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SUBORDINATION THEOREMS FOR SOME CLASSES OF STARLIKE FUNCTIONS

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Let $K_r = \{z : |z| < r\}, r > 0$. For given $\alpha, 0 < \alpha < \infty, d, 0 \leqq d < 1$, and $M, 1 < M \leqq \infty$, let $S(\alpha, d, M)$ denote the class of univalent and normalized α starlike functions f in K_1 with $K_d \subset f(K_1) \subset K_M$. The authors show the existence of a function $F \in S(\alpha, d, M)$ with the properties: (a) $\log F(z)/z$, $z \in K_1$, is univalent, (b) if $f \in S(\alpha, d, M)$, then $\log f(z)/z$, $z \in K_1$, is subordinate to $\log F(z)/z$, $z \in K_1$. Letting $\alpha \to 0$ they obtain a similar subordination result for normalized starlike univalent functions. They then point out that these subordination results solve and give uniqueness for a number of extremal problem in the above classes.

1. Introduction. Given $\alpha, 0 < \alpha < \infty$, let $S(\alpha)$ denote the class of normalized α starlike functions f in $K = \{z : |z| < 1\}$. That is, $f \in S(\alpha)$ if and only if $f(0) = 0, f'(0) = 1, z^{-1}f(z)f'(z) \neq 0\ (z \in K)$, and

$$\alpha \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} + (1-\alpha)\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geqq 0, \quad z \in K. \tag{1.1}$$

The class $S(\alpha)$ was first considered by Mocanu **[12]**. The following facts about $S(\alpha)$ are known (see Miller **[11]**),

(1.2a) Each $f \in S(\alpha)$ is starlike univalent,
(1.2b) $S(\alpha_2) \subset S(\alpha_1)$ whenever $0 < \alpha_1 \leqq \alpha_2 < \infty$,
(1.2c) If $f \in S(\alpha)$ and bounded, then f' is in the Hardy class H^1,
(1.2d) For given $f \in S(\alpha)$, there exists a starlike univalent function g satisfying $g(0) = 0$, $g'(0) = 1$, and

$$(f(z)/z)^{1/\alpha - 1} f'(z) = (g(z)/z)^{1/\alpha}, \ z \in K.$$

Here the $1/\alpha$ powers of the above functions in K are defined to be 1 at $z = 0$. We note that $S(1)$ is the class of normalized convex functions.

For given d, $0 \leqq d < 1, M, 1 < M \leqq \infty$, and $\alpha, 0 < \alpha < \infty$, let $S(\alpha, d, M)$ denote the subclass of functions $f \in S(\alpha)$ that satisfy:

$$d \leqq |f(z)/z| \leqq M, \ z \in K. \tag{1.3}$$

We observe that $S(\alpha, d, M)$ is compact, as follows easily from (1.1) and (1.3). Then in this paper we shall prove the following theorem:

THEOREM 1. *Let α, d, and M be fixed nonnegative numbers satisfying $0 < \alpha < \infty$, $0 \leqq d < 1$, and $1 < M \leqq \infty$. Then there exists a function $F = F(\cdot, \alpha, d, M) \in S(\alpha, d, M)$ with the following properties:*

(A) *The function $g(z) = \log F(z)/z$, $z \in K$, $(g(0) = 0)$ is univalent and convex in the direction of the imaginary axis,*

(B) *If $f \in S(\alpha, d, M)$, then $\log f(z)/z$, $z \in K$, is subordinate to g.*

In order to describe F we first make the following definition.

DEFINITION 1. Let α be given, $0 < \alpha < \infty$. Then γ is said to be an α curve in the w plane, if there exists a line in the ζ plane, not containing $\zeta = 0$, which is mapped onto γ by a continuous α power of ζ.

Second, we let ∂E denote the boundary of a set E, and $K_r = \{z : |z| < r\}$, $0 < r < \infty$, $r \neq 1$. Third, we let $\delta(M, \alpha)$ denote the radius of the largest disk with center at the origin contained in $f(K)$ for all $f \in S(\alpha, 0, M)$. Here α and M are fixed numbers satisfying $0 < \alpha < \infty$ and $1 < M \leqq \infty$. It is easily seen that $S(\alpha, d, M) = S(\alpha, \delta(M, \alpha), M)$ for $0 \leqq d \leqq \delta(M, \alpha)$. Hence in describing F we assume for given α and M as above that $\delta(M, \alpha) \leqq d < 1$. For such values of α, d, and M we now describe $\partial F(K)$. If $d = \delta(M, \alpha)$, then $\partial F(K)$ contains

(i) An arc with endpoints $C, \bar{C}$, of the α curve tangent to ∂K_d at $-d$.

If $\delta(M, \alpha) < d < 1$, then $\partial F(K)$ contains

(ii) An arc of ∂K_d through $-d$ with endpoints A, $\bar{A}$,

(iii) Two arcs with endpoints A, C, and $\bar{A}, \bar{C}$, of the two α curves tangent to ∂K_d at A and $\bar{A}$ respectively.

Either (a) $0 < C = \bar{C} \leqq M$ or (b) $C \neq \bar{C}$, $M < \infty$, and $|C| = M$. If (a) occurs, then $\partial F(K)$ is the arc in (i) for $d = \delta(M, \alpha)$, and the union of the arcs in (ii) and (iii) for $\delta(M, \alpha) < d < 1$. If (b) occurs, then $\partial F(K)$ contains

(iv) The arc of ∂K_M through M with endpoints $C, \bar{C}$.

$\partial F(K)$ is now the union of the arcs in (i) and (iv) for $d = \delta(M, \alpha)$, and the union of the arcs in (ii)-(iv) for $\delta(M, \alpha) < d < 1$. This completes the description of $\partial F(K)$.

The function F is uniquely determined by the above description of $\partial F(K)$ and the requirement that $F \in S(\alpha, d, M)$, as we show in §3.

We remark that Theorem 1 is well known in the simple case $\alpha = 1$, $M = \infty$, $d = \frac{1}{2}$. In this case it is a simple consequence of the fact that a normalized convex function is starlike of order $\frac{1}{2}$ (see Suffridge **[15]** for a proof of this fact). However, in all other cases Theorem 1 is new. The subordination result in (B) implies the following corollary (see for example Golusin **[4**, Ch.8, §8**]**).

COROLLARY 1. *Let α, d, M, and F be as in Theorem* 1. *Let Φ be a given nonconstant entire function. If $f \in S(\alpha, d, M)$, then*

(A) *For given $z \in K - \{0\}$*

$$\operatorname{Re}\left\{\Phi\left[\log\frac{f(z)}{z}\right]\right\} \leqq \max_{0<\theta\leqq 2\pi} \operatorname{Re}\left\{\Phi\left[\log\frac{F(e^{i\theta}z)}{e^{i\theta}z}\right]\right\},$$

(B) *For given $r, 0 < r < 1$, and $\lambda > 0$,*

$$\int_0^{2\pi} |f(re^{i\theta})|^\lambda d\theta \leqq \int_0^{2\pi} |F(re^{i\theta})|^\lambda d\theta,$$

(C) *For a given positive integer $N \geqq 2$,*

$$\sum_{k=2}^{N} |a_k|^2 \leqq \sum_{k=2}^{N} |A_k|^2,$$

where $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $F(z) = z + \sum_{k=2}^{\infty} A_k z^k$, $z \in K$.

Equality holds in any one of (A), (B), *or* (C) *only if for some real θ, $f(z) = e^{-i\theta}F(e^{i\theta}z)$, $z \in K$.*

We note that with the appropriate choice of Φ in Corollary 1, some of the classical extremal problems follow for $S(\alpha, d, M)$. For example, the quantities $|f(z)/z|$, $|\operatorname{Arg} f(z)/z|$, $\operatorname{Re}\{[f(z)/z]^p\}$, where $|z| = r, 0 < r < 1$, $f \in S(\alpha, d, M)$, and $p > 0$, are all maximized or minimized on ∂K_r by F. We remark that Krzyz **[10]** proved (A) of Corollary 1 for $S(1, 0, M)$, and Barnard **[1]** proved (A) of Corollary 1 for $S(1, d, M)$. However, they did not show the F in their respective classes was the unique function with property (A). Also Miller **[11]** proved (A) of Corollary 1 for $S(\alpha, 0, \infty)$, $0 < \alpha < \infty$, and $\Phi(w) = \pm w$.

Next for fixed d, $0 \leqq d < 1$, and $M, 1 < M \leqq \infty$, let $S^*(d, M)$ denote the class of normalized starlike univalent functions f in K which satisfy (1.3). We observe for given r, $0 < r < 1$, and $f \in S^*(d, M)$ that $f(rz)/r, z \in K$, is in $S(\alpha, d, M)$ for $\alpha > 0$ small enough. Moreover if f, $g \in S^*(d, M)$ and $0 < r < 1$, then the function $z[f(z)/z]^r[g(z)/z]^{1-r}$, $z \in K$, is in $S^*(d, M)$. Using these observations and Theorem 1, we easily obtain in §9, the following theorem.

(A) *The function $g(z) = \log F^*(z)/z$, $z \in K$, $(g(0) = 0)$ is convex univalent,*

(B) *If $f \in S^*(d, M)$, then $\log f(z)/z$, $z \in K$, is subordinate to g.*

Theorem 2 implies, as in the discussion after Theorem 1, the following corollary.

COROLLARY 2. *Let d and M be as in Corollary* 1. *Replace F by F^* and $S(\alpha, d, M)$ by $S^*(d, M)$ in Corollary* 1. *Then Corollary* 1 *is valid for F^*.*

We remark that Theorem 2 and Corollary 2 are well known in the simple case $d = \frac{1}{4}$, $M = \infty$ (see Goluzin **[4**, Thm. 1, p. 531**]**). Moreover, Suffridge **[16]** proved (C) of Corollary 2 for $S^*(d, \infty)$ and $N = 2$. Barnard **[2]** proved (A) of Corollary 2 for $S^*(d, M)$ and $\Phi(w) = \pm w$. With these exceptions, Theorem 2 and Corollary 2 are new results for starlike functions.

For given M, α, d, as in Theorem 1, let f be in $S(\alpha, d, M)$ and put $D = f(K)$. Then the proof of Theorem 1 is based upon a geometric description of ∂D and a use of the Julia variational formula similar to Krzyz **[10]** and Barnard **[1]**. This geometric description of ∂D is obtained in Lemmas 1–3 of §2. In §3 we use Lemmas 1–3 to determine $\delta(M, \alpha)$ and show $F \in S(\alpha, d, M)$ is uniquely defined by (i)-(iv).

In §4 we define our variations of D when ∂D contains an arc of an α curve. In §5 we show that the Hadamard variational formula holds for the Greens functions of our varied domains. In §6 we deduce the Julia variational formula from the Hadamard variational formula, and show how it can be used to solve an extremal problem. In §7 we prove Lemmas 4–7. We use these lemmas in §8 to prove Theorem 1. In §9 we deduce Theorem 2 from Theorem 1 and describe $\partial F^*(K, d, M)$.

