THE PADÉ TABLE OF FUNCTIONS HAVING A FINITE NUMBER OF ESSENTIAL SINGULARITIES

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Let $f(z)$ be regular at the origin and let it be single-valued and regular except for poles and $s + 1 < +\infty$ essential singularities $\alpha_0, \alpha_1, \alpha_2, \cdots, \alpha_s$. The α's may be limit-points of poles and $\alpha_0 = \infty$ is permissible. Assume that α_k is of finite order λ_k and let $\Lambda = \Sigma \lambda_k$.

The author obtains a convergence theorem for the Padé table of $\Sigma a_j z^j = f(z)$. The simplest consequence of his result may be stated as follows: if $\Lambda < 2$, the diagonal approximants of the Padé table converge almost everywhere in the plane.

Introduction. Let

$$(1) \qquad a_0 + a_1 z + a_2 z^2 + \cdots = f(z) \qquad (a_0 \neq 0),$$

have radius of convergence $\rho_0 > 0$. It is well known that given two integers $m \geq 0$, $n \geq 0$, it is always possible to find two polynomials

$$(2) \qquad Q_{mn}(z) = q_0 + q_1 z + \cdots + q_n z^n,$$

$$(3) \qquad P_{mn}(z) = p_0 + p_1 z + \cdots + p_m z^m,$$

such that

$$(4) \qquad Q_{mn}(z) \not\equiv 0,$$

and

$$(5) \qquad f(z)Q_{mn}(z) - P_{mn}(z) = \sum_{j=m+n+1}^{x} p_j z^j.$$

Although P and Q are not unique, the rational function

$$(6) \qquad R_{mn}(z) = \frac{P_{mn}(z)}{Q_{mn}(z)},$$

does not depend on the particular choice of the polynomials P and Q which satisfy (4) and (5) [6; pp. 235–237]. We place $R_{mn}(z)$ in the nth row and mth column of an infinite matrix. The resulting array is, by definition, the Padé table of $f(z)$.

We propose to study the convergence of infinite sequences of "approximants" defined as follows: with each $k = 1, 2, 3, \cdots$, we associate an ordered pair (m, n)

$$m = m(k) \quad (m \gtrless 0), \quad n = n(k) \qquad (n \geqq 0), \tag{7}$$

and examine the behavior of the "error"

$$f(z) - R_{m,n}(z) = \Delta_{mn}(z) = \Delta_k(z), \tag{8}$$

as $k \to \infty$. The functions $m(k)$ and $n(k)$ in (7) enable us to simplify our notation and, from this point on, we write

$$P_m(z), \quad Q_n(z), \tag{9}$$

instead of

$$P_{mn}(z), \quad Q_{mn}(z).$$

Our results concern the class of

Quasi-meromorphic functions of finite order. We say that $f(z)$ is quasi-meromorphic, of order Λ, if
  (i) $f(z)$ is single-valued; its only singularities are poles and essential singularities

$$\alpha_0, \alpha_1, \cdots, \alpha_s \qquad (s < +\infty),$$

which may be limit-points of poles;
  (ii) one of the α's may be ∞;
  (iii) the order of α_j is λ_j and

$$\lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_s = \Lambda < +\infty.$$

THEOREM 1. *Let $f(z)$ be quasi-meromorphic of order Λ.*
Assume that $m(k) > 0$ and $n(k) > 0$ are integers such that

$$\chi k^\beta < m + n \qquad (k = 1, 2, 3, \cdots; \chi > 0) \tag{10}$$

for some fixed χ and

$$2\beta > \Lambda. \tag{11}$$

Assume also that, as $k \to \infty$,

(12) $$m = o(n \log n), \quad n = o(m \log m).$$

Then, given $\rho > 0$ and $\delta > 0$, it is possible to find a measurable set $\Omega = \Omega(\rho, \delta)$ such that

(13) $$\text{meas } \Omega < \delta,$$

and such that

(14) $$R_{mn}(z) \rightarrow f(z) \qquad (k \rightarrow +\infty),$$

uniformly for all z restricted by the conditions

(15) $$|z| \leqq \rho, \quad z \notin \Omega.$$

As an immediate consequence, we obtain

COROLLARY 1. *If $\Lambda < 2$, then*

(16) $$R_{nn}(z) \rightarrow f(z) \qquad (n \rightarrow \infty),$$

for almost all z.

The connection between the uniform convergence in (14) and the pointwise convergence in (16) is readily established by Egoroff's theorem.   An appeal to the latter result does not simplify our proofs and we shall therefore not concern ourselves with this aspect of the question.

The introduction, in Theorem 1, of the exceptional set Ω cannot be avoided unless the class of functions under consideration is severely restricted [1], [4].

We study this situation in the following Theorem 2.   For simplicity we confine our attention to sequences of diagonal elements of the table.

THEOREM 2. *Let $f(z)$ be quasi-meromorphic, of order $\Lambda \leq +\infty$.*
I. *Consider the Hankel determinants*

(17) $$A_n = \begin{vmatrix} a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \\ a_{n+1} & a_n & a_{n-1} & \cdots & a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{2n-1} & a_{2n-2} & a_{2n-3} & \cdots & a_n \end{vmatrix} \quad (n = 1, 2, 3, \cdots),$$

and let $A_n \neq 0$ *for all n belonging to some infinite sequence* $\mathcal{N}$ *of positive, strictly increasing integers.*

Assume that, in some neighborhood of the origin, the approximants $R_{nn}(z)$ $(n \in \mathcal{N})$ *have no poles.*

Then

$$(18) \qquad\qquad \limsup_{\substack{n \to \infty \\ n \in \mathcal{N}}} \left| \frac{A_{n+1}}{A_n} \right|^{1/n \log n} \leqq e^{-1/\Lambda}.$$

II. *Assume that the disk*

$$|z - z_0| < \eta \qquad (\eta > 0)$$

contains no singularities of $f(z)$ *and no poles of* $R_{nn}(z)$.

Then, given η' $(0 < \eta' < \eta)$ *and* $\delta > 0$ *we have*

$$|f(z) - R_{nn}(z)| \leqq \exp\left(-\frac{n \log n}{\Lambda + \delta} \right),$$

for all z and n satisfying the conditions

$$(19) \qquad\qquad |z - z_0| \leqq \eta', \quad n \geqq N_0(\eta', \delta), \quad A_n \neq 0.$$

The method which leads to Theorems 1 and 2 may be applied to situations significantly more general than the one considered in this paper.

