$ for $i = 1, 2, \cdots, n$,

(2) $\Pi_{i=1}^{n}(1 + |G_i|) < B$,

(3) $\Pi_{i=1}^{n}(1 + \Sigma_{j=1}^{\infty} |(-1)^j G_i^j|) < B$, and

(4) $|{}_a\Pi^b(1+G) - \Pi_{i=1}^{n}(1+G_i)| < \epsilon(3B)^{-1}$.

Suppose $\{x_i\}_{i=0}^{n}$ is a refinement of D and $0 \leqq p < q \leqq n$. Let $Y = \{y_i\}_{i=0}^{r}$ and $Z = \{z_i\}_{i=0}^{s}$ be refinements of $\{x_i\}_{i=0}^{p}$ and $\{x_i\}_{i=q}^{n}$, respectively, such that

$$\left| \prod_{Y(I)} (1+G) - {}_a\Pi^{x_p}(1+G) \right| < \epsilon(3B^3)^{-1}$$

and

$$\left| -{}_{x_q}\Pi^b(1+G) + \prod_{Z(I)} (1+G) \right| < \epsilon(3B^2)^{-1}.$$

Further, let P and P' denote

$$\prod_{Y(I)} (1+G) \quad \text{and} \quad {}_a\Pi^{x_p}(1+G),$$

respectively, and let Q and Q' denote

$$\prod_{Z(I)} (1+G) \quad \text{and} \quad {}_{x_q}\Pi^b(1+G),$$

respectively. Note that P^{-1} and Q^{-1} exist and are

$$\prod_{i=1}^{r} \left[1 + \sum_{j=1}^{\infty} (-1)^j G^j (y_{r-i}, y_{r+1-i}) \right]$$

and

$$\prod_{i=1}^{s} \left[1 + \sum_{j=1}^{\infty} (-1)^j G^j (z_{s-i}, z_{s+1-i}) \right],$$

respectively.

Let W denote the subdivision $D \cup Y \cup Z$ of $[a,b]$. Thus,

$$\begin{aligned}
&\left| {}_{x_p}\Pi^{x_q}(1+G) - \prod_{i=p+1}^{q} (1+G_i) \right| \\
&\quad = \left| P^{-1}P\left[{}_{x_p}\Pi^{x_q}(1+G) - \prod_{i=p+1}^{q} (1+G_i) \right] Q Q^{-1} \right| \\
&\quad \leqq |P^{-1}| \left| P[{}_{x_p}\Pi^{x_q}(1+G)]Q - P\left[\prod_{i=p+1}^{q} (1+G_i) \right] Q \right| |Q^{-1}| \\
&\quad \leqq B \left| P[{}_{x_p}\Pi^{x_q}(1+G)]Q - \prod_{W(I)} (1+G) \right| \\
&\quad = B \left| [P - P' + P'][{}_{x_p}\Pi^{x_q}(1+G)][Q' - Q' + Q] - \prod_{W(I)} (1+G) \right|
\end{aligned}$$

$$\leqq B\,|P-P'|\,|{}_{x_p}\Pi^{x_q}(1+G)|\,|Q|+B\,|{}_a\Pi^{x_q}(1+G)|\,|-Q'+Q|$$

$$+B\left|{}_a\Pi^b(1+G)-\prod_{W(I)}(1+G)\right|$$

$$< B^3[\epsilon(3B^3)^{-1}]+B^2[\epsilon(3B^2)^{-1}]+B[\epsilon(3B)^{-1}]=\epsilon.$$

LEMMA 1.2. *If G is a function from $R\times R$ to N, $G\in OB^\circ$ on $[a,b]$ and ${}_x\Pi^y(1+G)$ exists for $a\leqq x<y\leqq b$, then $G(a,a^+)$ and $G(b^-,b)$ exist.*

Proof. We initially show that $G(a,a^+)$ exists. Let $\epsilon>0$. There exist numbers c and B such that $a<c<b$ and if $\{x_i\}_{i=0}^n$ is a subdivision of $[a,c]$, then

$$|-1|\left[\prod_{i=1}^n(1+|G_i|)\right]<B \quad\text{and}\quad \sum_{i=2}^n|G_i|<\epsilon(4B^2)^{-1}.$$

Further, there exists a subdivision $D=\{z_i\}_{i=0}^r$ of $[a,c]$ such that if J and K are refinements of D, then

$$\left|\prod_{J(I)}(1+G)-\prod_{K(I)}(1+G)\right|<\epsilon/2.$$

We now suppose $a<x<y<z_1$ and show that

$$|G(a,x)-G(a,y)|<\epsilon.$$

Let $\{x_i\}_{i=0}^m$ and $\{y_j\}_{j=0}^n$ denote $D\cup\{x\}$ and $D\cup\{y\}$, respectively. Thus,

$$\begin{aligned}
\epsilon/2&>\left|\prod_{i=1}^m(1+G_i)-\prod_{j=1}^n(1+G_j)\right|\\
&=\left|[1+G(a,x)]\left[\prod_{i=2}^m(1+G_i)\right]-[1+G(a,y)]\left[\prod_{j=2}^n(1+G_j)\right]\right|\\
&=\Bigg|[1+G(a,x)]\left[1+\sum_{i=2}^m G_i\prod_{k=i+1}^m(1+G_k)\right]\\
&\qquad-[1+G(a,y)]\left[1+\sum_{j=2}^n G_j\prod_{k=j+1}^n(1+G_k)\right]\Bigg|\\
&\geqq|G(a,x)-G(a,y)|-B\sum_{i=2}^m|G_i|\left|\prod_{k=i+1}^m(1+G_k)\right|\\
&\qquad-B\sum_{j=2}^n|G_j|\left|\prod_{k=j+1}^n(1+G_k)\right|
\end{aligned}$$

$$> |G(a,x) - G(a,y)| - B^2[\epsilon(4B^2)^{-1}] + B^2[\epsilon(4B^2)^{-1}],$$

and hence,

$$\epsilon > |G(a,x) - G(a,y)|.$$

Since the existence of $G(b^-, b)$ can be established in a similar manner, Lemma 1.2 follows.