As motivation for the proof of Theorem 1, we first remark that it turns out (A) of Corollary 1 implies Theorem 1. Second, we remark that our geometric description implies ∂D is made up of a finite number of arcs with the following property: each arc is the image, under ζ^α, of an arc contained in the boundary of a convex domain. Since the Julia variation is a local boundary variation, it follows that the solution to (A) of Corollary 1 in $S(\alpha, d, M)$ should be obtainable from a local use of conformal mapping and arguments similar to those of Krzyz **[10]** and Barnard **[1]**. Furthermore, a general description of $\partial F(K, \alpha, d, M)$ should follow from considering local α powers of ζ on $\partial F(K, 1, d, M)$. This is indeed the case, as we see from (i)-(iv). We emphasize, though, that the extremal functions in Theorem 1, corresponding to different values of α, do not bear such a simple relationship. Even though the bounds on $F(K, \alpha, d, M)$ make it quite difficult to obtain an explicit representation formula for F, this function is completely described by its geometric properties. Since the Julia variational method allows to preserve both bounds and the class, it seems the most natural way to prove Theorem 1.

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2. A geometric description of the image domains of α starlike functions. Given $w \neq 0$, let $\operatorname{Arg} w$, $-\pi \leqq \operatorname{Arg} w \leqq \pi$, denote the principal argument of w. Let γ be an α curve as in Definition 1. Since γ is the image of a line, not containing $\zeta = 0$, under a continuous α power of ζ, it follows for $0 < \alpha \leqq 2$, that γ divides the w plane into two disjoint domains. Moreover the domain containing $w = 0$ is starlike. However for $\alpha > 2$, γ intersects itself, and consequently there exist rays through $w = 0$ which intersect γ more than once. Since we shall be studying starlike domains in which part of the boundary is an arc of γ, it is necessary to make the following definition for fixed α, $0 < \alpha < \infty$.

DEFINITION 2. Let β denote a closed arc of an α curve γ. Then we shall call β an α arc of γ, if each ray through $w = 0$ intersects β in at most one point.

We shall determine the number of α arcs with endpoints $A, B (A \neq B)$ in the w plane. Clearly the number is zero if either $\operatorname{Arg}(\bar{A}B) = 0$, or one of A and B is zero. Hence we assume $A \neq 0$, $B \neq 0$, and $\operatorname{Arg}(\bar{A}B) \neq 0$. Next we draw the rays from $w = 0$ through A and B. These rays divide the w plane into two sectors, T_1 and T_2, with angular openings θ_1 and θ_2 respectively. We may suppose that $0 < \theta_1 \leqq \theta_2 < 2\pi$, since otherwise we renumber. We observe that if β is an α arc with endpoints A and B, then either $\beta \subset T_1 \cup \{A, B\}$, or $\beta \subset T_2 \cup \{A, B\}$, as follows from Definition 2. We claim for fixed α, $0 < \alpha < \infty$, that

(2.1a) Let i be fixed, $i = 1$ or 2. Then if $0 < \theta_i < \pi\alpha$, there exists exactly one α arc β with endpoints A and B for which $\beta \subset T_i \cup \{A, B\}$. If $\pi\alpha \leqq \theta_i$, there does not exist an α arc β with endpoints A and B for which $\beta \subset T_i \cup \{A, B\}$.

To prove (2.1a), let h_i denote an analytic $1/\alpha$ power of w in $T_i (i = 1$ or $2)$ which is continuous on ∂T_i. Then the line segment with endpoints $h_i(A)$ and $h_i(B)$ is contained in $h_i(T_i) \cup \{h_i(A), h_i(B)\}$, if and only if $0 < \theta_i < \pi\alpha$. Using this fact and considering the inverse mapping to h_i, we get (2.1a).

From (2.1a) we see for $0 < \alpha \leqq 1$ that if $0 < |\operatorname{Arg}(\bar{A}B)| < \pi\alpha$, then there exists exactly one α arc with endpoints A, B. For $1 < \alpha < \infty$, it follows from (2.1a) that there is at least one α arc with endpoints $A, B (\operatorname{Arg}(\bar{A}B) \neq 0)$. Also for $2 < \alpha < \infty$, there are exactly two α arcs with endpoints $A, B (\operatorname{Arg} \bar{A}B \neq 0)$.

Next we determine a geometric criterion for a bounded domain to be a magnification of the image domain of an α starlike function. This criterion is given by Lemma 1. In Lemma 1, β denotes the α arc with endpoints A, B, satisfying $\beta \subset T_1 \cup \{A, B\}$.

LEMMA 1. *Let D be a bounded domain containing $w = 0$ with the property that each ray through $w = 0$ intersects ∂D in exactly one point. Let α be a fixed positive number and suppose there exists a sufficiently small $\eta > 0$ such that whenever $A, B \in \partial D$ and $0 < |\operatorname{Arg}(\bar{A}B)| < \eta < \pi\alpha$, then either $\beta \subset D \cup \{A, B\}$ or $\beta \subset \partial D$. Then there is a function $g \in S(\alpha)$ and a number $t > 0$ such that $tg(K) = D$.*

Proof. Let A, B, be any two points of ∂D with $0 < |Arg\,(\bar{A}B)| < \eta \leqq \pi\alpha$. Define T_1 and $\beta \subset T_1 \cup \{A, B\}$ relative to A and B as in (2.1a). Let h_1 be an analytic $1/\alpha$ power of w on T_1 which is continuous on ∂T_1. Put $D_1 = D \cap T_1$, $\lambda = \partial D \cap T_1$, and suppose that E, F, $E \neq F$, are any two points of λ. Then from the hypotheses of Lemma 1 (with E, F, replacing A, B), either the line segment connecting $h_1(E)$ to $h_1(F)$ is contained in $h_1(\lambda)$ or it is contained in $h_1(D_1) \cup \{h_1(A), h_1(B)\}$. Since $\partial h_1(D_1)$ consists of $h_1(\lambda)$ and segments of two rays from $w = 0$ forming an angle less than π, it follows that $h_1(D_1)$ is convex. Hence $h_1(\lambda)$ may be approximated by a polygonal arc τ, made up of chords connecting points on $h_1(\lambda)$, with endpoints $h_1(A)$, $h_1(B)$. If n a positive integer is given, then τ can be chosen such that each point of τ lies within $1/n$ distance of a point of $h(\lambda)$. Also, τ can be chosen in such a way that a piecewise continuous argument of the tangent to τ does not decrease as τ is described in the counterclockwise direction with respect to $w = 0$.

Taking the preimage of τ under h_1, we find that λ may be approximated by an arc $\sigma_1 \subset D_1 \cup \lambda \cup \{A, B\}$, made up of α arcs, with endpoints A, B. Moreover each point of σ_1 is within C/n of a point of λ, where C is a positive constant which depends only on α and D. Also the tangent to σ_1 rotates counterclockwise as we pass from one α arc to another in the counterclockwise direction. Since ∂D may be written as a finite union of sets of the form λ, we see that ∂D may be approximated by a Jordan curve σ with the same properties as σ_1. The bounded domain $D(n)$, with $\partial D(n) = \sigma$, is clearly starlike with respect to $w = 0$. Let g_n denote the Riemann mapping function satisfying $g_n(0) = 0$, $g_n'(0) > 0$, and $g_n(K) = D(n)$. Then g_n is continuous in $K \cup \partial K$ and a continuous $1/\alpha$ power of g_n maps $\partial K - \{1\}$ onto a polygonal arc. Moreover, as $\partial K - \{1\}$ is described in the counterclockwise direction, a piecewise continuous argument of the tangent to this polygonal arc does not decrease. Using this fact and a Schwarz-Christophel type argument we deduce that

$$\alpha \operatorname{Re}\{1 + zg_n''(z)/g_n'(z)\} + (1-\alpha)\operatorname{Re}\{zg_n'(z)/g_n(z)\}$$
$$= \sum_{k=1}^{m} b_k \operatorname{Re}\left(\frac{1 + e^{-i\theta_k}z}{1 - e^{-i\theta_k}z}\right), \quad z \in K,$$

where b_k and θ_k are positive, and m is a positive integer. Hence

$$g_n/g_n'(0) \in S(\alpha). \tag{2.2}$$

The sequence $(g_n)_1^\infty$ is a uniformly bounded sequence of univalent functions in K. Moreover from the construction of $D(n)$, we see that $g_n(K) = D(n) \to D$ in the sense of kernel convergence. Using these facts and applying a theorem of Carathéodory (see Goluzin [4, Thm. 1, p. 55]), we deduce that $\lim_{n\to\infty} g_n = \hat{g}$, $\hat{g}'(0) > 0$, and $\hat{g}(K) = D$. Using the compactness of $S(\alpha)$ and (2.2), we further deduce that $\hat{g}/\hat{g}'(0) = g \in S(\alpha)$. Hence Lemma 1 is true.

To continue our geometric description of the image domains of α starlike functions we prove

LEMMA 2. *Let $f \in S(\alpha, 0, M)$ for some $M < \infty$ and put $D = f(K)$. Then each ray through $w = 0$ intersects ∂D in exactly one point. If A, B, $A \neq B$, are in ∂D and if β is an α arc with endpoints A and B, then*

(a) *either $\beta \subset \partial D$ or $\beta \subset D \cup \{A, B\}$,*

(b) *if Ω denotes the component of $D - \beta$ containing $w = 0$, then there exists a $g \in S(\alpha)$ and $t > 0$ such that $tg(K) = \Omega$.*

Proof. Let $g_r(z) = f(rz)$ for $z \in K$ and $0 < r < 1$. Put $D_r = g_r(K)$, and $\Gamma_r(\theta) = g_r(e^{i\theta})$, $0 \leqq \theta < 2\pi$. Then from (1.1) we see that

$$\alpha \operatorname{Re}\{1 + zg_r''(z)/g_r'(z)\} + (1-\alpha)\operatorname{Re}\{zg_r'(z)/g_r(z)\} \geqq 0, \quad z \in K \cup \partial K.$$

Let $\log \Gamma_r$ and $\log \Gamma_r'$ be continuous logarithms of Γ_r and $\Gamma_r'(\Gamma_r'(\theta) = d/d\theta\, \Gamma_r(\theta))$. Then the above inequality implies that

$$\alpha \operatorname{Im} \frac{d}{d\theta} \log [\Gamma_r^{1/\alpha - 1}(\theta)\, \Gamma_r'(\theta)] = \alpha \operatorname{Im} \frac{d}{d\theta} \log \Gamma_r'(\theta)$$
$$+ (1-\alpha) \operatorname{Im} \frac{d}{d\theta} \log \Gamma_r(\theta) \geqq 0.$$

Geometrically this inequality means

(2.3a) The argument of the tangent to $\Gamma_r^{1/\alpha}$ does not decrease as θ increases for a continuous $1/\alpha$ power of Γ_r.