Functions with singularities of transfinite measure zero. We say that the singularities of $f(z)$ have transfinite measure zero if $f(z)$ satisfies the two following conditions.

I. The analytic function $f(z)$ is single-valued and regular at the origin and throughout the complement CE of a closed set E of the extended z-plane. The origin lies in CE.

II. Consider, in the plane of the complex variable ζ, the image $\mathcal{E}$ of E given by the inversion

$$\zeta = z^{-1}.$$

Assume

$$\tau(\mathcal{E}) = 0,$$

where $\tau(\mathcal{E})$ is the transfinite diameter [5; pp. 268–273] of $\mathcal{E}$.

We state without proof the following analogue of assertion I of Theorem 2.

THEOREM 3. *Let the singularities of* $f(z)$ *have transfinite measure zero.*

Let A_n *and* $\mathcal{N}$ *have the same meaning as in Theorem 2. Assume that, in some neighborhood of the origin, the approximants* $R_{nn}(z)$ *$(n \in \mathcal{N})$ have no poles.*
        Then

$$
(20) \qquad\qquad \lim_{\substack{n \to \infty \\ n \in \mathcal{N}}} \left| \frac{A_{n+1}}{A_n} \right|^{1/n} = 0.
$$

An analogous extension of assertion II of Theorem 2 is also possible. We do not state it here because, in its present form, our proof introduces restrictions which are more complex than those given by (19).

The analogue of Theorem 1, for functions with singularities of transfinite measure zero, differs little from a recent result of Pommerenke [8]; it is consequently omitted from this note.

If $f(z)$ has singularities of transfinite measure zero, a classical result of Pólya [7; pp. 688–689] asserts that

$$
(21) \qquad\qquad \lim_{n \to \infty} |A_n|^{1/n^2} = 0.
$$

For the class of quasi-meromorphic functions of order Λ, I have shown [3; pp. 36–49] that

$$
(22) \qquad\qquad \limsup_{n \to \infty} |A_n|^{1/n^2 \log n} \leqq e^{-1/\Lambda},
$$

and that this relation is "best possible".
        From (21) and (22) we deduce, respectively,

$$
\liminf_{n \to \infty} |A_{n+1}/A_n|^{1/n} = 0,
$$

and

$$
\liminf_{n \to \infty} |A_{n+1}/A_n|^{1/2n \log n} \leqq e^{-1/\Lambda}.
$$

The corresponding relations with lim inf replaced by lim sup require, in addition, that the sequence $\{|A_n|\}_n$ be decreasing with a regularity that has little analytic significance.

On the other hand (18) and (20) show that, unless the determinants A_n display this type of regularity, the sequence of diagonal approximants $\{R_{nn}(z)\}_n$ will have poles accumulating at the origin. Following Chisholm [2], we say that such poles are "spurious"; they are accidents of the Padé method and do not reflect the presence of nearby singularities of $f(z)$. To eliminate spurious poles, we would have to impose severe and unnatural conditions: within the class of quasi-meromorphic functions of order Λ,

$$\limsup_{n \to \infty} |A_{n+1}/A_n|^{1/n \log n} \leqq e^{-1/\Lambda}$$

represents as serious a restriction as would be the requirement that, within the class of entire functions of finite order (for which $\limsup_{n \to \infty} |a_n|^{1/n \log n} < 1$), we only consider those for which

$$\limsup_{n \to \infty} |a_{n+1}/a_n|^{1/\log n} < 1.$$

Focusing our attention on the class of quasi-meromorphic functions of order $\Lambda < 2$, we see that the perturbations introduced by spurious poles are to some extent compensated by the fact that the convergence of the approximants is unaffected by the radius of convergence of (1) (or the radius of meromorphy of $f(z)$): in an obvious sense, the sequence

$$\{R_{nn}(z)\}_n$$

of diagonal approximants "overconverges" almost everywhere in the plane.

We conclude this introduction by some remarks about the notations K, K_0, N_0, which we use systematically. We always assume that $K > 1$; the value of K may depend on several parameters and is not necessarily the same one each time it occurs. The quantities K_0, N_0 denote positive integers with the same indeterminate character as K. They always appear as conditions such as $k \geqq K_0$, $n \geqq N_0$, and restrict the validity of some relation to sufficiently large values of k and n. Whenever they qualify expressions such as (19), involving the complex variable z, it is understood that the bounds K_0 or N_0 which they indicate hold uniformly for all z under consideration.

2. Integral representation of the error term $\Delta_{mn}(z)$. Using the notation (9), we rewrite (5) as

$$(2.1) \qquad f(z)Q_n(z) - P_m(z) = \sum_{j=m+n+1}^{\infty} p_j z^j \qquad (p_j = p_j(k)).$$

Set

$$(2.2) \qquad a_l = 0 \cdot \quad (l = -1, -2, -3, \cdots);$$

consider the Hankel matrix

$$(2.3) \qquad \begin{pmatrix} a_{m+1} & a_m & a_{m-1} & \cdots & a_{m-n+1} \\ a_{m+2} & a_{m+1} & a_m & \cdots & a_{m-n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m+n} & a_{m+n-1} & a_{m+n-2} & \cdots & a_m \end{pmatrix},$$

and assume that its rank is n. We may then obtain a "solution" of (2.1) by defining

$$(2.4) \qquad Q_n(z) = \begin{vmatrix} 1 & z & z^2 & \cdots & z^n \\ a_{m+1} & a_m & a_{m-1} & \cdots & a_{m-n+1} \\ a_{m+2} & a_{m+1} & a_m & \cdots & a_{m-n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m+n} & a_{m+n-1} & a_{m+n-2} & \cdots & a_m \end{vmatrix},$$

$$(2.5) \quad p_j = \begin{vmatrix} a_j & a_{j-1} & a_{j-2} & \cdots & a_{j-n} \\ a_{m+1} & a_m & a_{m-1} & \cdots & a_{m-n+1} \\ a_{m+2} & a_{m+1} & a_m & \cdots & a_{m-n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m+n} & a_{m+n-1} & a_{m+n-2} & \cdots & a_n \end{vmatrix} \quad (j = 0, 1, 2, 3, \cdots),$$