LEMMA 1.3. *If $\beta > 0$, G is a function from $R \times R$ to N, $|G| < 1 - \beta$ on (a, b), $G \in OB^\circ$ on $[a, b]$ and ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$, then $G \in OM^*$ on $[a, b]$.*

Proof. Let $\epsilon > 0$. There exist a subdivision E_1 of $[a, b]$ and a number $B > 1$ such that if $\{x_i\}_{i=1}^m$ is a refinement of E_1, then

$$\prod_{i=1}^{m} (1 + |G_i|) < B$$

and

$$\left| {}_a\Pi^b(1+G) - \prod_{i=1}^{m} (1 + G_i) \right| < \epsilon.$$

Let H be the function defined on $[a, b]$ such that

$$H(x, y) = \begin{cases} G(x, y) & \text{if } x \neq a \text{ and } y \neq b \\ 0 & \text{if } x = a \text{ or } y = b. \end{cases}$$

Thus, H satisfies the hypothesis of Lemma 1.1, and hence, there exists a subdivision E_2 of $[a, b]$ such that if $\{x_i\}_{i=0}^m$ is a refinement of E_2 and $0 \leqq p < q \leqq m$, then

$$\left| {}_{x_p}\Pi^{x_q}(1+H) - \prod_{i=p+1}^{q} (1 + H_i) \right| < \epsilon(3B)^{-1}.$$

It follows from Lemma 1.2 that $G(a, a^+)$ and $G(b^-, b)$ exist. Hence, there exists a point x, where $a < x < b$, such that if $\{x_i\}_{i=0}^m$ and $\{y_j\}_{j=0}^n$ are subdivisions of $[a, x]$, $1 \leqq r \leqq m$ and $1 \leqq s \leqq n$, then

$$\left| \prod_{i=1}^{r} (1 + G_i) - \prod_{j=1}^{s} (1 + G_j) \right| < \epsilon(3B)^{-1}.$$

Also, there exists a point y, where $a < y < b$, such that if $\{x_i\}_{i=0}^m$ and $\{y_j\}_{j=0}^n$ are subdivisions of $[y, b]$, $1 \leqq r \leqq m$ and $1 \leqq s \leqq n$, then

$$\left| \prod_{i=r}^{m} (1+G_i) - \prod_{j=s}^{n} (1+G_j) \right| < \epsilon(3B)^{-1}.$$

Let D denote the subdivision

$$E_1 \cup E_2 \cup \{x\} \cup \{y\}$$

of $[a, b]$. Further, suppose $\{x_i\}_{i=0}^m$ is a refinement of D and $0 \leqq p < q \leqq m$. If $p = 0$ and $q = m$, then the desired inequality follows from the existence of ${}_a\Pi^b(1+G)$. If $p \neq 0$ and $q \neq m$, then the inequality follows from the properties of the function H. Suppose $p = 0$ and $q \neq m$. There exists a subdivision J of $[a, x_1]$ such that

$$\left| {}_a\Pi^{x_1}(1+G) - \prod_{J(I)} (1+G) \right| < \epsilon(3B)^{-1}.$$

Thus,

$$\begin{aligned} &\left| {}_a\Pi^{x_q}(1+G) - \prod_{i=1}^{q} (1+G_i) \right| \\ &\quad < \left| {}_a\Pi^{x_1}(1+G) - (1+G_1) \right| \left| {}_{x_1}\Pi^{x_q}(1+G) \right| + B[\epsilon(3B)^{-1}] \\ &\quad < B \left| \prod_{J(I)} (1+G) - (1+G_1) \right| + B[\epsilon(3B)^{-1}] + \epsilon/3 \\ &\quad < B[\epsilon(3B)^{-1}] + 2\epsilon/3 = \epsilon. \end{aligned}$$

If $p \neq 0$ and $q = n$, then a similar argument establishes the inequality. Therefore, Lemma 1.3 follows.

THEOREM 1. *If G is a function from $R \times R$ to $N, G \in OB^\circ$ on $[a, b]$ and ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$, then $G \in OM^*$ on $[a, b]$.*

Proof. Since $G \in OB^\circ$ on $[a, b]$, there exists a subdivision $\{x_i\}_{i=0}^m$ of $[a, b]$ such that if $1 \leqq i \leqq m$ and $x_{i-1} < x < y < x_i$, then $|G(x, y)| < 1/2$. Hence, this theorem can be established by using Lemma 1.3 and the identity

$$\prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i = \sum_{i=1}^{n} \left(\prod_{j=1}^{i-1} b_j \right) (a_i - b_i) \left(\prod_{k=i+1}^{n} a_k \right),$$

where $\Pi_{j=1}^0 b_j = \Pi_{k=n+1}^n a_k = 1$.

We now use the approximation theorem to establish an existence theorem for sum integrals. In particular, we show that if G has bounded variation on $[a, b]$ and ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$, then $\int_a^b G$ exists. Several lemmas are required.

LEMMA 2.1. *If G is a function from $R \times R$ to N, $G \in OB^\circ$ on $[a, b]$ and ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$, then*

$$\int_a^b G(u, v)\,{}_v\Pi^b(1+G)$$

exists and is $-1 + {}_a\Pi^b(1+G)$.

Proof. Let $\epsilon > 0$. There exist a subdivision E_1 of $[a, b]$ and a number B such that if $\{x_i\}_{i=0}^m$ is a refinement of E_1, then

(1) $\Sigma_{i=1}^m |G_i| < B$, and

(2) $|\Pi_{i=1}^m (1+G_i) - {}_a\Pi^b(1+G)| < \epsilon/2$.