Using (2.3a) we now prove Lemma 2. Let A, B, and β be as in Lemma 2. Choose a sector V containing β in its interior and of angle opening ϕ, $0 < \phi < \pi\alpha$. This choice is possible by (2.1a). Let p be an analytic $1/\alpha$ power of w on V. Then (2.3a) implies that $p(V \cap D_r)$ is convex, as is easily seen. Since $D = \cup_{0<r<1} D_r$ and $D_s \subset D_r$, $s < r$, it follows that

(2.3b) $p(V \cap D)$ is convex.

Hence the line segment l with end points $p(A)$, $p(B)$, is either contained in $p(V \cap D) \cup \{p(A), p(B)\}$ or in $p(\partial D \cap V)$. Using this fact and the inverse mapping to p, we deduce that (a) of Lemma 2 is true. Also since each ray through the origin intersects $p(V \cap \partial D)$ in exactly one point, we see that $V \cap \partial D$ likewise has this property. Hence each ray through $w = 0$ intersects ∂D in exactly one point. To prove (b) of Lemma 2 we observe that $p(V \cap \Omega)$ is equal to the component of $p(V \cap D) - l$ containing zero in its boundary. Hence $p(V \cap \Omega)$ is convex. Using the inverse of p, it follows that the boundary points of Ω in a sufficiently small neighborhood of A satisfy the hypotheses of Lemma 1. A similar statement holds for the boundary points in a small neighborhood of B. Since $\eta > 0$ may be arbitrarily small in Lemma 1, and since $\partial\Omega$ consists of a part of ∂D and β, we find from the above discussion and (a) of Lemma 2 that $\partial\Omega$ satisfies the conditions of Lemma 1. Applying this lemma we deduce that (b) is valid. This proves Lemma 2.

Again suppose that $f \in S(\alpha, 0, M)$ for some $M < \infty$. Then $f' \in H^1$ (see (1.2c)) and hence $\Gamma(\theta) = f(e^{i\theta})$, $0 \leqq \theta < 2\pi$ is a bounded rectifiable curve in the w plane. (see for example Goluzin [**4**, Thm. 1, p. 409]). Let $w \in \Gamma$ and suppose that Γ has unique left and right hand tangents at w. If γ is an α curve through w, then we shall say γ is tangent to Γ from the right (left) at w, provided the tangent to γ coincides with the right hand (left hand) tangent of Γ at w. With this understanding we prove

LEMMA 3. *Let f and D be as in Lemma 2 and put $\Gamma = \partial D$. Then Γ has a unique right (left) hand tangent at each $w \in \Gamma$. Consequently, there exists exactly one α curve γ which is tangent to Γ at w from the right (left). If $\beta \subset \gamma$ is an α arc with one endpoint w, then $\beta \cap D = \{\phi\}$.*

Proof. Lemma 3 follows easily from (2.3b) and geometric properties of convex domains. We omit the details.

3. Applications of Lemmas 1–3. We now determine $\delta(M, \alpha)$ (see §1) for fixed M and α satisfying $1 < M < \infty$ and $0 < \alpha < \infty$. To do this we let f, D, and Γ be as in Lemma 3 and put $d(f) = \min\{|w|: w \in \Gamma\}$. We shall use the following remark which also will be used in §4 and §8.

REMARK 1. If $w_0 \in \Gamma$ is such that $|w_0| = d(f)$, then there is exactly one α curve γ tangent to Γ at w_0 from either the left or right. Furthermore, γ is tangent to $\partial K_{d(f)}$ at w_0.

Remark 1 follows easily from Lemma 3 by way of contradiction. Let γ_1, γ_2, be the α curves tangent to Γ at w_0 from the right and left, respectively. If γ_1 were not tangent to $\partial K_{d(f)}$ at w_0, then γ_1 would contain points of $K_{d(f)}$ arbitrarily near w_0. Hence there would exist an α arc $\beta \subset \gamma_1$ with endpoint w_0 and $\beta \cap D \neq \{\phi\}$. This inequality contradicts Lemma 3. Therefore γ_1 is tangent to $\partial K_{d(f)}$ at w_0. Repeating the argument we see that γ_2 is tangent to $\partial K_{d(f)}$ at w_0. Since the definition of an α curve implies there is exactly one α curve tangent to $\partial K_{d(f)}$ at w_0, we must have $\gamma_1 = \gamma_2$.

To continue the determination of $\delta(M, \alpha)$, we need some notation. First, given a simply connected domain G containing $w = 0$, we shall let m.r. G denote the mapping radius of G (see Hayman [5, p. 78] for a definition). Also, we shall say G is α *starlike*, if there exists $h \in S(\alpha)$ and $t > 0$ such that $th(K) = G$. Second, for given M and α as above, and given $s, 0 < s < M$, we draw the α curve γ tangent to ∂K_s at $-s$. From the definition of γ we see that either γ intersects itself at a point $t = t(s)$, $0 < t \leqq M$, or γ does not intersect itself in $K_M \cup \partial K_m$, and γ intersects ∂K_M at $Me^{i\phi}$, $Me^{-i\phi}$, for some $\phi = \phi(s)$, $0 < \phi < \pi$. In the first case we let $\Omega(s)$ denote the bounded domain containing $w = 0$ whose boundary is the two α arcs of γ with endpoints $-s, t$. In the second case we let $\Omega(s)$ denote the bounded domain containing $w = 0$ whose boundary consists of the α arc of γ with endpoints $Me^{i\phi}$, $Me^{-i\phi}$, and the arc of ∂K_M with endpoints $Me^{i\phi}$, $Me^{-i\phi}$, which contains M. We claim that $\Omega(s)$ is α starlike. Indeed, it is obvious that $\partial\Omega(s)$ satisfies the hypotheses of Lemma 1 except possibly in a small disk about t in the first case or in small neighborhoods of $Me^{i\phi}$, $Me^{-i\phi}$, in the second case considered above. Using (2.3b) with A and B properly defined, it is easily checked that $\partial\Omega(s)$ also satisfies the hypotheses of Lemma 1 at these boundary points. Hence $\Omega(s)$ is α starlike for $0 < s < M$. Next we observe that $\Omega(s_1) \underset{\neq}{\subset} \Omega(s_2) \underset{\neq}{\subset} K_M$ for $0 < s_1 < s_2 < M$, as can be seen by examining $\partial\Omega(s_i)$ ($i = 1$ or 2). Using elementary properties of subordination, it follows that

$$0 = \lim_{s \to 0} \text{m.r. } \Omega(s_1) < \text{m.r. } \Omega(s_2) < \lim_{s \to M} \text{m.r. } \Omega(s) = M,$$

for $0 < s_1 < s_2 < M$. From the above inequality we see there exists a unique s_0, $0 < s_0 < M$, for which

$$\text{m.r.}\,\Omega(s_0) = 1. \tag{3.1}$$

Finally we determine $\delta(M,\alpha)$. Let f, Γ, D, and $d(f)$ be as previously defined in §3. We may assume $-d(f) \in \Gamma$, since otherwise we rotate D. Then from Remark 1 and Lemma 3, we deduce that $\Omega[d(f)] \supset D$ and thereupon, m.r., $\Omega[d(f)] \geqq \text{m.r.}\,D = 1$. Hence, $\Omega(s_0) \subset \Omega[d(f)]$, and so, $s_0 \leqq d(f)$. Since $\Omega(s_0)$ is the image domain of a function $F \in S(\alpha, 0, M)$, we conclude that $\delta(M,\alpha) = s_0$.

Next in §3 we show for fixed α, M, and d satisfying $0 < \alpha < \infty$, $1 < M < \infty$, and $\delta(M,\alpha) \leqq d < 1$, that $F(\cdot, \alpha, d, M) \in S(\alpha, d, M)$ is uniquely defined by (i)-(iv) of §1. To do this for given θ, $0 < \theta < \pi$, draw the α curves γ, $\bar{\gamma}$, tangent to ∂K_d at $de^{i\theta}$, $de^{-i\theta}$, respectively. Then either γ intersects $\bar{\gamma}$ at a point $u = u(\theta)$, $0 < u \leqq M$, or γ and $\bar{\gamma}$ intersect ∂K_M at points $P = P(\theta)$, $\bar{P} = \bar{P}(\theta)$, respectively with $P \neq \bar{P}$. In the first case we let $\Lambda(d,\theta)$ denote the bounded domain containing $w = 0$ whose boundary consists of

(+) the arc of ∂K_d with endpoints $de^{i\theta}$, $de^{-i\theta}$, containing $-d$, and α arcs of γ and $\bar{\gamma}$ with end points $de^{i\theta}$, u, and $de^{-i\theta}$, u, respectively.

In the second case we let $\Lambda(d,\theta)$ denote the bounded domain containing $w = 0$ whose boundary consists of the arc of ∂K_d in (+), the α arcs of γ and $\bar{\gamma}$ with end points $de^{i\theta}$, P, and $de^{-i\theta}$, $\bar{P}$, respectively, and the arc of ∂K_M with end points P, $\bar{P}$, containing M. We also put $\Lambda(d,\pi) = \Omega(d)$, where $\Omega(d)$ is as defined previously in §3.

Again using (2.3b) and Lemma 1, we see that $\Lambda(d,\theta)$ is α starlike for $0 < \theta \leqq \pi$. Furthermore $\Lambda(d,\theta_1) \underset{\neq}{\subset} \Lambda(d,\theta_2)$ for $0 < \theta_1 < \theta_2 \leqq \pi$. Hence,

$$d = \lim_{\theta \to 0} \text{m.r.}\,\Lambda(d,\theta) < \text{m.r.}\,\Lambda(d,\theta_1) < \text{m.r.}\,\Lambda(d,\theta_2) < \text{m.r.}\,\Lambda(d,\pi), \tag{3.2}$$

for $0 < \theta_1 < \theta_2 < \pi$.

Now suppose that $F \in S(\alpha, d, M)$ is a function for which $\partial F(K)$ satisfies (i)-(iv) of §1. If $d = \delta(M,\alpha)$, then from (3.1) we see that $F(K) = \Lambda(d,\pi) = \Omega(s_0)$. If $\delta(M,\alpha) < d < 1$, then from (3.1), (3.2), and the fact that $\Lambda[\delta(M,\alpha),\pi] \underset{\neq}{\subset} \Lambda(d,\pi)$, we see there exists exactly one $\theta_0 = \theta_0(d)$ satisfying $0 < \theta_0 < \pi$ and for which $F(K) = \Lambda(d,\theta_0)$. We conclude for a fixed α, $0 < \alpha < \infty$, M, $1 < M < \infty$, and d, $\delta(M,\alpha) \leqq d < 1$, that $F \in S(\alpha, d, M)$ is uniquely defined by (i)-(iv) of §1. The situation $M = \infty$, $0 < \alpha < \infty$, $\delta(\infty,\alpha) \leqq d < 1$, can be handled by treating it as a limiting case, as $M \to \infty$, of the previous cases considered. We omit the details.