and

$$(2.6) \qquad P_m(z) = \sum_{j=0}^{m} p_j z^j.$$

Whenever the rank of (2.3) is n, the polynomials P_m and Q_n will be defined by (2.6) and (2.4). Then

$$(2.7) \qquad f(z)Q_n(z) - P_m(z) = (-1)^n A_{m+1}^{(n+1)} z^{m+n+1} + z^{m+n+2} \mathcal{S}(z),$$

where $\mathscr{S}(z)$ is a series of nonnegative powers of z and $A_{m+1}^{(n+1)}$ is the Hankel determinant of the matrix obtained by adding to (2.3), as last row

$$a_{m+n+1}, a_{m+n}, \cdots a_{m+1}.$$

If the rank of (2.3) is $< n$, (2.4) becomes $Q_n(z) \equiv 0$; it violates our condition (4) and hence cannot be used. For sake of definiteness we shall then select, for P_m and Q_n, some specific solution of (2.1) with

$$(2.8) \qquad \text{degree } P_m \leqq m, \quad \text{degree } Q_n \leqq n, \quad Q_n(z) \not\equiv 0.$$

A closer characterization of these polynomials is unnecessary.
Denote by

$$V_n(z) = V_{nm}(z)$$

a polynomial such that

$$(2.9) \qquad\qquad \text{degree } V_n \leqq n, \qquad V_n(z) \not\equiv 0,$$

otherwise arbitrary.
From (2.1) we deduce that

$$(2.10) \qquad \frac{f(z)Q_n(z)V_n(z) - P_m(z)V_n(z)}{z^{m+n+1}} = F_n(z)$$

is holomorphic for $|z| < \rho_0$ and hence by Cauchy's theorem

$$(2.11) \; F_n(z) = \frac{1}{2\pi i} \oint \frac{f(\zeta)Q_n(\zeta)V_n(\zeta)}{\zeta^{m+n+1}(\zeta - z)} \, d\zeta - \frac{1}{2\pi i} \oint \frac{P_m(\zeta)V_n(\zeta)}{\zeta^{m+n+1}(\zeta - z)} \, d\zeta,$$

where $|z| < \rho_0$ and the contour of integration is a circumference

$$(2.12) \qquad \zeta = r_0 e^{i\omega} \qquad (0 \leqq \omega < 2\pi, \, |z| < r_0 < \rho_0).$$

Since $P_m(z)V_n(z)$ is of degree $\leqq m + n$, the second integral in (2.11) is zero and hence we obtain the fundamental relation

$$(2.13) \quad f(z) - \frac{P_m(z)}{Q_n(z)} = \Delta_{mn}(z)$$

$$= \frac{z^{m+n+1}}{Q_n(z)V_n(z)} \frac{1}{2\pi i} \oint \frac{f(\zeta)Q_n(\zeta)V_n(\zeta)}{\zeta^{m+n+1}(\zeta - z)} \, d\zeta.$$

We note that (2.1) and (2.11) also imply

$$(2.14) \qquad F_n(0) = p_{m+n+1} V_n(0) = \frac{1}{2\pi i} \oint \frac{f(\zeta) Q_n(\zeta) V_n(\zeta)}{\zeta^{m+n+2}} \, d\zeta.$$

We shall use the above relation in the special case

$$(2.15) \qquad m = n, \quad A_n \neq 0, \quad V_n(0) \neq 0.$$

Since

$$A_n = A_n^{(n)} = Q_n(0),$$

we may rewrite (2.14) in the equivalent form

$$(2.16) \qquad (-1)^n \frac{A_{n+1}}{A_n} = \frac{1}{2\pi i \, V_n(0)} \oint \frac{f(\zeta) V_n(\zeta) \{ Q_n(\zeta) / Q_n(0) \}}{\zeta^{2n+2}} \, d\zeta.$$

We also note that (2.7) (with $m = n$) leads at once to

$$Q_{n-1}(z) P_n(z) - Q_n(z) P_{n-1}(z) = (-1)^{n-1} A_n^2 z^{2n-1}.$$

Hence, if $A_n \neq 0$, $P_n(z)$ and $Q_n(z)$ have no common zeros and the poles of $R_{nn}(z)$ coincide exactly with the zeros of $Q_n(z)$.

3. Factors of a quasi-meromorphic function.

From our definitions it follows that $f(z)$, in Theorem 1, may be represented in the form

$$(3.1) \quad f(z) = h_0(z) h_1 \left(\frac{1}{z - \alpha_1} \right) h_2 \left(\frac{1}{z - \alpha_2} \right) \cdots h_s \left(\frac{1}{z - \alpha_s} \right) \quad (s < +\infty),$$

where

$$(3.2) \qquad h_\nu(t) \qquad (\nu = 0, 1, 2, \cdots, s)$$

is a meromorphic function of t, of order $\lambda_\nu < +\infty$. Meromorphic functions correspond to the special case $s = 0$.

The possibility of a decomposition such as (3.1) and the uniqueness of the orders λ_ν ($\nu = 0, 1, \cdots, s$) are readily deducible from the definition of order of an essential singularity.

This aspect of the question need not concern us here and we shall be content with the naive view that Theorem 1 is applicable to all functions representable in the form (3.1).

It is obvious that a finite number of zeros and poles can always be transferred from one factor h to another. This remark leads us to a preliminary clarification of what we mean by zeros and poles of $h_\nu(t)$.

Select γ_0 so as to satisfy the conditions

$$(3.3) \qquad\qquad 0 < 2\gamma_0 < \min(|\alpha_1|, |\alpha_2|, \cdots, |\alpha_s|, 1),$$

and

$$(3.4) \qquad 2\gamma_0 < \min_{\nu \neq \mu}(|\alpha_\nu - \alpha_\mu|) \qquad (1 \leqq \nu \leqq s, 1 \leqq \mu \leqq s).$$

The poles of $f(z)$ which lie in the region

$$(3.5) \qquad\qquad 0 < |z - \alpha_\nu| < \gamma_0$$

will be denoted by

$$(3.6) \qquad\qquad b_1^{(\nu)}, b_2^{(\nu)}, b_3^{(\nu)}, \cdots$$

and arranged so that

$$(3.7) \qquad \gamma_0 > |b_1^{(\nu)} - \alpha_\nu| \geqq |b_2^{(\nu)} - \alpha_\nu| \geqq |b_3^{(\nu)} - \alpha_\nu| \geqq \cdots.$$