Theorem 1 implies that $G \in OM^*$ on $[a, b]$, and hence, there exists a subdivision E_2 of $[a, b]$ such that if $\{x_i\}_{i=0}^m$ is a refinement of E_2 and $0 \leqq p < q \leqq m$, then

$$\left| {}_{x_p}\Pi^{x_q}(1+G) - \prod_{i=p+1}^{q} (1+G_i) \right| < \epsilon(2B)^{-1}.$$

Let D denote the subdivision $E_1 \cup E_2$ of $[a, b]$ and suppose $\{x_i\}_{i=0}^m$ is a refinement of D. Thus,

$$\begin{aligned}
&\left| \sum_{i=1}^m G_i[{}_{x_i}\Pi^b(1+G)] - [-1 + {}_a\Pi^b(1+G)] \right| \\
&\quad < \left| \sum_{i=1}^m G_i[{}_{x_i}\Pi^b(1+G)] + 1 - \prod_{i=1}^m (1+G_i) \right| + \epsilon/2 \\
&\quad = \left| \sum_{i=1}^m G_i[{}_{x_i}\Pi^b(1+G)] + 1 - \left[1 + \sum_{i=1}^m G_i \prod_{k=i+1}^m (1+G_k) \right] \right| + \epsilon/2 \\
&\quad \leqq \sum_{i=1}^m |G_i| \left| {}_{x_i}\Pi^b(1+G) - \prod_{k=i+1}^m (1+G_k) \right| + \epsilon/2 \\
&\quad < B[\epsilon(2B)^{-1}] + \epsilon/2 = \epsilon.
\end{aligned}$$

LEMMA 2.2. *If H and G are functions from $R \times R$ to N, $H \in OL^\circ$ on $[a, b]$, $G \in OB^\circ$ on $[a, b]$ and $\int_a^b G$ exists, then $\int_a^b HG$ exists and $\int_a^b GH$ exists.*

Proof. B. W. Helton [**2,** Theorem 2, p. 494] proves that HG and GH are in $OA°$ on $[a, b]$ with the hypothesis of Lemma 2.2 and the additional restriction that $G \in OA°$ on $[a, b]$. This lemma follows by essentially the same argument.

Observe that weakening the hypothesis of Helton's result by requiring only the existence of $\int_a^b G$ produces a corresponding weakening of the conclusion since we now have that $\int_a^b HG$ and $\int_a^b GH$ exist rather than that HG and GH are in $OA°$ on $[a, b]$.

Lemma 2.2 is not true for functions defined on a linearly ordered set [**4,** p. 149]. For example, consider

$$S = [0, 1) \cup (1, 2],$$

with the usual ordering for the real numbers. Let G be the function defined on $S \times S$ such that

$$G(x, y) = \begin{cases} 1 & \text{if } x < 1 \text{ and } y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $G \in OA° \cap OB°$ on $S \times S$. Let H be the function defined on $S \times S$ such that

$$H(x, y) = \begin{cases} 1 & \text{if } x < 1,\ y > 1 \text{ and } x \text{ rational} \\ -1 & \text{if } x < 1,\ y > 1 \text{ and } x \text{ irrational} \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $H \in OL°$ on $S \times S$. However, $\int_a^b HG$ does not exist.

LEMMA 2.3. *If $\beta > 0$, G is a function from $R \times R$ to N, $|G| < 1 - \beta$ on $[a, b]$, $G \in OB°$ on $[a, b]$ and ${}_a\Pi^b(1 + G)$ exists, then ${}_b\Pi^a(1 + H)$ exists and is $[{}_a\Pi^b(1 + G)]^{-1}$, where*

$$H(y, x) = \sum_{j=1}^{\infty} (-1)^j G^j(x, y)$$

for $a \leqq x < y \leqq b$.

Proof. We initially show that ${}_b\Pi^a(1 + H)$ exists. Let $\epsilon > 0$. There exist a subdivision D of $[a, b]$ and a number B such that if $\{x_i\}_{i=0}^m$ and $\{y_j\}_{j=0}^n$ are refinements of D, then

(1) $|G_i| < 1-\beta$ for $i = 1, 2, \cdots, m$,
(2) $|\Pi_{i=1}^m (1 + H_{m+1-i})| < B$, and
(3) $|\Pi_{i=1}^m (1 + G_i) - \Pi_{j=1}^n (1 + G_j)| < \epsilon B^{-2}$.

Note that we are using H_{m+1-i} to denote $H(x_{m+1-i}, x_{m-i})$. Suppose $\{x_i\}_{i=0}^m$ and $\{y_j\}_{j=0}^n$ are refinements of D. Thus,

$$\begin{aligned}
&\left|\prod_{i=1}^m (1 + H_{m+1-i}) - \prod_{j=1}^n (1 + H_{n+1-j})\right| \\
&\quad \leqq \left|\prod_{i=1}^m (1 + H_{m+1-i})\right| \left|1 - \left[\prod_{i=1}^m (1 + H_{m+1-i})\right]^{-1}\left[\prod_{j=1}^n (1 + H_{n+1-j})\right]\right| \\
&\quad \leqq B\left|1 - \left[\prod_{i=1}^m (1 + G_i)\right]\left[\prod_{j=1}^n (1 + H_{n+1-j})\right]\right| \\
&\quad \leqq B\left|\prod_{j=1}^n (1 + G_j) - \prod_{i=1}^m (1 + G_i)\right| \left|\prod_{j=1}^n (1 + H_{n+1-j})\right| \\
&\qquad + B\left|1 - \left[\prod_{j=1}^n (1 + G_j)\right]\left[\prod_{j=1}^n (1 + H_{n+1-j})\right]\right| \\
&\quad < B^2(\epsilon B^{-2}) + B(0) = \epsilon.
\end{aligned}$$

We now show that $[{}_a\Pi^b(1+G)]^{-1}$ exists and is ${}_b\Pi^a(1+H)$. Let $\epsilon > 0$. There exists a subdivision $\{x_i\}_{i=0}^m$ of $[a, b]$ such that

$$\left|[{}_a\Pi^b(1+G)][{}_b\Pi^a(1+H)] - \left[\prod_{i=1}^m (1 + G_i)\right]\left[\prod_{i=1}^m (1 + H_{m+1-i})\right]\right| < \epsilon.$$

Hence,

$$\begin{aligned}
&|[{}_a\Pi^b(1+G)][{}_b\Pi^a(1+H)] - 1| \\
&\quad < \left|\left[\prod_{i=1}^m (1 + G_i)\right]\left[\prod_{i=1}^m (1 + H_{m+1-i})\right] - 1\right| + \epsilon \\
&\quad = 0 + \epsilon = \epsilon.
\end{aligned}$$

LEMMA 2.4. *If $\beta > 0$, G is a function from $R \times R$ to N, $|G| < 1-\beta$ on $[a, b]$, $G \in OB^\circ$ on $[a, b]$ and ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$, then $\int_a^b G$ exists.*

Proof. It follows from Lemma 2.1 that

$$\int_a^b G(u, v)\,{}_v\Pi^b(1 + G)$$

exists. Let H be the function defined on $[a, b]$ such that

$$H(u, v) = [{}_v\Pi^b(1+G)]^{-1}.$$

The existence of H follows from Lemma 2.3. Further, $H \in OL^\circ$ on $[a, b]$. Hence, the existence of $\int_a^b G$ can be established by using Lemma 2.2.