4. Boundary variations. Again we assume that M, α, d, are fixed numbers satisfying $1 < M < \infty$, $0 < \alpha < \infty$, and $\delta(M, \infty) \leqq d < 1$. If $f \in S(\alpha, d, M)$, we also put $D = f(K)$, $\Gamma = \partial D$. Let $A, B (A \neq B)$ and E, $F (E \neq F)$ be in Γ. We suppose that Γ contains an α arc β with end points A, B, and an α arc μ with endpoints E, F. We further suppose that μ and β are disjoint, except possibly $B = F$ or $A = E$. Let V and N be sectors drawn from $w = 0$ which contain β and μ in their interiors, respectively. Thanks to (2.1a), we may choose V and N to each have angle opening less than $\pi\alpha$. We let p and ϕ be analytic $1/\alpha$ powers of w on V and N respectively. Then we shall define the following variations on Γ (see Barnard **[1]** for similar variations in the convex case).

I. An inward variation whenever μ is not tangent to ∂K_d, and the right and left hand tangents to Γ at F do not coincide.

II. An outward variation whenever the right and left hand tangents to Γ at A do not coincide, and B satisfies either (a) or (b):

(a) $|B| = M$,

(b) $|B| < M$, and the left and right hand tangents to Γ at B do not coincide.

III. An outward sliding of β when $\Gamma \cap \partial K_d$ contains a set of distinct points, $\{Q_n\}_1^\infty$, with $\lim_{n \to \infty} Q_n = A$, and B satisfies either (a) or (b) of II.

Variation I will be defined in terms of a parameter δ for $0 < \delta \leqq \delta_0$ (δ_0 small) in such a way that if $\Gamma_1(\delta)$ denotes the variation of Γ, then $\Gamma_1(\delta)$ is the boundary of an α starlike domain $D_1(\delta)$, and

$$\Gamma_1(\delta) \subset \{z : d \leqq |z| \leqq M\} = L(d, M). \tag{4.1}$$

Furthermore,

$$D_1(\delta_2) \underset{\neq}{\subset} D_1(\delta_1), \quad \text{whenever} \quad 0 < \delta_1 < \delta_2 \leqq \delta_0, \tag{4.2}$$

$$\bigcup_{0 \leqq \delta \leqq \delta_0} D_1(\delta) = D. \tag{4.3}$$

To define I let F_0 be a point on $(\Gamma - \mu) \cap N$ which is near F. Draw the α arc μ_0 whose endpoints are E and F_0 contained in N. It is possible to draw such an arc for F_0 near F by (2.1a). Since F is as in I it follows from (a) of Lemma 2 that $\mu_0 \subset D \cup \{E, F_0\}$. Hence the smallest angle between the tangents to μ and μ_0 at E is positive. Let $\delta_0 > 0$ denote this angle.

Now suppose that $F_1, F_2\,(F_1 \neq F_2)$ are points on the arc of $\Gamma \cap N$ with endpoints F_0, F. Also we suppose that $F_1 \neq F$, $F_2 \neq F$. Draw the α arcs μ_1 and μ_2 with endpoints E, F_1, and E, F_2, respectively. Let δ_i, $i = 1,2$, denote the smallest angle between μ_i and μ at E. As above we observe that $\mu_i \subset D \cup \{E, F_i\}$ and thereupon that $\delta_i > 0$. Also since $\mu_1 \subset D \cup \{E, F_1\}$, we must have $\mu_1 \cap (\mu_2 - \{E\}) = \{\phi\}$. Hence $\delta_1 \neq \delta_2$. Let $D_1(\delta_i)$, $i = 1, 2$, denote the component of $D - \mu_i$ containing $w = 0$. From Lemma 2 we see that $D_1(\delta_i)$ is an α starlike domain and $D_1(\delta_i) \underset{\neq}{\subset} D$. Furthermore if $0 < \delta_1 < \delta_2 \leqq \delta_0$, then $D_1(\delta_2) \underset{\neq}{\subset} D_1(\delta_1)$. To see this observe that μ_2 is an α arc connecting two boundary points of $D_1(\delta_1)$. Furthermore since $\delta_1 \neq \delta_2$, $\mu_2 \subset D_1(\delta_1) \cup \{E, F_2\}$. Since $D_1(\delta_2)$ is the bounded component of $D_1(\delta_1) - \mu_2$ containing $w = 0$, it follows that (4.2) is true.

We put $\delta = \delta_1$ and let F_1 vary subject to the above restrictions. For each δ, $0 < \delta \leqq \delta_0$, we obtain an α starlike domain $D_1(\delta) \underset{\neq}{\subset} D$ with boundary $\Gamma_1(\delta)$. Moreover, from the definition of $D_1(\delta)$ it is clear that (4.3) holds. To prove (4.1) it suffices to show that $K_d \subset D_1(\delta_0)$ since $D_1(\delta_0) \subset D(\delta) \subset D$ for $0 < \delta \leqq \delta_0$. To do this recall that by assumption μ is not tangent to ∂K_d. Then by Remark 1, μ has a positive distance from ∂K_d. Hence for $\delta_0 > 0$ small enough μ_0 also has a positive distance from ∂K_d and so, $K_d \subset D_1(\delta_0)$.

Variation II will be defined in terms of a parameter ϵ for $0 < \epsilon \leqq \epsilon_0$, while variation III will be defined for $\epsilon > 0$ in a sequence, $z = (\epsilon_j)$, with $\lim_{j\to\infty} \epsilon_j = 0$. The variations will be defined in such a way that if $\Gamma_2(\epsilon)$ denotes the variation of Γ, then $\Gamma_2(\epsilon)$ is the boundary of an α starlike domain $D_2(\epsilon)$, and

$$\Gamma_2(\epsilon) \subset L(d, M), \tag{4.4}$$

$$D_2(\epsilon_1) \underset{\neq}{\subset} D_2(\epsilon_2), \quad \text{whenever} \quad 0 < \epsilon_1 < \epsilon_2, \tag{4.5}$$

$$\bigcap D_2(\epsilon) = D. \tag{4.6}$$

We remark for later use that if the right and left hand tangents at A coincide, then our method of variation in II will still produce a starlike domain $D_2(\epsilon)$ satisfying (4.4)–(4.6).

To define II(a), choose a point $B_0 \in (\partial K_M - \partial D) \cap V$ near B with the property that the ray from the origin to B_0 intersects β. Draw the α arc β_0 whose endpoints are A and B_0 which is contained in V. Again it is possible to draw such an arc for B_0 near B by (2.1a). Let $\epsilon_0 > 0$ denote the smallest angle between β and β_0 at A.

Now suppose that B_1, $B_2(B_1 \neq B_2)$ are points on the arc of $\partial K_M \cap V$ with endpoints B, B_0. Also we suppose that $B_1 \neq B$, $B_2 \neq B$. Draw the α arcs β_1 and β_2 with endpoints A, B_1, and A, B_2, respectively. Let $\epsilon_i (i = 1,2)$ denote the smallest angle between β_i and β at A. Clearly $\epsilon_1, \epsilon_2 > 0$ and $\epsilon_1 \neq \epsilon_2$. Let $D_2(\epsilon_i)$, $i = 1,2$, denote the domain whose boundary is the union of the arcs: β_i, $\Gamma - \beta$, and the arc of ∂K_M with endpoints B, B_i, contained in V.

We claim that $D_2(\epsilon_i)$, $i = 1,2$, is α starlike when B_0 is near B. To see this note from (2.3b) that $p(V \cap D)$ is convex. Also $\partial p(V \cap D)$ contains the line segment l with endpoints $p(A)$, $p(B)$. Since A is as in II(a), we see that the left and right hand tangents to $\partial p(V \cap D)$ at $p(A)$ do not coincide. Using these observations and well known geometric properties of convex domains we deduce for given $i = 1$ or 2 that the bounded domain with boundary,

(i) the line segment with endpoints $p(A)$, $p(B_i)$,
(ii) the arc of $p(K_M \cap V)$ with endpoints $p(B)$, $p(B_i)$,
(iii) $\partial p(V \cap D) - l$,

is convex. Also the boundary of this domain is contained in $p[K_M \cap V]$. Since this domain is also equal to $p[D_2(\epsilon_i) \cap V]$, it follows, upon taking the inverse of p, that $D_2(\epsilon_i)$ satisfies the hypotheses of Lemma 1 and that $\partial D_2(\epsilon_i) \subset L(d, M)$. Hence $D_2(\epsilon_i)$ is α starlike and $\Gamma_2(\epsilon_i) = \partial D_2(\epsilon_i)$ satisfies (4.4).

Next we prove (4.5). If $0 < \epsilon_1 < \epsilon_2 \leqq \epsilon_0$, then from Lemma 1 we see that $\beta_1 \subset D_2(\epsilon_2) \cup \{A, B_1\}$. It follows that $D_2(\epsilon_1)$ is the bounded component of $D_2(\epsilon_2) - \beta_1$ containing $w = 0$. Hence (4.5) is valid. Put $\epsilon = \epsilon_1$ and let B_1 vary subject to the previous restrictions. For each ϵ, $0 < \epsilon \leqq \epsilon_0$, we obtain an α starlike domain $D_2(\epsilon)$ which satisfies (4.4)–4.5). From the definition of $D_2(\epsilon)$ and (4.5) we also see that (4.6) holds.

To define II (b), let γ be the α curve tangent to Γ at B which does not contain β. Let B_0, $|B_0| < M$, be a point on γ near B with the property that the ray from the origin through B_0 intersects β. Let β_0 denote the α arc with endpoints A and B_0 which is contained in V. Let $\epsilon_0 > 0$ be the smallest angle between β and β_0 at A. Now let $B_1 \neq B$ be a point on the arc of $\gamma \cap V$ with endpoints B_0, B. Draw the α arc β_1 with endpoints A, B_1. Let $\epsilon > 0$ denote the smallest angle between β and β_1 at A. Let $D_2(\epsilon)$ denote the domain whose boundary is β_1, $\Gamma - \beta$, and the α arc of γ with endpoints B, B_1, which is contained in V. Then $D_2(\epsilon)$ is an α starlike domain with boundary $\Gamma_2(\epsilon)$ for $0 < \epsilon \leqq \epsilon_0$. Furthermore, (4.4)–(4.6) are true. The proof of these facts is similar to the proof used in II (a). We omit the details.