The "poles of $h_\nu(t)$" are, by definition, the quantities

$$(3.8) \qquad\qquad t_1^{(\nu)}, t_2^{(\nu)}, t_3^{(\nu)}, \cdots$$

deduced from (3.6) by the transformation

$$(3.9) \qquad\qquad t_j^{(\nu)} = \frac{1}{b_j^{(\nu)} - \alpha_\nu} \qquad (j = 1, 2, 3, \cdots).$$

These operations are performed for all $\nu = 1, 2, \cdots, s$. The poles of $f(z)$ which do not belong to one of the sequences $\{b_j^{(\nu)}\}_j$ $(\nu = 1, 2, \cdots, s)$, have no finite point of accumulation. We shall denote them by b_j and arrange them in a sequence

$$(3.10) \qquad\qquad b_1, b_2, b_3, \cdots$$

so that

$$0 < \rho_0 \leqq |b_1| \leqq |b_2| \leqq |b_3| \leqq \cdots.$$

We also include, in (3.10), elements such as α_ν (repeated a suitable number of times). They enable us to take into account factors $(z - \alpha_\nu)^{-\beta}$ $(\beta > 0)$ which may be present in the decomposition (3.1). The sequence (3.10) is by definition the sequence of poles of $h_0(z)$.

In view of the conditions (3.4), (3.5), and of the regularity of $f(z)$ at the origin, every pole of $f(z)$ belongs to one and only one of the factors h_ν in (3.1).

The zeros of $f(z)$ (which do not explicitly appear in our proofs) are to be distributed in exactly the same way among the $s + 1$ functions $h_\nu(t)$. The assumption $a_0 \neq 0$ implies the regularity of $1/f(z)$ at the origin. Hence there is complete symmetry between the functions $f(z)$ and $1/f(z)$. Both have the same essential singularities α_ν with the same orders λ_ν $(\nu = 0, 1, 2, \cdots, s)$.

4. Estimates from the theory of meromorphic function.

LEMMA 1. *Let* $h(z)(h(0) \neq \infty)$ *be meromorphic, of finite order* λ *and let*

$$b_1, b_2, \cdots, b_j, \cdots \qquad (0 < |b_1| \leq |b_2| \leq |b_3| \leq \cdots)$$

be the sequence of its poles.

Take $\sigma > \lambda$. *Then, if* n *is large enough, the function* $h(z)$ *has fewer than* n *poles in the disk*

$$(4.1) \qquad\qquad |z| \leq n^{1/\sigma}.$$

The function

$$(4.2) \qquad\qquad g_n(z) = h(z) \prod_{|b_j| \leq n^{1/\sigma}} \left(1 - \frac{z}{b_j}\right),$$

which is regular in the disk (4.1), *satisfies the inequality*

$$(4.3) \qquad\qquad |g_n(z)| \leq e^n \qquad (|z| \leq \tfrac{1}{2} n^{1/\sigma}).$$

Proof. Denote as usual by $n(x, \infty)$ [9; p. 284b] the number of poles of $h(z)$ in $|z| \leq x$. By the elements of the theory

$$(4.4) \qquad n(x, \infty) < x^{\sigma'} \qquad (\lambda < \sigma' < \sigma; \; x > x_0(\sigma')).$$

Hence there are, in the disk (4.1), fewer than

(4.5) $n^{\sigma'/\sigma} < n$ $(n \geqq N_0)$

poles of $h(z)$.

From the Poisson-Jensen-Nevanlinna identity [9; p. 129] we deduce the well-known inequality

(4.6) $$\log |h(z)| \leqq \frac{x + |z|}{x - |z|}\, m(x, h) + \sum_{|b_j| \leqq x} \log \left(\frac{2x}{|z - b_j|}\right)$$

where $|z| < x$ and $m(x, h)$ is Nevanlinna's mean.   Applying (4.6) with

$$x = n^{1/\sigma}, \qquad |z| \leqq \tfrac{1}{2} n^{1/\sigma},$$

we find

(4.7) $$\log |g_n(z)| \leqq 3m(n^{1/\sigma}, h) + n(n^{1/\sigma}, \infty) \log 2 + N(n^{1/\sigma}, \infty),$$

where $N(x, \infty)$ has the meaning assigned to it in Nevanlinna's theory.

The inequality (4.3) follows immediately from (4.7) and the well-known behavior of $m(x, h), n(x, \infty), N(x, \infty)$ associated with functions of finite order.

5.   Construction of the polynomials $V_n(z)$ and of the contours $\mathscr{C}_n$.   Choose $\eta > 0$ such that

(5.1) $\sigma_\nu = \lambda_\nu + \eta$ $(\nu = 0, 1, 2, \cdots, s)$

satisfy the condition

(5.2) $$\sum_{\nu=0}^{s} \sigma_\nu = \Sigma < 2\beta.$$

This is possible by (11).

Put

(5.3) $$n_\nu = \left[\frac{\sigma_\nu n}{\Sigma}\right]   (\nu = 0, 1, 2, \cdots, s),$$

which imply

(5.4) $$n_\nu \sim \frac{\sigma_\nu n}{\Sigma}   (\nu = 0, 1, 2, \cdots, s ; n \to \infty)$$

and

(5.5)
$$\sum_{\nu=0}^{s} n_\nu \leqq n.$$

The contour $\mathscr{C}_n$ $(n \geqq N_0)$ is formed by $s + 1$ circumferences:

(5.6) $\mathscr{C}_n^{(0)} = \{\zeta : \zeta = \frac{1}{2} e^{i\omega} n_0^{1/\sigma_0} \quad (0 \leqq \omega < 2\pi)\}$,

(5.7) $\mathscr{C}_n^{(\nu)} = \{\zeta : \zeta = 2 e^{-i\omega} n_\nu^{-1/\sigma_\nu} + \alpha_\nu \quad (0 \leqq \omega < 2\pi)\} \quad (\nu = 1, 2, \cdots, s)$.