Lemma 2.5. *If $\beta > 0$, G is a function from $R \times R$ to N, $|G| < 1 - \beta$ on (a, b), $G \in OB^\circ$ on $[a, b]$ and ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$, then $\int_a^b G$ exists.*

Proof. Lemma 2.5 follows by using Lemma 1.2 and Lemma 2.4.

Theorem 2. *If G is a function from $R \times R$ to N, $G \in OB^\circ$ on $[a, b]$ and ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$, then $\int_a^b G$ exists.*

Proof. There exists a subdivision $\{x_i\}_{i=0}^m$ of $[a, b]$ such that if $1 \leqq i \leqq m$ and $x_{i-1} < x < y < x_i$, then $|G(x, y)| < 1/2$. Hence, the theorem follows from Lemma 2.5.

An existence theorem for product integrals is now established. In particular, we show that if G has bounded variation on $[a, b]$ and $\int_a^b G$ exists, then ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$.

Lemma 3.1. *If G is a function from $R \times R$ to N such that $G \in OB^\circ$ on $[a, b]$, then there exists $\alpha \in OA^+$ on $[a, b]$ such that*

$$|G(x, y)| \leqq \alpha(x, y)$$

for $a \leqq x < y \leqq b$.

Proof. There exist a subdivision $\{x_i\}_{i=0}^n$ of $[a, b]$ and a number B such that if H is a refinement of $\{x_i\}_{i=0}^n$, then $\Sigma_{H(I)}|G| < B$. Let g be the function such that for $x_{p-1} < x \leqq x_p$, $g(x) = \operatorname{lub} \Sigma_{H(I)}|G|$ for all refinements H of $\{x_i\}_{i=0}^{p-1} \cup \{x\}$. Let $\alpha(x, y) = \int_x^y dg$. This produces the desired function.

Theorem 3. *If G is a function from $R \times R$ to N, $G \in OB^\circ$ on $[a, b]$ and $\int_a^b G$ exists, then ${}_x\Pi^y(1+G)$ exists for $a \leqq x < y \leqq b$.*

Proof. Suppose $a \leqq x < y \leqq b$. In the following we show that ${}_x\Pi^y(1+G)$ exists and is $\Sigma_{p=0}^{\infty} G_p(x, y)$, where $G_0(x, y) = 1$ and

$$G_p(x, y) = (R)\int_x^y G \cdot G_{p-1}(\quad, y)$$

for $p = 1, 2, \cdots$. The existence of these integrals follows from Lemma 2.2.

It follows from Lemma 3.1 that there exists $\alpha \in OA^+$ such that if $x \leqq r < s \leqq y$, then

$$|G(r, s)| \leqq \alpha(r, s).$$

Further, from a result of MacNerney [**4**, Theorem 6.2, p. 160], $\Sigma_{p=0}^{\infty} g_p(x, y)$ exists, where $g_0(x, y) = 1$ and

$$g_p(x, y) = (R)\int_x^y \alpha \cdot g_{p-1}(\quad, y)$$

for $p = 1, 2, \cdots$.

It can be established by induction that if $\{x_i\}_{i=0}^n$ is a subdivision of $[x, y]$, then

$$\prod_{i=1}^n (1 + G_i) = 1 + \sum_{k_1=1}^n G_{k_1} + \sum_{k_1=1}^n \sum_{k_2=k_1+1}^n G_{k_1} G_{k_2} + \cdots + \sum_{k_1=1}^n \sum_{k_2=k_1+1}^n \cdots \sum_{k_n=k_{n-1}+1}^n G_{k_1} G_{k_2} \cdots G_{k_n},$$

where $\Sigma_{i=p}^q G_i = 0$ if $p > q$. Further, it can also be established by induction that

$$\left| \sum_{k_1=1}^n \sum_{k_2=k_1+1}^n \cdots \sum_{k_p=k_{p-1}+1}^n G_{k_1} G_{k_2} \cdots G_{k_p} \right| \leqq g_p(x, y)$$

for $p = 1, 2, \cdots$.

Let $\epsilon > 0$. There exists a positive integer N such that

$$\sum_{p=N+1}^{\infty} g_p(x, y) < \epsilon/3.$$

Further, there exists a subdivision D of $[x, y]$ such that if $\{x_i\}_{i=0}^n$ is a refinement of D, then

$$\left| \left[1 + \sum_{k_1=1}^{n} G_{k_1} + \sum_{k_1=1}^{n} \sum_{k_2=k_1+1}^{n} G_{k_1} G_{k_2} + \cdots \right. \right.$$

$$\left. \left. + \sum_{k_1=1}^{n} \sum_{k_2=k_1+1}^{n} \cdots \sum_{k_N=k_{N-1}+1}^{n} G_{k_1} G_{k_2} \cdots G_{k_N} \right] - \sum_{p=0}^{N} G_p(x, y) \right| < \epsilon/3.$$

Suppose $\{x_i\}_{i=0}^{n}$ is a refinement of D. Thus,

$$\begin{aligned}
&\left| \prod_{i=1}^{n} (1 + G_i) - \sum_{p=0}^{\infty} G_p(x, y) \right| \\
&\quad = \left| \left[1 + \sum_{k_1=1}^{n} G_{k_1} + \sum_{k_1=1}^{n} \sum_{k_2=k_1+1}^{n} G_{k_1} G_{k_2} + \cdots \right. \right. \\
&\qquad \left. \left. + \sum_{k_1=1}^{n} \sum_{k_2=k_1+1}^{n} \cdots \sum_{k_n=k_{n-1}+1}^{n} G_{k_1} G_{k_2} \cdots G_{k_n} \right] - \sum_{p=0}^{\infty} G_p(x, y) \right| \\
&\quad < \left| \left[1 + \sum_{k_1=1}^{n} G_{k_1} + \sum_{k_1=1}^{n} \sum_{k_2=k_1+1}^{n} G_{k_1} G_{k_2} + \cdots \right. \right. \\
&\qquad \left. \left. + \sum_{k_1=1}^{n} \sum_{k_2=k_1+1}^{n} \cdots \sum_{k_N=k_{N-1}+1}^{n} G_{k_1} G_{k_2} \cdots G_{k_N} \right] - \sum_{p=0}^{N} G_p(x, y) \right| \\
&\qquad + \epsilon/3 + \epsilon/3 \\
&\quad < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.
\end{aligned}$$

THEOREM 4. *If G is a function from $R \times R$ to N and $G \in OB^\circ$ on $[a, b]$, then $\int_a^b G$ exists if and only if ${}_x\Pi^y(1 + G)$ exists for $a \leqq x < y \leqq b$.*

Proof. This theorem follows as a corollary to Theorems 2 and 3.