To define III when B satisfies II (a), we first note from Remark 1 that β is tangent to ∂K_d at A. Let $A_0 \in \partial K_d \cap \Gamma \cap V$ be near

$A(A_0 \neq A)$. Let γ be the α curve containing β, and let γ_0 be the α curve tangent to $\partial K_d \cap \Gamma$ at A_0. Let P_0 be the point of intersection in V of γ and γ_0 which is nearest A. Let $\epsilon_0 > 0$ denote the smallest angle between the tangents to γ and γ_0 at P_0.

Now suppose that $A_1 \neq A$ is a point on the arc of $\partial K_d \cap V \cap \Gamma$ with endpoints A_0, A. Draw the α curve γ_1 tangent to $\partial K_d \cap \Gamma$ at A_1. Let P_1 denote the point of intersection of γ_1 and γ in V which is nearest A. Let $\epsilon, 0 < \epsilon \leqq \epsilon_0$, be the smallest angle between γ and γ_1 at P_1. The bounds on ϵ may be established using the function p and elementary geometry. Let B_1 be the point of $\partial K_M \cap \gamma_1$ which is nearest B. We claim for ϵ_0 small enough that there exists an α arc β_1 of $\gamma_1 \cap V$ with endpoints A_1, B_1. Again this is easily seen using (2.3b) and the function p. Let σ_1 be the arc of Γ with endpoints A, A_1, which is contained in V. Finally let $D_2(\epsilon)$ be the domain whose boundary is the union of the arcs, β_1, $\Gamma - \{\beta \cup \sigma_1\}$, and the arc of ∂K_M with endpoints B, B_1, which is contained in V. Put $\Gamma_2(\epsilon) = \partial D_2(\epsilon)$. Next we let ϵ vary subject to the above restrictions. Since $\{Q_n\}_1^\infty \subset \partial K_d \cap \Gamma$, we obtain a sequence, $\langle D_2(\epsilon)\rangle_{\epsilon \in z}$, of domains with boundaries, $\Gamma_2(\epsilon)$, $\epsilon \in z$. We assert that $D_2(\epsilon)$, $\epsilon \in z$, is α starlike and (4.4)–(4.6) are true. The assertion that $D_2(\epsilon)$ is α starlike may be proved using (2.3b) and Lemma 1. (4.4) follows from the definition of $\Gamma_2(\epsilon)$. (4.5) is a consequence of Lemma 3 and the fact that the γ_1 corresponding to ϵ_2 is tangent to $\Gamma_2(\epsilon_1)$ for $0 < \epsilon_1 < \epsilon_2 \leqq \epsilon_0$. (4.6) then follows from (4.5), the definition of $D_2(\epsilon)$, and the fact that $\lim_{n\to\infty} Q_n = A$ (see III).

To define III when B satisfies II (b) we choose a point $A_0 \in \Gamma \cap \partial K_d$ near A, $A_0 \neq A$, and let A_1 be a point on the arc of $\partial K_d \cap \Gamma$ with endpoints A_0, A. With this notation γ, γ_0, and γ_1 are defined as in III (a). Let γ^* be the α curve tangent to Γ at B which does not contain β. Let B_1 be the point nearest B in V where γ_1 and γ^* intersect. With this notation we define β_1 relative to A_1, B_1, and σ_1 relative to A, A_1, as in III (a). P_1 and $\epsilon > 0$ are also as in III (a). Let $D_2(\epsilon)$ be the domain whose boundary is the union of the arcs β_1, $\Gamma - \{\beta \cup \sigma_1\}$, and the α arc of $\gamma^* \cap V$ with end points B, B_1. Then $D_2(\epsilon)$ is α starlike and (4.4)–(4.6) are true, as follows from an argument similar to our previous arguments. We omit the details.

We now consider the effect on D of applying an outward variation of Γ, as in II or III, followed by an inward variation of the form I. To simplify our notation we put $Y = (0, \epsilon_0]$, if D is varied as in II, and $Y = z$ if D is varied as in III. First applying variation II or III we obtain for each ϵ, $\epsilon \in Y$, an α starlike domain $D_2(\epsilon)$ with boundary $\Gamma_2(\epsilon)$. Also, $\Gamma_2(\epsilon)$ contains an α arc $\mu(\epsilon)$ with one endpoint E, and $\mu \subset \mu(\epsilon)$ ($\mu = \mu(\epsilon)$ unless $B = F$). Next we apply variation I with $\mu(\epsilon)$, $\Gamma_2(\epsilon)$, replacing μ, Γ, in I. This is permissible if $\epsilon_0 > 0$ is small enough. Applying variation I, we obtain for each δ, $0<\delta \leqq \delta_0(\epsilon)$, an α starlike domain $D(\epsilon,\delta)$ with boundary $\Gamma(\epsilon,\delta)$. We claim that $\delta_0(\epsilon)$ does not depend on ϵ. This claim is clearly true if $B \neq F$, since in this case the inward and outward variations are independent for small $\epsilon_0>0$. If $B=F$, then it is easily checked that $D(\epsilon,\delta)$ is well defined for $\epsilon \in Y$ and $0<\delta \leqq \delta_0(\epsilon_0)$, when $\epsilon_0 \in Y$ is small. Hence our claim is true and we may take $\delta_0(\epsilon)=\delta_0(\epsilon_0)$.

Finally in this section we consider the equation

$$\text{(4.7)} \qquad \text{m.r.}\, D(\epsilon,\delta)=1$$

for $0<\delta \leqq \delta_0(\epsilon_0)$ and $\epsilon \in Y$. Here m.r. $D(\epsilon,\delta)$, as previously defined, denotes the mapping radius of $D(\epsilon,\delta)$. We claim that the ordered pairs (ϵ,δ) satisfying (4.7) define a decreasing function $\delta=\delta(\epsilon)$ for $\epsilon \in Y \cap (0,\epsilon_1]$, $0<\epsilon_1 \leqq \epsilon_0$. Also $\delta(\epsilon)\to 0$ as $\epsilon \to 0$ in Y. This claim is verified using (4.1)–(4.6), and the monoticity of the mapping radius. We omit the details.

We put $D(\epsilon)=D[\epsilon,\delta(\epsilon)]$, $\epsilon \in Y\cap(0,\epsilon_1]$, $\Gamma(\epsilon)=\partial D[\epsilon,\delta(\epsilon)]$, $\epsilon \in Y \cap (0,\epsilon_1]$. We also put $D(0)=D$, $\Gamma(0)=\Gamma$. Then $D(\epsilon)$ is α starlike and from (4.1), (4.4), (4.7), we have

$$\text{(4.8)} \qquad \Gamma(\epsilon)\subset L(d,M), \epsilon \in Y_1=[Y\cup\{0\}]\cap[0,\epsilon_1],$$

$$\text{(4.9)} \qquad \text{m.r.}D(\epsilon)=1,\ \epsilon \in Y_1.$$

5. The Hadamard variational formula. From the definition of $\Gamma(\epsilon)$ for $\epsilon \in Y \cap (0,\epsilon_1]$ we see that $\Gamma(\epsilon)$ contains an α arc $\beta_1=\beta_1(\epsilon)$ with ϵ the smallest angle between β_1 and the α arc containing β at A or P_1. Also $\Gamma(\epsilon)$ contains an α arc $\mu_1=\mu_1(\epsilon)$ with $\delta(\epsilon)$ the smallest angle between μ_1 and μ at E. This observation will be used throughout §5. In the sequel the symbols, $\epsilon\to 0$, $\lim_{\epsilon\to 0}$, apply only to $\epsilon \in Y_1$.

Given ϵ, $\epsilon \in Y_1$, let $g_\epsilon(\cdot, w_1)$ denote Green's function for $D(\epsilon)$ with pole at $w_1 \in D(\epsilon)$. If $w_1 \in D(0)$ is fixed, then

$$\text{(5.1)} \qquad \lim_{\epsilon\to 0}[g_\epsilon(\cdot,w_1)-g_0(\cdot,w_1)]=0$$

uniformly on compact subsets of $D(0)$. This inequality follows from the fact that $D(\epsilon)\to D(0)$ as $\epsilon\to 0$ in the sense of kernel convergence. We remark for fixed $w_1 \in D(\epsilon)$ that the outer normal derivative of $g_\epsilon(\cdot,w_1)$ exists at each $s \in \mu_1 \cup \beta_1$, except possibly at the endpoints of these arcs. We denote this derivative by $\partial g_\epsilon/\partial n(s,w_1)$. We wish to show for fixed $w_1 \in D(0)$ that

$$g_0(w_1,0)-g_\epsilon(w_1,0)=\frac{q\epsilon}{2\pi}\int_\mu \frac{\partial g_0}{\partial n}(s,0)\frac{\partial g_0}{\partial n}(s,w_1)\frac{|\phi(s)-\phi(E)|}{|\phi'[s)|}|ds|$$

(5.2)

$$-\frac{\epsilon}{2\pi}\int_\beta \frac{\partial g_0}{\partial n}(s,0)\frac{\partial g_0}{\partial n}(s,w_1)\frac{|p(s)-p(A)|}{|p'(s)|}|ds|$$
$$+o(\epsilon)$$

as $\epsilon \to 0$. Here p and ϕ are analytic $1/\alpha$ powers of w in V and N respectively (see §4). Also, $q = I_2/I_1$, where

$$I_1=\int_\mu \left[\frac{\partial g_0}{\partial n}(s,0)\right]^2 \frac{|\phi(s)-\phi(E)|}{|\phi'(s)|}|ds|,$$

$$I_2=\int_\beta \left[\frac{\partial g_0}{\partial n}(s,0)\right]^2 \frac{|p(s)-p(A)|}{|p'(s)|}|ds|.$$

For $w_1=0$ the lefthand side of (5.2) is to be interpreted as the value of the harmonic function $g_0(w,0)-g\epsilon(w,0)$, $w\in D(0)\cap D(\epsilon)$, at $w=0$. The term $o(\epsilon)$ in (5.2) is independent of w_1 when w_1 lies in a compact subset of $D(0)$. Formula (5.2) is essentially just the Hadamard variational formula (see Bergman [3, Ch.8]). However since our variations are not strictly normal and since Γ need not be twice continuously differentiable, we shall give the proof of (5.2).

Let w_1 be given in $\cup_{\epsilon\in Y_1} D(\epsilon)$, and let Δ, $\Delta(w_1)$, denote disks about $w=0$, $w=w_1$, of radius $r>0$ respectively, which are contained in each $D(\epsilon)$ for $\epsilon\in Y_1$. Let $\rho(w,\epsilon)$ denote the distance of $w\in D(\epsilon)$ from $\Gamma(\epsilon)$ for $\epsilon\in Y_1$. We also let C denote a positive constant, not necessarily the same at each occurence, which may depend on α, r, $D(0)$, and ϵ_1 (see (4.8)), but not on ϵ or $w\in D_\epsilon-\{\Delta\cup\Delta(w_1)\}$. Then as a first step in proving (5.2) we show

$$\max\{g_\epsilon(w,w_1),g_\epsilon(w,0)\}\leqq C\rho(w,\epsilon) \tag{5.3}$$

for $w\in D(\epsilon)-\{\Delta\cup\Delta(w_1)\}$ and $\epsilon\in Y_1$.