We introduce, for later use, the compact set

(5.8)
$$\mathscr{D}_n = \{z : |z| \leqq \tfrac{1}{2} n_0^{1/\sigma_0}\} - \bigcup_{\nu=1}^{s} \{z : |z - \alpha_\nu| < 2 n_\nu^{-1/\sigma_\nu}\},$$

which is bounded by

$$\mathscr{C}_n = \bigcup_{\nu=0}^{s} \mathscr{C}_n^{(\nu)}.$$

Consider now the points $b_j^{(\nu)}$ which appear in (3.6) and (3.7); let there be $l^{(\nu)}$ of them which satisfy the condition

(5.9)
$$|b_j^{(\nu)} - \alpha_\nu| \geqq n_\nu^{-1/\sigma_\nu}.$$

In view of Lemma 1

(5.10)
$$l^{(\nu)} < n_\nu.$$

Let

(5.11)
$$V_{n\nu}(z) = (z - \alpha_\nu)^{n_\nu - l^{(\nu)}} \prod_{j=1}^{l^{(\nu)}} (z - b_j^{(\nu)})$$

$$= (z - \alpha_\nu)^{n_\nu} \prod_{j=1}^{l^{(\nu)}} \frac{z - b_j^{(\nu)}}{z - \alpha_\nu} = (z - \alpha_\nu)^{n_\nu} W_{n\nu}(z).$$

We thus define $V_{n\nu}(z) (\nu = 1, 2, \cdots, s)$ and finally

(5.12)
$$V_{n0}(z) = \prod_{|b_j| \leqq n_0^{1/\sigma_0}} \left(1 - \frac{z}{b_j}\right),$$

where the b_j are members of (3.10). By Lemma 1, the degree $l^{(0)}$ of $V_{n0}(z)$ satisfies the condition

(5.13)
$$l^{(0)} < n_0.$$

Our polynomial $V_n(z)$ is now defined as

$$(5.14) \qquad\qquad V_n(z) = \prod_{\nu=0}^{s} V_{n\nu}(z);$$

its degree is

$$(5.15) \qquad\qquad l^{(0)} + n_1 + n_2 + \cdots + n_s < n.$$

In view of (3.1), (5.11), (5.12) and (5.14)

$$(5.16) \quad f(z)V_n(z) = h_0(z)V_{n0}(z)\left\{\prod_{\nu=1}^{s} h_\nu\left(\frac{1}{z-\alpha_\nu}\right) W_{n\nu}(z)\right\} \prod_{\nu=1}^{s}(z-\alpha_\nu)^{n_\nu}.$$

6. Estimates for $f(z)V_n(z)$ and $V_n(z)$. If $n \geqq N_0$, Lemma 1 shows that $h_0(z)V_{n0}(z)$ is regular at all points of the disk $|z| \leqq n_0^{1/\sigma_0}$. Moreover

$$(6.1) \qquad\qquad |h_0(z)V_{n0}(z)| \leqq e^{n_0} \qquad \left(|z| \leqq \frac{n_0^{1/\sigma_0}}{2}\right).$$

Similarly, the substitution

$$t = \frac{1}{z - \alpha_\nu}$$

transforms

$$(6.2) \qquad\qquad h_\nu\left(\frac{1}{z-\alpha_\nu}\right) W_{n\nu}(z)$$

into

$$h_\nu(t) \prod_{j=1}^{l^{(\nu)}} \left(1 - \frac{t}{t_j^{(\nu)}}\right).$$

Again, by Lemma 1, the product (6.2) is regular for

$$|z - \alpha_\nu| \geqq n_\nu^{-1/\sigma_\nu}$$

and satisfies the inequality

$$(6.3) \qquad \left| h_\nu\left(\frac{1}{z-\alpha_\nu}\right) W_{n\nu}(z) \right| \leqq e^{n_\nu} \qquad (|z-\alpha_\nu| \geqq 2\,n_\nu^{-1/\sigma_\nu}).$$

LEMMA 2. *If n is large enough*

(6.4)
$$f(z)V_n(z)$$

is regular at all points of $\mathscr{D}_n$.
    Moreover

(6.5) $\quad |f(\zeta)V_n(\zeta)| \leqq e^n(\tilde{K})^n \exp\left(-\dfrac{n_\nu \log n_\nu}{\sigma_\nu}\right) \quad (\zeta \in \mathscr{C}_n^{(\nu)} \ \nu \neq 0)$,

with

$$\tilde{K} = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_s| + 2$$

and

(6.6) $\quad |f(\zeta)V_n(\zeta)| \leqq (2e)^n |\zeta|^{n_1+n_2+\cdots+n_s} \quad (\zeta \in \mathscr{C}_n^{(0)})$.

Proof. The regularity of the product (6.4) in $\mathscr{D}_n$ is an immediate consequence of the decomposition (5.16) and of the regularity of each factor (6.2) and of $h_0(z)V_{n0}(z)$.
    The estimates (6.5) and (6.6) are obvious consequences of (5.16), of the estimates (6.1) and (6.3), and of the definitions of the contours $\mathscr{C}_n^{(\nu)}$.
    This proves Lemma 2. We also need the lower bound for $V_n(z)$ contained in

LEMMA 3. *Consider the point sets*

(6.7) $\quad D(\rho, \gamma) = \{z : |z| \leqq \rho\} - \bigcup_{\nu=1}^{s} \{z : |z - \alpha_\nu| < \gamma\}$

and

(6.8) $\quad D\left(2\rho, \dfrac{\gamma}{2}\right) = \{z : |z| \leqq 2\rho\} - \bigcup_{\nu=1}^{s} \left\{z : |z - \alpha_\nu| < \dfrac{\gamma}{2}\right\}$,

where

(6.9) $\quad \rho \geqq 1 + \max_{1 \leqq \nu \leqq s} |\alpha_\nu|, \quad 0 < \gamma < \gamma_0$,

and γ_0 is the constant in (3.3) and (3.4).
    Let there be $\mathscr{S} = \mathscr{S}(2\rho, \gamma/2)$ poles of $f(z)$ in $D(2\rho, \gamma/2)$; rename them

$$c_1, c_2, \cdots, c_{\mathscr{P}}$$

and let

$$H_n = \bigcup_{j=1}^{\mathscr{P}} \{z : |z - c_j| < e^{-n}\}.$$

Then, the condition

(6.10) $$z \in D(\rho, \gamma) - H_n,$$

implies

(6.11) $$|V_n(z)| \geqq (\gamma/2\rho)^n e^{-\mathscr{P}n}.$$

Proof. An inspection of (5.14), (5.11) and (5.12), shows that each one of the linear factors $(z - c_j)$ or $(1 - z/c_j)$ which may appear in $V_n(z)$ has a modulus $\geqq e^{-n}/2\rho$, provided $z \notin H_n$. For $z \in D(\rho, \gamma)$, all other factors have a modulus $\geqq \gamma/2\rho$. The Lemma is now obvious.