THEOREM 5. *If H and G are functions from $R \times R$ to N, $H \in OL^\circ$ on $[a, b]$, $G \in OB^\circ$ on $[a, b]$ and either $\int_a^b G$ exists or ${}_x\Pi^y(1 + G)$ exists for $a \leqq x < y \leqq b$, then $\int_a^b HG$ and $\int_a^b GH$ exist and ${}_x\Pi^y(1 + HG)$ and ${}_x\Pi^y(1 + GH)$ exist for $a \leqq x < y \leqq b$.*

Proof. This theorem follows as a corollary to Theorem 4 and Lemma 2.2.

We now show that if G has bounded variation on $[a, b]$, then $G \in OA^\nu$ on $[a, b]$ if and only if $G \in OM^\nu$ on $[a, b]$. This is a generalization of a result of B. W. Helton [**1**, Theorem 3.4, p. 301].

LEMMA 6.1. *If $\epsilon > 0$ and G is a function from $R \times R$ to N such that $G \in OB^\circ$ and S_2 on $[a, b]$, then there exists a subdivision D of $[a, b]$ such that if $\{x_i\}_{i=0}^n$ is a refinement of D, $1 \leqq i \leqq n$ and $\{x_{ij}\}_{j=0}^{n(i)}$ is a subdivision of $[x_{i-1}, x_i]$, then*

$$\left| \prod_{j=1}^{n(i)} (1 + G_{ij}) - \left(1 + \sum_{j=1}^{n(i)} G_{ij}\right) \right| < \epsilon.$$

Proof. Since $G \in OB^\circ \cap S_2$ on $[a, b]$, this lemma can be established by applying the covering theorem.

LEMMA 6.2. *If $\epsilon > 0$ and G is a function from $R \times R$ to N such that $G \in OB^\circ$ and S_2 on $[a, b]$, then there exists a subdivision D of $[a, b]$ such that if $\{x_i\}_{i=0}^n$ is a refinement of D and $\{x_{ij}\}_{j=0}^{n(i)}$ is a subdivision of $[x_{i-1}, x_i]$ for $1 \leqq i \leqq n$, then*

$$\sum_{i=1}^{n} \left| \prod_{j=1}^{n(i)} (1 + G_{ij}) - \left(1 + \sum_{j=1}^{n(i)} G_{ij}\right) \right| < \epsilon.$$

Proof. There exist a subdivision $\{r_i\}_{i=0}^r$ of $[a, b]$ and a number B such that if $\{y_i\}_{i=0}^m$ is a refinement of $\{r_i\}_{i=0}^r$, then

(1) $\Sigma_{i=1}^m |G_i| < B$, and

(2) $\Pi_{i=1}^m (1 + |G_i|) < B$.

It follows by applying the covering theorem that there exists a subdivision $\{s_i\}_{i=0}^s$ of $[a, b]$ such that if $1 \leqq i \leqq s$ and $\{x_{ij}\}_{j=0}^{s(i)}$ is a subdivision of $[s_{i-1}, s_i]$, then

$$\sum_{j=2}^{s(i)-1} |G_{ij}| < \epsilon (2B^2)^{-1}.$$

Further, it follows from Lemma 6.1 that there exists a subdivision $\{t_i\}_{i=0}^t$ of $[a, b]$ such that if $\{x_i\}_{i=0}^n$ is a refinement of $\{t_i\}_{i=0}^t$, $1 \leqq i \leqq n$ and $\{x_{ij}\}_{j=0}^{n(i)}$ is a subdivision of $[x_{i-1}, x_i]$, then

$$\left| \prod_{j=1}^{n(i)} (1 + G_{ij}) - \left(1 + \sum_{j=1}^{n(i)} G_{ij}\right) \right| < \epsilon (4s)^{-1}.$$

Let D denote the subdivision

$$\{r_i\}_{i=0}^r \cup \{s_i\}_{i=0}^s \cup \{t_i\}_{i=0}^t$$

of $[a, b]$ and suppose $\{x_i\}_{i=0}^n$ is a refinement of D. Further, suppose $\{x_{ij}\}_{j=0}^{n(i)}$ is a subdivision of $[x_{i-1}, x_i]$ for $1 \leqq i \leqq n$. Let P be the subset of $\{i\}_{i=1}^n$ such that $i \in P$ only if $x_i \in \{s_i\}_{i=0}^s$ or $x_{i-1} \in \{s_i\}_{i=0}^s$. Finally, let

$$Q = \{i\}_{i=1}^{n} - P.$$

In the following manipulations, we use the identity

$$\prod_{i=1}^{n}(1+b_i) = 1 + \sum_{i=1}^{n} b_i + \sum_{i=1}^{n} b_i\left\{\sum_{j=i+1}^{n} b_j\left[\prod_{k=j+1}^{n}(1+b_k)\right]\right\},$$

where $\Sigma_{j=n+1}^{n} b_j = 0$ and $\Pi_{k=n+1}^{n}(1+b_k) = 1$. This result can be established by induction.