To prove (5.3), let f_ϵ, $\epsilon\in Y_1$, be the function in $S(\alpha,d,M)$ for which $f_\epsilon(K)=D(\epsilon)$. The existence of f_ϵ is guaranteed by (4.8) and (4.9). Let k_ϵ denote the inverse of f_ϵ and note that

(5.4)
$$g_\epsilon(w,0)=-\log|k_\epsilon(w)|,$$
$$g_\epsilon(w,w_1)=-\log\left|\frac{k_\epsilon(w)-k_\epsilon(w_1)}{1-\bar{k}_\epsilon(w_1)k_\epsilon(w)}\right|$$

for $w\in D(\epsilon)-\{\Delta\cup\Delta(w_1)\}$. We assert that

$$(5.5)\qquad |f_\epsilon(a)-f_\epsilon(b)|\geqq C_1|b-a|$$

whenever $a, b \in K$ and $\{f_\epsilon(b), f_\epsilon(a)\}\subset D(\epsilon)-\{\Delta\cup\Delta(w_1)\}$. Here C_1 is a positive constant which has the same dependence as C defined previously. (5.3) is then an easy consequence of (5.4) and (5.5).

If $|\operatorname{Arg}[\overline{f_\epsilon(b)}f_\epsilon(a)]|\geqq\pi\alpha/2$ and $f_\epsilon(a)$, $f_\epsilon(b)$, are in $D(\epsilon)-[\Delta\cup\Delta(w_1)]$, then clearly

$$|f_\epsilon(b)-f_\epsilon(a)|\geqq C\geqq\frac{C}{2}|b-a|.$$

Hence we assume $0\leqq|\operatorname{Arg}[\overline{f_\epsilon(a)}f_\epsilon(b)]|<\pi\alpha/2$. Let R be a sector drawn from $w=0$ which contains $f_\epsilon(a)$, $f_\epsilon(b)$, in its interior. We also choose R to be of angle-opening less than $\pi\alpha/2$. Let h be an analytic $1/\alpha$ power of w on R. Let λ_ϵ be the α arc contained in R with endpoints $f_\epsilon(a)$, $f_\epsilon(b)$. Then since $h[R\cap D(\epsilon)]$ is convex, the line segment σ_ϵ, with endpoints $h[f_\epsilon(a)]$, $h[f_\epsilon(b)]$, is contained in $h[R\cap D(\epsilon)]$. Since $|\operatorname{Arg}[\overline{h(f_\epsilon(a))}h(f_\epsilon(b))]|<\pi/2$ and $\min(|f_\epsilon(a)|, |f_\epsilon(b)|)\geqq r$, it follows from elementary geometry that

$$\min_{\zeta\in\sigma_\epsilon}|\zeta|\geqq\frac{\sqrt{2}}{2}r^{1/\alpha}.$$

Moreover since h maps λ_ϵ onto σ_ϵ, we deduce that

$$(5.6)\qquad \min_{w\in\lambda_\epsilon}|w|\geqq\left(\frac{\sqrt{2}}{2}\right)^{\alpha}r.$$

If τ_ϵ denotes the preimage in K of σ_ϵ under $h\circ f_\epsilon$, then τ_ϵ has endpoints a, b, and

$$(5.7)\qquad |h[f_\epsilon(b)]-h[f_\epsilon(a)]|=\int_{\tau_\epsilon}\left|\frac{d}{dz}h\circ f_\epsilon(z)\right||dz|.$$

Since $f_\epsilon\in S(\alpha, d, M)$, and we have (1.2d), we may write for $z\in K-\{0\}$ that

$$(5.8)\qquad \left|\frac{d}{dz}h\circ f_\epsilon(z)\right|=\frac{1}{\alpha}|f_\epsilon(z)|^{\frac{1}{\alpha}-1}|f_\epsilon'(z)|=\frac{1}{\alpha}|\psi_\epsilon(z)/z|^{\frac{1}{\alpha}}\cdot|z|^{\frac{1}{\alpha}-1},$$

where ψ_ϵ is starlike univalent and $\psi_\epsilon(0)=0$, $\psi_\epsilon'(0)=1$. It is well known that $|\psi_\epsilon(z)/z|\geqq\frac{1}{4}$. It also follows from well known estimates for normalized univalent functions and (5.6) that $|z| \geqq C$, $z \in \tau_\epsilon$ (see Goluzin [4, p. 52, (10)] for these estimates). Using (5.8) and the above facts we get,

$$\left|\frac{d}{dz} h \circ f_\epsilon(z)\right| \geqq \frac{1}{\alpha}\left(\frac{1}{4}\right)^{1/\alpha} C^{1/\alpha - 1} = C$$

for $z \in \tau_\epsilon$. From this inequality and (5.7) we deduce $|h(f_\epsilon(b)) - h(f_\epsilon(a))| \geqq C|b-a|$. Since clearly $|h(f_\epsilon(b)) - h(f_\epsilon(a))| \leqq C|f_\epsilon(b) - f_\epsilon(a)|$ whenever $\{f_\epsilon(a), f_\epsilon(b)\} \subset D(\epsilon) - [\Delta \cup \Delta(w_1)]$ and $\operatorname{Arg}[f_\epsilon(a)\overline{f_\epsilon(b)}] < \pi\alpha/2$, it follows that (5.5) is true.

We now prove (5.3). We first note that

$$\max\{g_\epsilon(w, w_1), g_\epsilon(w, 0)\} \leqq C, \tag{5.9}$$

when $\epsilon \in Y_1$ and $w \in D(\epsilon) - [\Delta \cup \Delta(w_1)]$, as follows from the maximum principle for harmonic functions. Consider the case when $\rho(w, \epsilon) \geqq \frac{C_1}{16}(1 - |k_\epsilon(w_1)|)$, [$C_1$ as in (5.5)]. Then from (5.9) we have

$$\max\{g_\epsilon(w, w_1), g_\epsilon(w, 0)\} \leqq \frac{16\, C\rho(w, \epsilon)}{C_1(1 - |k_\epsilon(w_1)|)} = C\rho(w, \epsilon).$$

Here we have used the fact that $\sup_{\epsilon \in Y_1} |k_\epsilon(w_1)| < 1$. Next consider the case when $\rho(w, \epsilon) < \frac{(1 - |k_\epsilon(w_1)|)C_1}{16}$. In this case let $w_0 = w_0(\epsilon)$ in $\Gamma(\epsilon)$ be such that $|w - w_0| = \rho(w, \epsilon)$. Choose $w^* = w^*(\epsilon) \in D(\epsilon)$ near enough w_0 such that

$$\min\left\{|k_\epsilon(w^*)|, \left|\frac{k_\epsilon(w^*) - k_\epsilon(w_1)}{1 - \overline{k_\epsilon(w^*)}k_\epsilon(w_1)}\right|\right\} \geqq \frac{1}{2} \tag{5.10}$$

and such that

$$|w - w^*| \leqq 2\rho(w, \epsilon). \tag{5.11}$$

Then from (5.5) with $a = k_\epsilon(w)$, $b = k_\epsilon(w^*)$, and (5.11) we deduce

$$|k_\epsilon(w) - k_\epsilon(w^*)| \leqq C_1^{-1}|w - w^*| \leqq 2C_1^{-1}\rho(w, \epsilon). \tag{5.12}$$

Using (5.12) and (5.10) it follows for

$$u = \frac{k_\epsilon(w) - k_\epsilon(w_1)}{1 - \overline{k_\epsilon(w_1)}\, k_\epsilon(w)}, \quad v = \frac{k_\epsilon(w^*) - k_\epsilon(w_1)}{1 - \overline{k_\epsilon(w_1)}k_\epsilon(w^*)},$$

that

$$|v|^{-1}|u-v| \leqq 2|v|^{-1}\frac{|k_\epsilon(w)-k_\epsilon(w^*)|}{1-|k_\epsilon(w_1)|} \leqq \frac{8C_1^{-1}\rho(w,\epsilon)}{1-|k_\epsilon(w_1)|} < \frac{1}{2}.$$

Using this inequality, (5.4), and the fact that $-\log(1-x) \leqq 2x$ for $0 \leqq x \leqq \frac{1}{2}$, we get

$$g_\epsilon(w,w_1) = -\log|u| = -\log\left|\frac{u-v}{v}+1\right| - \log|v| \leqq 2\left|\frac{u-v}{v}\right|$$

$$-\log|v| \leqq \frac{16C_1^{-1}\rho(w,\epsilon)}{1-|k_\epsilon(w_1)|} - \log|v| = C\rho(w,\epsilon) - \log|v|.$$

Letting $w^* \to w_0$, we obtain since $\log|v| \to 0$ that $g_\epsilon(w,w_1) \leqq C\rho(w,\epsilon)$, $w \in D(\epsilon) - \{\Delta \cup \Delta(w_1)\}$. Similarly from (5.10) and (5.12), we get $g_\epsilon(w,0) \leqq C\rho(w,\epsilon)$ for $w \in D(\epsilon) - \{\Delta \cup \Delta(w_1)\}$. We conclude that (5.3) is true.

Next we use (5.3) to prove (5.2). We first claim for given ϵ, $\epsilon \in Y_1$, that

$$g_0(w_1,0) - g_\epsilon(w_1,0) = J_1 + J_2 + J_3 + o(\epsilon) \tag{5.13}$$

as $\epsilon \to 0$ where

$$J_1 = \frac{1}{2\pi}\int_{\beta \cap D(\epsilon)} g_\epsilon(s,0)\frac{\partial g_0}{\partial n}(s,w_1)\,|ds|,$$

$$J_2 = -\frac{1}{2\pi}\int_{\mu_1 \cap D(0)} g_0(s,w_1)\frac{\partial g_\epsilon}{\partial n}(s,0)\,|ds|,$$

$$J_3 = \frac{1}{2\pi}\int_{\mu_1 \cap D(0)} \left[g_0(s,w_1)\frac{\partial}{\partial n}g_0(s,0) - g_0(s,0)\frac{\partial g_0}{\partial n}(s,w_1)\right]|ds|.$$

To verify this claim we consider two cases. First suppose that β is as in variation III. In this case we let $D^*(\epsilon) \subset D(\epsilon) \cap D(0)$, be the domain whose boundary is the union of the arcs: $\partial D(\epsilon) \cap \partial D(0)$, $\beta \cap D(\epsilon)$, $\mu_1 \cap D(0)$, and the arc of ∂K_d with endpoints A_1, A, which is contained in V. Here A_1 is as in variation III. Let v denote the above arc of ∂K_d. We observe that $D^*(\epsilon)$ is α starlike. This observation is verified using (2.3b) and Lemma 1.