7. Proof of assertion I of Theorem 2. Consider

$$R_{nn}(z) = \frac{P_n(z)}{Q_n(z)},$$

where $q = q(n) \leqq n$ is the exact degree of $Q_n(z)$.

If $n \in \mathscr{N}$, we have

(7.1) $$A_n = Q_n(0) \neq 0$$

and hence, if $Q_n(z)$ has zeros z_{nj},

$$\frac{Q_n(z)}{Q_n(0)} = \left(1 - \frac{z}{z_{n1}}\right)\left(1 - \frac{z}{z_{n2}}\right) \cdots \left(1 - \frac{z}{z_{nq}}\right).$$

In view of the remark at the end of §2, the poles of $R_{nn}(z)$ coincide exactly with the zeros of $Q_n(z)$ so that by one of the assumptions of Theorem 2, we have

(7.2) $$\left|\frac{Q_n(z)}{Q_n(0)}\right| \leqq \left(1 + \frac{|z|}{\delta}\right)^n \qquad (\delta > 0, \, n \in \mathscr{N})$$

where δ is suitably small and fixed.

By (7.1), (2.16), Lemma 2, and the elements of contour integration,

$$(7.3) \qquad (-1)^n \frac{A_{n+1}}{A_n} = \frac{1}{V_n(0)} \sum_{\nu=0}^{s} I_{n\nu}$$

with

$$(7.4) \qquad I_{n\nu} = \frac{1}{2\pi i} \int_{\mathscr{C}_n^{(\nu)}} \frac{f(\zeta) V_n(\zeta)\{Q_n(\zeta)/Q_n(0)\}}{\zeta^{2n+2}} \, d\zeta.$$

By (7.2) and (6.5)

$$(7.5) \qquad |I_{n\nu}| \leqq K^n \exp\left(-\frac{n_\nu \log n_\nu}{\sigma_\nu}\right)\left(1 + \frac{\tilde{K}}{\delta}\right)^n \gamma_0^{-2n-1}$$

$$(n \geqq N_0; \ \nu = 1, 2, \cdots, s),$$

where γ_0 satisfies (3.3).

Similarly, using (6.6) instead of (6.5), we find

$$(7.6) \qquad |I_{n0}| \leqq K^n \exp\left(-\frac{n_0 \log n_0}{\sigma_0}\right)\left\{\frac{2}{n_0^{1/\sigma_0}} + \frac{1}{\delta}\right\}^n \qquad (n \geqq N_0).$$

By Lemma 3,

$$(7.7) \qquad \frac{1}{|V_n(0)|} \leqq K^n.$$

We now use (7.5), (7.6) and (7.7) in (7.3) and obtain

$$\left|\frac{A_{n+1}}{A_n}\right| \leqq K^n \sum_{\nu=0}^{s} \exp\left(-\frac{n_\nu \log n_\nu}{\sigma_\nu}\right) \qquad (n \in \mathscr{N}, \ n \geqq N_0).$$

In view of (5.4), this leads to

$$(7.8) \qquad \limsup_{\substack{n \to \infty \\ n \in \mathscr{N}}} \left|\frac{A_{n+1}}{A_n}\right|^{1/n \log n} \leqq e^{-1/\Sigma}.$$

Since

$$\Sigma = \Lambda + (s+1)\eta \qquad (\eta > 0),$$

and η is arbitrary, (18) follows from (7.8) and assertion I of Theorem 2 is proved.

8.   A consequence of the Boutroux-Cartan lemma.

LEMMA 4.   *Let*

$$(8.1) \qquad Q_n(z) = \kappa_n(z - z_{n1})(z - z_{n2}) \cdots (z - z_{nq}) \qquad (\kappa_n \neq 0)$$

be a polynomial of degree $q \leqq n$.

Then, given $\epsilon > 0$ and $\rho \geqq \epsilon$, it is possible to find a set G_n, depending only on ϵ, ρ, and on the zeros of $Q_n(z)$, formed by the union of no more than n disks with sum of radii $\leqq 2e\epsilon$ and such that

$$(8.2) \qquad |z| \leqq \rho, \qquad z \notin G_n$$

imply

$$(8.3) \qquad \left| \frac{Q_n(\zeta)}{Q_n(z)} \right| \leqq \left(\frac{|\zeta| + 2\rho}{\epsilon} \right)^n.$$

Each one of the disks of G_n contains one or more zeros of $Q_n(z)$.

Proof.   Let

$$z_{n\nu} \qquad\qquad\qquad (\nu = 1, 2, \cdots, l)$$

be all those zeros of $Q_n(z)$ which lie in $|z| \leqq 2\rho$.
    In view of (8.2)

$$|Q_n(z)| \geqq |\kappa_n| \prod_{\nu=l+1}^{q} \left| \frac{z_{n\nu}}{2} \right| \prod_{\nu=1}^{l} |z - z_{n\nu}|,$$

and by the Boutroux-Cartan lemma [**10**; p. 60]

$$(8.4) \qquad |Q_n(z)| \geqq |\kappa_n| \epsilon^l 2^{l-q} \prod_{\nu=l+1}^{q} |z_{n\nu}| \qquad (z \notin G_n).$$

A classical proof of the Boutroux-Cartan lemma [**10**] shows that each exceptional disk of G_n must contain zeros of $Q_n(z)$.
    The upper bound

$$(8.5) \qquad |Q_n(\zeta)| \leqq |\kappa_n| (|\zeta| + 2\rho)^l \prod_{\nu=l+1}^{q} \left(1 + \frac{|\zeta|}{2\rho} \right) |z_{n\nu}|$$

is obvious so that (8.3) follows immediately from (8.4) and (8.5).

9.   Analytic continuation of the error term.   We start from (2.13) with $|z| < r_0 < \rho_0$ and with the contour of integration (2.12).