We now establish the desired inequality:

$$\begin{aligned}
&\sum_{i=1}^{n}\left|\prod_{j=1}^{n(i)}(1+G_{ij}) - \left(1+\sum_{j=1}^{n(i)} G_{ij}\right)\right| \\
&\quad = \sum_{i\in Q}\left|\prod_{j=1}^{n(i)}(1+G_{ij}) - \left(1+\sum_{j=1}^{n(i)} G_{ij}\right)\right| \\
&\qquad + \sum_{i\in P}\left|\prod_{j=1}^{n(i)}(1+G_{ij}) - \left(1+\sum_{j=1}^{n(i)} G_{ij}\right)\right| \\
&\quad < \sum_{i\in Q}\left|1+\sum_{j=1}^{n(i)} G_{ij} + \sum_{j=1}^{n(i)} G_{ij}\left\{\sum_{u=j+1}^{n(i)} G_{iu}\left[\prod_{v=u+1}^{n(i)}(1+G_{iv})\right]\right\}\right. \\
&\qquad \left. - \left(1+\sum_{j=1}^{n(i)} G_{ij}\right)\right| + 2s[\epsilon(4s)^{-1}] \\
&\quad = \sum_{i\in Q}\left|\sum_{j=1}^{n(i)} G_{ij}\left\{\sum_{u=j+1}^{n(i)} G_{iu}\left[\prod_{v=u+1}^{n(i)}(1+G_{iv})\right]\right\}\right| + \epsilon/2 \\
&\quad \leqq \sum_{i\in Q}\sum_{j=1}^{n(i)} |G_{ij}|\left\{\sum_{u=j+1}^{n(i)} |G_{iu}|\left[\prod_{v=u+1}^{n(i)}(1+|G_{iv}|)\right]\right\} + \epsilon/2 \\
&\quad \leqq B\sum_{i\in Q}\sum_{j=1}^{n(i)} |G_{ij}|\left\{\sum_{u=j+1}^{n(i)} |G_{iu}|\right\} + \epsilon/2 \\
&\quad \leqq B[\epsilon(2B^2)^{-1}]\sum_{i\in Q}\sum_{j=1}^{n(i)} |G_{ij}| + \epsilon/2 \\
&\quad < B[\epsilon(2B^2)^{-1}]B + \epsilon/2 = \epsilon.
\end{aligned}$$

LEMMA 6.3. *If G is a function from $R \times R$ to N, $G \in OB^\circ$ on $[a,b]$ and $\int_a^b G$ exists, then*

$$\int_a^b \left|\Pi(1+G) - \left(1+\int G\right)\right| = 0.$$

Proof. The existence of ${}_x\Pi^y(1+G)$ for $a \leqq x < y \leqq b$ follows from Theorem 3. Also, since $G \in OB^\circ$ on $[a, b]$ and $\int_a^b G$ exists, $G \in S_2$ on $[a, b]$.

Let $\epsilon > 0$. It follows from Lemma 6.2 that there exists a subdivision D of $[a, b]$ such that if $\{x_i\}_{i=0}^n$ is a refinement of D and $\{x_{ij}\}_{j=0}^{n(i)}$ is a subdivision of $[x_{i-1}, x_i]$ for $1 \leqq i \leqq n$, then

$$\sum_{i=1}^n \left| \prod_{j=1}^{n(i)} (1+G_{ij}) - \left(1 + \sum_{j=1}^{n(i)} G_{ij}\right)\right| < \epsilon/3.$$

Suppose $\{x_i\}_{i=0}^n$ is a refinement of D. For $1 \leqq i \leqq n$, let $\{x_{ij}\}_{j=0}^{n(i)}$ be a subdivision of $[x_{i-1}, x_i]$ such that

$$\left| {}_{x_{i-1}}\Pi^{x_i}(1+G) - \prod_{j=1}^{n(i)} (1+G_{ij}) \right| < \epsilon/3n$$

and

$$\left| \sum_{j=1}^{n(i)} G_{ij} - \int_{x_{i-1}}^{x_i} G \right| < \epsilon/3n.$$

Thus,

$$\begin{aligned}
\sum_{i=1}^n & \left| {}_{x_{i-1}}\Pi^{x_i}(1+G) - \left(1 + \int_{x_{i-1}}^{x_i} G\right)\right| \\
& \leqq \sum_{i=1}^n \left| {}_{x_{i-1}}\Pi^{x_i}(1+G) - \prod_{j=1}^{n(i)} (1+G_{ij})\right| \\
& \quad + \sum_{i=1}^n \left| \prod_{j=1}^{n(i)} (1+G_{ij}) - \left(1 + \sum_{j=1}^{n(i)} G_{ij}\right)\right| \\
& \quad + \sum_{i=1}^n \left| \sum_{j=1}^{n(i)} G_{ij} - \int_{x_{i-1}}^{x_i} G\right| \\
& < n(\epsilon/3n) + \epsilon/3 + n(\epsilon/3n) = \epsilon.
\end{aligned}$$

THEOREM 6. *If ν is a nonnegative number, G is a function from $R \times R$ to N and $G \in OB^\circ$ on $[a, b]$, then $G \in OA^\nu$ on $[a, b]$ if and only if $G \in OM^\nu$ on $[a, b]$.*

Proof. Suppose $G \in OM^\nu$ on $[a, b]$. It follows from Theorem 2 that $\int_a^b G$ exists. Hence, it is only necessary to show that

$$\int_a^b \left| G - \int G \right| = \nu.$$

Let $\epsilon > 0$. There exists a subdivision D_1 of $[a, b]$ such that if $\{x_i\}_{i=0}^n$ is a refinement of D_1, then

$$\nu - \epsilon/2 < \sum_{i=1}^n |1 + G_i - {}_{x_{i-1}}\Pi^{x_i}(1+G)| < \nu + \epsilon/2.$$

Further, it follows from Lemma 6.3 that there exists a subdivision D_2 of $[a, b]$ such that if $\{x_i\}_{i=0}^n$ is a refinement of D_2, then

$$\sum_{i=1}^n \left| {}_{x_{i-1}}\Pi^{x_i}(1+G) - \left(1 + \int_{x_{i-1}}^{x_i} G\right)\right| < \epsilon(2|-1|)^{-1}.$$

Let $D = D_1 \cup D_2$. Suppose $\{x_i\}_{i=0}^n$ is a refinement of D. Now,

$$\begin{aligned}&\sum_{i=1}^n \left| G_i - \int_{x_{i-1}}^{x_i} G \right| \\ &\quad = \sum_{i=1}^n \Bigg| [1 + G_i - {}_{x_{i-1}}\Pi^{x_i}(1+G)] \\ &\qquad + \left[{}_{x_{i-1}}\Pi^{x_i}(1+G) - \left(1 + \int_{x_{i-1}}^{x_i} G\right)\right]\Bigg|.\end{aligned}$$