From the above observation and Lemma 2, it now follows that $D^*(\epsilon)$ can be approximated by a sequence of α starlike domains $\langle\Omega(n)\rangle_1^\infty$ with the property that

(i) the sets $\beta \cap D(\epsilon)$, $\mu_1 \cap D(0)$, and v, are contained in $\partial\Omega(n)$,

(ii) $\Omega(n) \subset D^*(\epsilon)$,

(iii) Each point of $\partial\Omega(n)$ is within $1/n$ distance of a point of $\partial D^*(\epsilon)$,

(iv) $\partial\Omega(n)$ consists of a finite number of α arcs and v.

Since $D^*(\epsilon) \subset D \cap D(\epsilon)$ we clearly may apply Green's second identity in $\Omega(n)$-$\{w : |w - w_1| \leqq \eta\}$, η small, to the functions: $g_0(w, w_1)$, $g_0(w, 0) - g_\epsilon(w, 0)$, $w \in D(\epsilon) \cap D$ (see Nehari **[13**, p. 9] for this identity). Doing this, letting $\eta \to 0$, using (iii) and (5.3), we get

$$g_0(w_1, 0) - g_\epsilon(w_1, 0) = J_1 + J_2 + J_3 + J_4 + O\left(\frac{1}{n}\right),$$

where

$$J_4 = \frac{1}{2\pi}\int_v [g_\epsilon(s, 0) - g_0(s, 0)] \frac{\partial g_0}{\partial n}(s, w_1)\,|ds|$$
$$-\frac{1}{2\pi}\int_v g_0(s, w_1)\frac{\partial}{\partial n}[g_\epsilon(s, 0) - g_0(s, 0)]\,|ds|.$$

We note that each point of v is $O(\epsilon)$ distance from A. Furthermore the arc length of v is $O(\epsilon)$ as $\epsilon \to 0$. Hence from (5.3), $J_4 = O(\epsilon^2)$ as $\epsilon \to 0$. Using this fact and letting $n \to \infty$ in the above equality, we get (5.13) when β is as in variation III. The proof of (5.13) when β is as in variation II is similar. We omit the details.

To continue the proof of (5.2) we show

$$\text{(5.14)} \qquad \lim_{\epsilon \to 0}\frac{J_1}{\epsilon} = -\frac{1}{2\pi}\int_\beta \frac{\partial g_0}{\partial n}(s, 0)\frac{\partial g_0}{\partial n}(s, w_1)\frac{|p(s) - p(A)|}{|p'(s)|}\,|ds|,$$

$$\text{(5.15)} \qquad \lim_{\delta \to 0}\frac{J_2}{\delta} = \frac{1}{2\pi}\int_\mu \frac{\partial g_0}{\partial n}(s, 0)\frac{\partial g_0}{\partial n}(s, w_1)\frac{|\phi(s) - \phi(E)|}{|\phi'(s)|}\,|ds|$$

$$\text{(5.16)} \qquad \lim_{\delta \to 0}\frac{J_3}{\delta} = 0.$$

To prove (5.14) let $V \supset \beta$ be the domain of definition of p, as defined at the beginning of §4. Let l_ϵ, $\epsilon \in Y_1$-$\{0\}$, and l_0, denote the line segments which are images of $\beta_1 = \beta_1(\epsilon)$ and β under p respectively. We also put $H_\epsilon(\zeta) = g_\epsilon(w, 0)$ when $p(w) = \zeta$, $w \in V \cap D(\epsilon)$, $\epsilon \in Y_1$. Then H_ϵ is harmonic in $p(V \cap D(\epsilon))$, vanishes on l_ϵ, and from (5.3), (5.1), it follows that

$$H_\epsilon(\zeta) \leqq C, \ \zeta \in p(V \cap [D(\epsilon) - \Delta]), \tag{5.17}$$

$$\lim_{\epsilon \to 0} H_\epsilon(\zeta) = H_0(\zeta), \ \zeta \in p[V \cap D(0)]. \tag{5.18}$$

Now suppose that s is a fixed number in $\beta - \{A, B\}$ and $t = p(s)$. Then if $\epsilon_2 > 0$ is small enough, we have $s \in D(\epsilon) \cap \beta$ for $\epsilon \in Y \cap (0, \epsilon_2]$, $0 < \epsilon_2 \leqq \epsilon_1$, as follows from the definition of $D(\epsilon)$. Let $R = R(\epsilon)$ denote the point of intersection of β_1 and the α arc containing β in V (Either $R = A$ or $R = P_1$). Then we may write $t = p(R) + xe^{i\theta}$, where $x = x(\epsilon) > 0$ and $e^{i\theta}$ denotes the direction of l_0. Since the angle between l_ϵ and the line containing l_0 at $p(R)$ is ϵ, either the point $u(\epsilon) = p(R) + xe^{i(\theta+\epsilon)}$ or the point $t(\epsilon) = p(R) + xe^{i(\theta-\epsilon)}$, is in l_ϵ for $\epsilon_2 > 0$ small, $\epsilon \in Y_1 \cap [0, \epsilon_2]$. We first assume that $t(\epsilon)$ is in $l(\epsilon)$. Then if ϵ_2 is small enough there exists $\rho > 0$ and a semicircular disk, $Q(\epsilon)$, of radius ρ, center $t(\epsilon)$, and whose diameter is a line segment of l_ϵ, which is contained in $p[V \cap D(\epsilon)]$ for $\epsilon \in Y_1 \cap [0, \epsilon_2]$. Since H_ϵ vanishes on the diameter of $Q(\epsilon)$, it follows from the reflection principle, that

$$H_\epsilon(\zeta) = \operatorname{Im}\left\{\sum_{n=1}^{\infty} a_n(\epsilon) e^{in(\epsilon-\theta)}[\zeta - t(\epsilon)]^n\right\}, \tag{5.19}$$

$\zeta \in Q(\epsilon)$, where $a_n(\epsilon)$ is real, $n = 1, 2, \cdots$, and

$$|a_n(\epsilon)| \leqq C\rho^{-n}, \ n = 1, 2, \cdots, \tag{5.20}$$

$$a_1(\epsilon) = \frac{2}{\pi\rho} \int_{\theta-\epsilon}^{\theta-\epsilon+\pi} H_\epsilon[t(\epsilon) + \rho e^{i\phi}] \sin(\phi - \theta + \epsilon) d\phi. \tag{5.21}$$

From (5.19) and (5.20) it follows with $\zeta = t(0) = t$ that

$$\left|\frac{H_\epsilon(t) - a_1(\epsilon)\operatorname{Im}[e^{i(\epsilon-\theta)}(t - t(\epsilon))]}{\epsilon}\right| \leqq C\epsilon. \tag{5.22}$$

Furthermore from (5.18), (5.21), the bounded convergence theorem and the fact that $\lim_{\epsilon \to 0} R(\epsilon) = A$ we deduce, $\lim_{\epsilon \to 0} a_1(\epsilon) = a_1(0)$. Hence,

$$\lim_{\epsilon \to 0} g_\epsilon(s, 0)/\epsilon = \lim_{\epsilon \to 0} H_\epsilon(t)/\epsilon = a_1(0)\ |p(A) - t| = -\frac{\partial H_0}{\partial n}\ |p(A) - t|. \tag{5.23}$$

Here $\partial H_0/\partial n$ denotes the outer normal derivative of H_0 on l_0. If $u(\epsilon)$ is in l_ϵ for ϵ_2 small, then the above equality also holds, as is easily seen. We observe that, $\partial g/\partial n\ (s, 0) = \partial H_0/\partial n\ (t)\ |p'(s)|$. Using this observation, (5.3), (5.23), and the bounded convergence theorem we deduce that (5.14) is true.

The proof of (5.15) is similar to the proof of (5.14). Let N be the domain of definition of ϕ, as defined at the beginning of §4. Let y_δ, and y_0, denote the line segments which are images of $\mu_1 \cap D(0)$, and μ-$\{E, F\}$ under ϕ respectively. We also put $\Psi(\zeta) = g_0(w, w_1)$, $H_\epsilon(\zeta) = g_\epsilon(w, 0)$ when $\zeta = \phi(w)$, $w \in N \cap D(\epsilon)$, and $\epsilon \in Y_1$. Since the angle between y_δ and y_0 at $\phi(E)$ is δ, we may assume that

$$y_\delta = \{\phi(E) + xe^{i(\theta-\delta)} \colon 0 < x < x_1(\delta)\},$$

where $e^{i\theta}$ is the direction of y_0 and $\delta = \delta(\epsilon)$, $\epsilon \in Y_1$. From the definition of $D(\epsilon)$ and the fact that $\lim_{\epsilon\to 0} \delta(\epsilon) = 0$, we see that $\lim_{\delta\to 0} x_1(\delta) = x_1(0)$.

Using our new notation and changing variables in the integral defining J_2 we find that

$$\frac{J_2}{\delta} = \int_0^{x_1(\delta)} \frac{\Psi[\phi(E) + xe^{i(\theta-\delta)}]}{-2\pi\delta} \frac{\partial H_\epsilon}{\partial n} [\phi(E) + xe^{i(\theta-\delta)}]dx. \tag{5.24}$$

Here $\partial H_\epsilon / \partial n$ denotes the outer normal derivative of H_ϵ to y_δ. From (5.3) we note that $\delta^{-1}\Psi[\phi(E) + xe^{i(\theta-\delta)}]$ is bounded for $\delta = \delta(\epsilon)$, $\epsilon \in Y_1 - \{0\}$, and $0 < x < x_1(\delta)$. Moreover,

$$\lim_{\delta\to 0} \frac{\Psi[\phi(E) + xe^{i(\theta-\delta)}]}{\delta} = -\frac{\partial \Psi}{\partial n}[\phi(E) + xe^{i\theta}] \cdot x \tag{5.25}$$

when $\phi(E) + xe^{i\theta} \in y_0$. Also, as in (5.23) we find that

$$\lim_{\delta\to 0} \frac{\partial H_\epsilon}{\partial n}[\phi(E) + xe^{i(\theta-\epsilon)}] = \frac{\partial H_0}{\partial n}[\phi(E) + xe^{i\theta}], \tag{5.26}$$

when $\phi(E) + xe^{i\theta} \in y_0$. Using (5.24)–(5.26), (5.3), the bounded convergence theorem, and changing back to our original variables, we conclude that (5.15) is true.

The proof of (5.16) is essentially the same as the proof of (5.15). We omit the details. Hence (5.14)–(5.16) are true.