By Lemma 2 and the elements of contour integration

$$(9.1) \qquad f(z) - R_{mn}(z) = \frac{z^{m+n+1}}{Q_n(z)V_n(z)} \sum_{\nu=0}^{s} J_{n\nu} \qquad (n \geqq N_0)$$

where

$$(9.2) \qquad J_{n\nu} = \frac{1}{2\pi i} \int_{\mathscr{C}_n^{(\nu)}} \frac{f(\zeta)V_n(\zeta)Q_n(\zeta)}{\zeta^{m+n+1}(\zeta - z)} \, d\zeta.$$

An inspection of (6.7) and (5.8) shows that, if n is large enough, the identity (9.1) is valid for all points of $D(\rho, \gamma)$ other than the zeros of $V_n(z)Q_n(z)$.

10.   Proof of assertion II of Theorem 2.   We take $m = n$ in (9.1) and (9.2).

By assumption

$$\Gamma_0 = \{z : |z - z_0| < \eta\}$$

contains no poles of $R_{nn}(z)$ and, since $A_n \neq 0$, it also contains no zeros of $Q_n(z)$.

We use Lemma 4 with

$$4\epsilon e < \eta - \eta'.$$

Then, the exceptional disks do not intersect

$$\Gamma_1 = \{z : |z - z_0| \leqq \eta'\}$$

and consequently

$$(10.1) \qquad \left| \frac{Q_n(\zeta)}{Q_n(z)} \right| \leqq \left(\frac{|\zeta| + 2\rho}{\epsilon} \right)^n \qquad (z \in \Gamma_1).$$

Take in Lemma 3

$$\gamma < \frac{\eta - \eta'}{2},$$

ρ large enough to imply

$$\Gamma_0 \subset \{z : |z| \leqq \rho\},$$

and n large enough to imply

$$\Gamma_1 \subset D(\rho, \gamma) - H_n.$$

Then

$$(10.2) \qquad \left|\frac{1}{V_n(z)}\right| \leqq K^n \qquad (K > 1, z \in \Gamma_1),$$

for some suitable value of K.

From (9.2), (10.1) and (10.2) we deduce

$$\left|\frac{J_{n\nu}}{Q_n(z)V_n(z)}\right| \leqq \left(\frac{K}{\epsilon}\right)^n \frac{1}{\gamma_0(\eta - \eta')} \max_\zeta \left\{\left(1 + \frac{2\rho}{|\zeta|}\right)^n \left|\frac{f(\zeta)V_n(\zeta)}{\zeta^n}\right|\right\},$$

where

$$\zeta \in \mathscr{C}_n^{(\nu)}, \quad \nu = 0, 1, 2, \cdots, s, \quad n \geqq N_0.$$

Using these estimates, (6.5) and (6.6) in (9.1), we find

$$(10.3) \quad |f(z) - R_{nn}(z)| \leqq K^n \sum_{\nu=0}^{s} \exp\left(-\frac{n_\nu \log n_\nu}{\sigma_\nu}\right) \quad (z \in \Gamma_1, n \geqq N_0).$$

In view of (5.4), assertion II of Theorem 2 is an elementary consequence of (10.3).

11. Proof of Theorem 1. We first restrict our arguments to the approximants on the side of the diagonal characterized by

$$(11.1) \qquad\qquad\qquad m(k) \geqq n(k),$$

and assume that there are infinitely many approximants satisfying the above condition.

By (10), (12) and (11.1) we obtain

$$(11.2) \qquad\qquad \beta \log k < \log n + o(\log n) \qquad (k \to \infty),$$

and therefore $n \to \infty$ as $k \to \infty$.

We now use (5.2) to select $\xi > 0$, small **enough** to imply

$$(11.3) \qquad (\Sigma + 3\xi) < \frac{2\beta}{1 + \xi} \, ,$$

and deduce from (11.2)

$$(11.4) \qquad (\Sigma + 3\xi) \log (k^{(1/2)+(\xi/2)}) < \log n + o\,(\log n) \qquad (k \to \infty).$$

Let ρ and δ be the given quantities in the statement of Theorem 1. We increase, if necessary, the value of ρ so that the first inequality in (6.9) be satisfied; this does not affect the generality of Theorem 1. With ρ thus restricted, we select γ such that

$$0 < \gamma < \min(\gamma_0, \delta^{1/2}(12\pi s)^{-1/2}),$$

and consider Lemma 3. The number $\mathscr{S}$ which appears in this lemma is now determined and enables us to choose N_0 large enough to imply

$$(11.5) \qquad \text{meas } H_n \leqq \mathscr{S}\pi e^{-2n} < \frac{\delta}{12} \qquad (n \geqq N_0).$$

Clearly

$$H_{n+1} \subset H_n,$$

$$(11.6) \qquad \text{meas } (D(\rho, \gamma) - H_n) \geqq \text{meas } (D(\rho, \gamma) - H_{N_0}) > \pi\rho^2 - \frac{\delta}{6}.$$

Lemma 4 with $\epsilon = k^{(-1/2)-(\xi/2)}$ and ρ as above, yields

$$(11.7) \qquad \left|\frac{Q_n(\zeta)}{Q_n(z)}\right| \leqq (|\zeta| + 2\rho)^n\, k^{(1+\xi)n/2} \qquad (n = n(k)),$$

for

$$(11.8) \qquad |z| \leqq \rho, \qquad z \notin G_n,$$

with

$$(11.9) \qquad \text{meas } G_n \leqq 4\pi e^2 k^{-1-\xi}.$$

From (11.7) and (11.4) we deduce

$$(11.10) \left|\frac{Q_n(\zeta)}{Q_n(z)\zeta^n}\right| \leqq \left(1 + \frac{2\rho}{|\zeta|}\right)^n \exp\left(\frac{n \log n}{\Sigma + 2\xi}\right) \quad (n = n(k), k \geqq K_0).$$

Increasing if necessary the value of K_0, we assume

$$4\pi e^2 \sum_{k=K_0}^{\infty} k^{-1-\xi} < \frac{\delta}{6},$$

so that

$$\Gamma = \bigcup_{k=K_0}^{\infty} G_n \qquad (n = n(k))$$

implies

$$(11.11) \qquad\qquad \operatorname{meas} \Gamma < \frac{\delta}{6}.$$

From this point on, all our inequalities will be subject to the conditions

$$(11.12) \qquad\qquad z \in D(\rho, \gamma), \quad z \notin H_{N_0}, \quad z \notin \Gamma.$$

By (11.6) and (11.11) this means that we are excluding from the disk $|z| < \rho$, a measurable set $\Omega_1 = \Omega_1(\rho, \delta)$ with

$$(11.13) \qquad\qquad \operatorname{meas} \Omega_1 < \frac{\delta}{3}.$$

Lemma 3 implies

$$(11.14) \qquad \frac{1}{|V_n(z)|} \leqq K^n \qquad (z \in \{D(\rho, \gamma) - H_{N_0}\}),$$

where $K > 1$ is a bound which depends on several parameters but not on n.