Thus,

$$\begin{aligned}&\sum_{i=1}^n \left| G_i - \int_{x_{i-1}}^{x_i} G \right| \\ &\quad \leqq \sum_{i=1}^n |1 + G_i - {}_{x_{i-1}}\Pi^{x_i}(1+G)| \\ &\qquad + \sum_{i=1}^n \left| {}_{x_{i-1}}\Pi^{x_i}(1+G) - \left(1 + \int_{x_{i-1}}^{x_i} G\right)\right| \\ &\quad < \nu + \epsilon/2 + \epsilon/2 = \nu + \epsilon.\end{aligned}$$

Further,

$$\begin{aligned}&\sum_{i=1}^n \left| G_i - \int_{x_{i-1}}^{x_i} G \right| \\ &\quad \geqq \sum_{i=1}^n |1 + G_i - {}_{x_{i-1}}\Pi^{x_i}(1+G)|\end{aligned}$$

$$-|-1|\sum_{i=1}^{n}\left|{}_{x_{i-1}}\Pi^{x_i}(1+G)-\left(1+\int_{x_{i-1}}^{x_i}G\right)\right|$$

$$>\nu-\epsilon/2-\epsilon/2=\nu-\epsilon.$$

Hence,

$$\nu-\epsilon<\sum_{i=1}^{n}\left|G_i-\int_{x_{i-1}}^{x_i}G\right|<\nu+\epsilon.$$

Therefore, $G\in OA^{\nu}$ on $[a,b]$.

Suppose $G\in OA^{\nu}$ on $[a,b]$. It follows from Theorem 3 that ${}_x\Pi^y(1+G)$ exists for $a\leqq x<y\leqq b$. Hence, it is only necessary to show that

$$\int_a^b|1+G-\Pi(1+G)|=\nu.$$

Let $\epsilon>0$. There exists a subdivision D_1 of $[a,b]$ such that if $\{x_i\}_{i=0}^n$ is a refinement of D_1, then

$$\nu-\epsilon/2<\sum_{i=1}^{n}\left|G_i-\int_{x_{i-1}}^{x_i}G\right|<\nu+\epsilon/2.$$

Further, it follows from Lemma 6.3 that there exists a subdivision D_2 of $[a,b]$ such that if $\{x_i\}_{i=0}^n$ is a refinement of D_2, then

$$\sum_{i=1}^{n}\left|1+\int_{x_{i-1}}^{x_i}G-{}_{x_{i-1}}\Pi^{x_i}(1+G)\right|<\epsilon(2|-1|)^{-1}.$$

Let $D=D_1\cup D_2$. Suppose $\{x_i\}_{i=0}^n$ is a refinement of D. Now,

$$\sum_{i=1}^{n}|1+G_i-{}_{x_{i-1}}\Pi^{x_i}(1+G)|$$

$$=\sum_{i=1}^{n}\left|\left[G_i-\int_{x_{i-1}}^{x_i}G\right]+\left[1+\int_{x_{i-1}}^{x_i}G-{}_{x_{i-1}}\Pi^{x_i}(1+G)\right]\right|.$$

It follows as in the preceding argument that

$$\nu-\epsilon<\sum_{i=1}^{n}|1+G_i-{}_{x_{i-1}}\Pi^{x_i}(1+G)|<\nu+\epsilon.$$

Therefore, $G\in OM^{\nu}$ on $[a,b]$.

We now prove a theorem on the existence of integrals of products of functions. This result is related to a theorem by B. W. Helton [**2**, Theorem 2, p. 494].

LEMMA 7.1. *If* $\epsilon > 0, H$ *is a function from* $R \times R$ *to* N *and* $H \in OL^\circ$ *on* $[a, b]$, *then there exist a subdivision* $\{t_i\}_{i=0}^t$ *of* $[a, b]$ *and a sequence* $\{k_i\}_{i=1}^t$ *such that if* $1 \leqq i \leqq t$ *and* $t_{i-1} < x < y < t_i$, *then*

$$|H(x, y) - k_i| < \epsilon.$$

Proof. This lemma is a variation of a lemma used by B. W. Helton [**2**, Lemma, p. 498]. The proof presented there can be used to establish the lemma as we have stated it.

LEMMA 7.2. *Suppose* $|AB| = |A||B|$ *for* $A, B \in N$. *If* ν *is a nonnegative number,* $k \in N, G$ *is a function from* $R \times R$ *to* N *and* $G \in OA^\nu$ *on* $[a, b]$, *then* $kG \in OA^{|k|\nu}$ *on* $[a, b]$.

Proof. Since $|AB| = |A||B|$, the proof is readily constructed. If the preceding equality did not hold, the lemma would not necessarily follow. An example of such a situation is presented after the proof of Theorem 7.

THEOREM 7. *Suppose* $|AB| = |A||B|$ *for* $A, B \in N$. *If* ν *is a nonnegative number,* H *and* G *are functions from* $R \times R$ *to* N, $H \in OL^\circ$ *on* $[a, b]$, $G \in OB^\circ$ *on* $[a, b]$ *and either* $G \in OA^\nu$ *on* $[a, b]$ *or* $G \in OM^\nu$ *on* $[a, b]$, *then there exist nonnegative numbers* α *and* β *such that* HG *is in* OA^α *and* OM^α *on* $[a, b]$ *and* GH *is in* OA^β *and* OM^β *on* $[a, b]$.

Proof. We initially establish that there exists a nonnegative number α such that $HG \in OA^\alpha$ on $[a, b]$. It follows from Theorem 6 that $G \in OA^\nu$ on $[a, b]$. Hence, the existence of $\int_a^b HG$ follows from Theorem 5. We use the Cauchy criterion to establish the existence of

$$\int_a^b \left| HG - \int HG \right|.$$

Let $\epsilon > 0$. There exist a subdivision E_1 of $[a, b]$ and a number B such that if $\{x_i\}_{i=0}^n$ is a refinement of E_1, then

$$\sum_{i=1}^n |G_i| < B.$$

It follows from Lemma 7.1 that there exist a subdivision $E_2 = \{t_i\}_{i=0}^t$ of $[a, b]$ and a sequence $\{k_i\}_{i=1}^t$ such that if $1 \leqq i \leqq t$ and $t_{i-1} < x < y < t_i$, then

$$|H(x, y) - k_i| < \epsilon(8| - 1|B)^{-1}.$$

Since $G \in OB^\circ \cap OA^\nu$ on $[a, b]$, it follows that there exist subdivisions $\{r_i\}_{i=0}^{t+1}$ and $\{s_i\}_{i=0}^{t+1}$ of $[a, b]$ such that

(1) $t_{i-1} < r_i < s_i < t_i$ for $1 \leqq i \leqq t$, and

(2) $\sum_{j=1}^n \left| H_j G_j - \int_{x_{j-1}}^{x_j} HG \right| < \epsilon[8(t + 1)]^{-1}$ for $1 \leqq i \leqq t + 1$ and each refinement $\{x_j\}_{j=0}^n$ of $\{s_{i-1}, t_{i-1}, r_i\}$.

It follows from Lemma 7.2 that $k_i G \in OA^{|k_i|\nu}$ on $[r_i, s_i]$ for $1 \leqq i \leqq t$. Hence, for each i there exists a subdivision D_i of $[r_i, s_i]$ such that if J and K are refinements of D_i, then

$$\left| \sum_{J(I)} \left| k_i G - \int k_i G \right| - \sum_{K(I)} \left| k_i G - \int k_i G \right| \right| < \epsilon(4t)^{-1}.$$

Let D denote the subdivision $\cup_{i=1}^2 E_i \cup_{i=1}^t D_i$ of $[a, b]$. Suppose J_1 and J_2 are refinements of D, P_{1i} and P_{2i} are subdivisions of $[s_{i-1}, r_i]$ for $1 \leqq i \leqq t + 1$, Q_{1i} and Q_{2i} are subdivisions of $[r_i, s_i]$ for $1 \leqq i \leqq t$ and J_1 and J_2 are equal to

$$\bigcup_{i=1}^{t+1} P_{1i} \bigcup_{i=1}^{t} Q_{1i} \quad \text{and} \quad \bigcup_{i=1}^{t+1} P_{2i} \bigcup_{i=1}^{t} Q_{2i},$$

respectively. For convenience, suppose

$$\sum_{J_1(I)} \left| HG - \int HG \right| \geqq \sum_{J_2(I)} \left| HG - \int HG \right|.$$

Thus,

$$\begin{aligned}
&\left| \sum_{J_1(I)} \left| HG - \int HG \right| - \sum_{J_2(I)} \left| HG - \int HG \right| \right| \\
&\quad = \sum_{J_1(I)} \left| HG - \int HG \right| - \sum_{J_2(I)} \left| HG - \int HG \right| \\
&\quad = \sum_{i=1}^{t+1} \sum_{P_{1i}(I)} \left| HG - \int HG \right| + \sum_{i=1}^{t} \sum_{Q_{1i}(I)} \left| HG - \int HG \right| \\
&\qquad - \sum_{i=1}^{t+1} \sum_{P_{2i}(I)} \left| HG - \int HG \right| - \sum_{i=1}^{t} \sum_{Q_{2i}(I)} \left| HG - \int HG \right|
\end{aligned}$$

$$< (t+1)\{\epsilon[8(t+1)]^{-1}\} + \sum_{i=1}^{t} \sum_{Q_{1i}(I)} \left| HG - \int HG \right|$$

$$+ (t+1)\{\epsilon[8(t+1)]^{-1}\} - \sum_{i=1}^{t} \sum_{Q_{2i}(I)} \left| HG - \int HG \right|$$

$$= \sum_{i=1}^{t} \sum_{Q_{1i}(I)} \left| (H - k_i + k_i)G - \int (H - k_i + k_i)G \right|$$

$$- \sum_{i=1}^{t} \sum_{Q_{2i}(I)} \left| (H - k_i + k_i)G - \int (H - k_i + k_i)G \right| + \epsilon/4$$

$$\leqq |-1| \sum_{j=1}^{2} \sum_{i=1}^{t} \sum_{Q_{ji}(I)} |(H - k_i)G|$$

$$+ \sum_{j=1}^{2} \sum_{i=1}^{t} \sum_{Q_{ji}(I)} \left| \int (H - k_i)G \right|$$

$$+ \sum_{i=1}^{t} \sum_{Q_{1i}(I)} \left| k_i G - \int k_i G \right|$$

$$- \sum_{i=1}^{t} \sum_{Q_{2i}(I)} \left| k_i G - \int k_i G \right| + \epsilon/4$$

$$< 2B|-1|[\epsilon(8|-1|B)^{-1}] + 2B[\epsilon(8|-1|B)^{-1}] + t[\epsilon(4t)^{-1}] + \epsilon/4$$

$$\leqq \epsilon.$$

Therefore, $\int_a^b \left| HG - \int HG \right|$ exists. Hence, there exists a nonnegative number α such that $G \in OA^{\alpha}$ on $[a, b]$. Thus, it follows from Theorem 6 that $G \in OM^{\alpha}$ on $[a, b]$.

A similar argument can be used to establish the existence of β. Therefore, the theorem follows.

Theorem 7 does not remain true if the requirement that $|AB| = |A||B|$ is removed. In the following we establish this assertion by constructing a function G and a constant K such that $\int_0^1 G$ exists, $\int_0^1 \left| G - \int G \right|$ exists and $\int_0^1 \left| KG - \int KG \right|$ does not exist.

We consider the set of infinite diagonal matrices with bounded elements and $|M| = \text{lub}|m_{ij}|$. For $p = 1, 2, \cdots$, let A_p be the infinite diagonal matrix such that $a_{pp} = 1$ and $a_{qq} = 0$ if $q \neq p$. Let $A = \{A_p \mid p = 1, 2, \cdots\}$. There exists a reversible function f from the rational numbers in [0, 1] to A. Let G be an interval function defined on [0, 1] such that

$$G(u, v) = \begin{cases} (v-u)f(v) & \text{if } v \text{ is rational} \\ (v-u)f(r) & \text{where } r \text{ is a rational number in } (u, v) \text{ if } v \text{ is irrational.} \end{cases}$$

For each rational number r in $[0, 1]$, let $p(r)$ be the positive integer such that $f(r) = A_{p(r)}$. Let K be the infinite diagonal matrix such that if $r = m/n$ is a rational number contained in $[0, 1]$ and m and n have no common integral factors other than 1, then

$$k_{p(r),p(r)} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

We have now constructed a function G and a constant K such that $\int_0^1 G = 0$, $\int_0^1 \left| G - \int G \right| = 1$ and $\int_0^1 \left| KG - \int KG \right|$ does not exist. This example was suggested by an example in a previous paper by the author [3].

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