The relations (9.1) and (9.2) are valid for $k \geqq K_0$ and there only remains to estimate the integrals $J_{n\nu}$, using Lemma 2, (11.10), (11.14) and the inequality

$$|\zeta - z| \geqq \frac{\gamma}{2} \quad (z \in D(\rho, \gamma), \quad \zeta \in \mathscr{C}_n, \quad k \geqq K_0).$$

We find

$$(11.15) \quad \left| \frac{z^{m+n+1}}{V_n(z)Q_n(z)} J_{0n} \right| \leqq \frac{2\rho^{m+n+1} K^n}{\gamma |\zeta|^{m-n} |\zeta|^{n_0}} \left(1 + \frac{2\rho}{|\zeta|}\right)^n \exp\left(\frac{n \log n}{\Sigma + 2\xi}\right)$$

with

$$m \geqq n, \quad |\zeta| = \frac{n_0^{1/\sigma_0}}{2}, \quad n_0 \sim \frac{\sigma_0 n}{\Sigma} \quad (n = n(k), \, k \to \infty) \, ;$$

and hence, we deduce from (11.15),

$$(11.16) \quad \left| \frac{z^{m+n+1}}{V_n(z)Q_n(z)} J_{0n} \right| \leqq K^n \exp\left(n \log n \left\{ \frac{1}{\Sigma + 2\xi} - \frac{1}{\Sigma} + o(1) \right\} \right)$$

$$\leqq K^n \exp\left(\frac{-\xi n \log n}{\Sigma(\Sigma + 2\xi)} \right) \quad (n = n(k), \, k \geqq K_0).$$

Similarly, using (6.5) instead of (6.6), and taking (12) into account, we find that (11.16) holds with J_{0n} replaced by $J_{\nu n}$ ($\nu = 1, 2, \cdots, s$). Hence (9.1) yields the final estimate

$$(11.17) \quad |f(z) - R_{mn}(z)| = |\Delta_k(z)| \leqq (s+1)K^n \exp\left(\frac{-\xi n \log n}{\Sigma(\Sigma + 2\xi)} \right) = \eta_k$$

for

$$(11.18) \qquad |z| \leqq \rho, \quad z \notin \Omega_1, \quad n = n(k) \leqq m(k), \quad k \geqq K_0.$$

Clearly

$$(11.19) \qquad \eta_k \to 0 \qquad (k \to \infty),$$

and hence we have proved a "restricted Theorem 1" which requires the additional condition (11.1).

To obtain the general form of the result we consider all the remaining approximants, characterized by

$$(11.20) \qquad m = m(k) < n(k) = n,$$

and with m and n thus defined, we associate suitable approximants of $1/f$.

We observe that:

(i) $1/f$ is regular at the origin and $1/f(0) \neq 0$;

(ii) the function $1/f$ has $s+1$ essential singularities at $\alpha_0 = \infty, \alpha_1, \alpha_2, \alpha_s$, with respective orders $\lambda_0, \lambda_1, \cdots, \lambda_s$;

(iii) the integers m and n appear symmetrically in the conditions (10) and (12).

Consider the Padé approximant

$$R_{nm}^*(z) = \frac{Q_n^*(z)}{P_m^*(z)}$$

of $1/f$, where $Q_n^*(z)$ is of degree $\leqq n$, $P_m^*(z)$ is of degree $\leqq m$ and $P_m^*(z) \not\equiv 0$.

In view of (11.20), the "restricted Theorem 1" is applicable and yields

$$(11.21) \qquad \left| \frac{1}{f(z)} - \frac{Q_n^*(z)}{P_m^*(z)} \right| = |\Delta_k^*(z)| \leqq \eta_k^*,$$

provided

$$(11.22) \quad |z| \leqq \rho, \quad z \notin \Omega_1^*, \quad m = m(k) < n(k) = n, \quad k > K_0.$$

Again, we have

$$(11.23) \qquad \text{meas } \Omega_1^* < \frac{\delta}{3}, \quad \eta_k^* \to 0 \qquad (k \to \infty).$$

We now construct an open set $\mathscr{G}$ with

$$(11.24) \qquad \text{meas } \mathscr{G} < \frac{\delta}{3},$$

and such that $\mathscr{G}$ contains the s essential singularities α_ν $(\nu = 1, 2, \cdots, s)$ as well as all the poles of f which lie in $|z| \leqq \rho$.

If

$$(11.25) \qquad |z| \leqq \rho, \qquad z \notin \mathscr{G},$$

we have, for some suitable bound ω_0

$$(11.26) \qquad |f(z)| \leqq \omega_0.$$

Hence (11.21), (11.22), and (11.25) imply

$$(11.27) \qquad \left| \frac{Q_n^*(z)}{P_m^*(z)} \right| \geqq \frac{1}{2\omega_0}, \quad Q_n^*(z) \not\equiv 0.$$

We now set

$$\Omega_2 = \Omega_1^* \cup \mathscr{G}$$

so that

$$(11.28) \qquad \text{meas } \Omega_2 < \frac{2\delta}{3} .$$

By (11.21), (11.26) and (11.27), we see that

$$(11.29) \qquad |z| \leqq \rho, \quad z \notin \Omega_2, \quad m = m(k) < n(k), \quad k \geqq K_0,$$

now yield

$$(11.30) \qquad \left| f(z) - \frac{P_m^*(z)}{Q_n^*(z)} \right| \leqq 2\omega_0^2 \eta_k^*.$$

By the uniqueness of the Padé table

$$R_{mn}(z) = \frac{P_m(z)}{Q_n(z)} = \frac{P_m^*(z)}{Q_n^*(z)} .$$

An inspection of (11.17), (11.18), (11.29) and (11.30) shows that we have proved the general form of Theorem 1. The exceptional set Ω is

$$\Omega = \{\Omega_1 \cup \Omega_2\}$$

with meas $\Omega < \delta$ (by (11.13) and (11.28)).

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