Finally we show that

$$\lim_{\epsilon\to 0} \delta(\epsilon)/\epsilon = q \tag{5.27}$$

where q is as in (5.2). (5.2) is then an obvious consequence of (5.13)–(5.16) and (5.27). To prove (5.27) we put $w_1 = 0$ in (5.13). We obtain from (4.9) that

$$\frac{1}{2\pi}\int_{\beta\cap D(\epsilon)} g_\epsilon(s,0)\frac{\partial g_0}{\partial n}(s,0)\,|ds| = \frac{1}{2\pi}\int_{\mu_1\cap D(0)} g_0(s,0)\,\frac{\partial g_\epsilon}{\partial n}(s,0)\,|ds| \tag{5.28}$$

$$+\,o(\epsilon), \quad \text{as} \quad \epsilon\to 0.$$

From (5.14)–(5.15) with $w_1 = 0$, we see that

$$-\int_{\beta\cap D(\epsilon)} g_\epsilon(s,0)\frac{\partial g_0}{\partial n}(s,0)\,|ds| = [1+o(1)]\epsilon I_2,$$

$$-\int_{\mu_1\cap D(0)} g_0(s,0)\frac{\partial g_\epsilon(s,0)}{\partial n}|ds| = [1+o(1)]\delta I_1.$$

Using these equalities and (5.28), we find that (5.27) is true.

We have now shown that (5.2) holds with a $o(\epsilon)$ term that depends on w_1. To complete the proof of (5.2), we show this term does not depend on w_1 when w_1 lies in a compact subset, X, of $D(0)$. Clearly, it suffices to prove the above for given $w_0 \in D(0)$ and $X = \{w_2\colon |w_2 - w_0| \leqq r/2\}$, r small. Moreover, since a pointwise limit of uniformly bounded harmonic functions is uniform, it suffices to show that $\epsilon^{-1}[g_0(w_1,0) - g_\epsilon(w_1,0)]$ is uniformly bounded for $\epsilon \in Y_1 - \{0\}$ and $w_1 \in \{w_2\colon |w_2 - w_0| \leqq r/2\}$. From (5.13), its subsequent proof, and (5.27), we see that this statement will be true if we can show the constant in (5.3) does not depend on w_1 when $w_1 \in \{w_2\colon |w_2 - w_0| \leqq r/2\}$.

To argue the last statement we first assert that $[g_\epsilon(w,w_0)]^{-1}g_\epsilon(w,w_1)$ is uniformly bounded whenever $w_1 \in \{w_2\colon |w_2 - w_0| \leqq r/2\}$, $w \in \{w_2\colon |w_2 - w_0| = r\}$, and $\epsilon \in Y_1$. Indeed, since $D(\epsilon)\to D(0)$ in the sense of kernel convergence, we have $k_\epsilon \to k_0$ uniformly on $\{w_2\colon |w_2 - w_0| \leqq r\}$. Using this and (5.4), it follows that our assertion is true. If c denotes the uniform bound in our assertion, then from the maximum principle for harmonic functions we have, $g_\epsilon(w,w_1) \leqq c g_\epsilon(w,w_0)$ when $w \in D(\epsilon) - \{w_2\colon |w_2 - w_0| < r\}$, $w_1 \in \{w_2\colon |w_2 - w_0| \leqq r/2\}$ and $\epsilon \in Y_1$. Using (5.3) with $w_1 = w_0$, we conclude that

$$g_\epsilon(w,w_1) \leqq c g_\epsilon(w,w_0) \leqq cC\rho(w,\epsilon),$$

when w, w_1, and ϵ are in the above sets. Hence the constant in (5.3) does not depend on $w_1 \in \{w_2\colon |w_2 - w_0| \leqq r/2\}$. This completes the proof of (5.2).

6. The Julia variational formula. In this section we show how the Julia variational formula for the mapping functions f_ϵ, corresponding to $D(\epsilon)$, can be derived from the Hadamard variational formula for g_ϵ [see (5.2)]. We then show in a general way how the Julia variational formula can be used to solve some extremal problems. We use the same notation as in §5.

First note that $\Gamma(0)$ is a Jordan curve. Hence from the strong form of the Riemann mapping theorem (see Goluzin [4, Thm. 4, p. 44]), f_0 is a homeomorphism of $K \cup \partial K$ onto $D(0) \cup \Gamma(0)$. Consequently there exist arcs λ, τ, of ∂K, disjoint, except possibly for endpoints, such that $f_0(\lambda) = \mu$, and $f_0(\tau) = \beta$. Also from the reflection principle we see that f_0 can be extended analytically to a larger domain containing all of $\lambda \cup \tau$, except possibly the endpoints of these arcs. We denote this extension again by f_0.

Put $s = f_0(\zeta)$, $\zeta \in \lambda \cup \tau$, and choose $z \in K$ such that $w_1 = f_0(z)$. Furthermore, let $h(\zeta) = -q|\phi(s) - \phi(E)|/|\phi'(s)|$, when $s = f_0(\zeta) \in \mu$, and $h(\zeta) = |p(s) - p(A)|/|p'(s)|$ when $s = f_0(\zeta) \in \beta$. Here q is as in (5.2). Using (5.4), changing variables in (5.2), and arguing as in Julia [7], we get

$$f_\epsilon(z) = f_0(z) + \frac{\epsilon z f_0'(z)}{2\pi} \int_{\lambda \cup \tau} \left(\frac{\zeta + z}{\zeta - z}\right) \frac{h(\zeta)}{|f'(\zeta)|} |d\zeta| + o(\epsilon) \tag{6.1}$$

as $\epsilon \to 0$. Now let $d\Lambda(\zeta) = \dfrac{h(\zeta)|d\zeta|}{2\pi|f'(\zeta)|}$, when $\zeta \in \lambda \cup \tau$. Then from (6.1) we obtain

$$\log[f_\epsilon(z)/z] = \log[f_0(z)/z] + \epsilon \int_{\lambda \cup \tau} \frac{z f_0'(z)}{f_0(z)} \left(\frac{\zeta + z}{\zeta - z}\right) d\Lambda(\zeta) + o(\epsilon)$$

as $\epsilon \to 0$. If Φ is a given nonconstant entire function, then

$$\Phi\left[\log \frac{f_\epsilon(z)}{z}\right] = \Phi\left[\log \frac{f_0(z)}{z}\right] + \epsilon \int_{\lambda \cup \tau} \Phi'\left[\log \frac{f_0(z)}{z}\right] \frac{z f_0'(z)}{f_0(z)} \left(\frac{\zeta + z}{\zeta - z}\right) d\Lambda(\zeta) + o(\epsilon),$$

as $\epsilon \to 0$. Hence

$$\operatorname{Re}\left\{\Phi\left[\log \frac{f_\epsilon(z)}{z}\right]\right\} - \operatorname{Re}\left\{\Phi\left[\log \frac{f_0(z)}{z}\right]\right\} = \epsilon \int_{\lambda \cup \tau} \sigma(\zeta) d\Lambda(\zeta) + o(\epsilon) \tag{6.2}$$

where,

$$\sigma(\zeta) = \operatorname{Re}\left\{\Phi'\left[\log \frac{f_0(z)}{z}\right] \frac{z f_0'(z)}{f_0(z)} \left(\frac{\zeta + z}{\zeta - z}\right)\right\}, \quad \zeta \in \partial K. \tag{6.3}$$

Next let α, d, and M be fixed positive numbers satisfying $0 < \alpha < \infty$, $0 \leqq d < 1$, and $1 < M < \infty$. Let C denote a given compact subclass of

$S(\alpha, d, M)$. Let Φ be a given nonconstant entire function. Consider the following extremal problem:

Problem 1. *Find* $\max_{f \in C} \operatorname{Re}\{\Phi[\log f(z)/z]\}$ *for given* $z \in K\text{-}\{0\}$.

Assume that f_ϵ is in C for $\epsilon \in Y_1$. Then we shall outline the method in which (6.2) can be used to obtain information about an extremal function which solves Problem 1 in C. First observe that if $\Phi'[\log f_0(z)/z] \neq 0$, then σ defined by (6.3) is the real part of an analytic function which maps ∂K onto a circle. Hence ∂K can be divided into two arcs, disjoint except for endpoints, such that σ is increasing on one arc, and decreasing on the other. It follows from this monotonic property of σ that if we are given any three arcs of ∂K (disjoint except possibly for endpoints), then we can choose two of the arcs, say λ and τ, such that

$$\min_{\zeta \in \tau} \sigma(\zeta) \geqq \max_{\zeta \in \lambda} \sigma(\zeta). \tag{6.4}$$

If (6.4) holds, we claim that either f_0 is not an extremal function for Problem 1 or $\Phi'[\log f_0(z)/z] = 0$. To verify this claim observe that $d\Lambda(\zeta) < 0$, $\zeta \in \lambda$, and $d\Lambda(\zeta) > 0$, $\zeta \in \tau$, except possibly at endpoints of these arcs. Also, $\int_{\lambda \cup \tau} d\Lambda(\zeta) = 0$. Using these facts and (6.4), we obtain from (6.2) that either

$$\operatorname{Re}\left\{\Phi\left[\log \frac{f_\epsilon(z)}{z}\right]\right\} > \operatorname{Re}\left\{\Phi\left[\log \frac{f_0(z)}{z}\right]\right\} \tag{6.5}$$

for $\epsilon > 0$ small or

$$\Phi'\left[\log \frac{f_0(z)}{z}\right] = 0. \tag{6.6}$$

If (6.5) occurs clearly f_0 is not an extremal function for Problem 1. Hence our claim is true.

7. Preliminary lemmas. Let α, d, and M be fixed positive numbers satisfying $0 < \alpha < \infty$, $1 < M < \infty$, and $0 \leqq d < 1$. Then in this section we first consider Problem 1 in some subclasses of $S(\alpha, d, M)$. Using this information, we then consider Problem 1 in $S(\alpha, d, M)$. Our goal is to show that a rotation of F defined by (i)-(iv) of §1 solves Problem 1 in $S(\alpha, d, M)$ (Lemma 8). We use the method of §6. To begin, let $\mathscr{A}$ denote the class of α starlike domains Ω with Ω in $\mathscr{A}$ if and only if $f(K) = \Omega$ for some $f \in S(\alpha, d, M)$. Let $\mathscr{A}_n$ denote the subclass of $\mathscr{A}$ consisting of all domains Ω whose boundary is the union of a finite number of α arcs with at most n nondegenerate vertices. By a vertex we mean of course the intersection point of two α arcs. The vertex is nondegenerate if the smallest angle θ between the two α arcs at this vertex satisfies $0<\theta<\pi$. If β is an α arc connecting two nondegenerate vertices of Ω, then we shall call β an α *side* of Ω. If the vertices of an α side β lie on ∂K_M, then we shall call β an α *chord* of ∂K_M. The following lemma shows that $\cup_{1\leqq n\leqq\infty}\mathscr{A}_n$, is dense in $\mathscr{A}$.

LEMMA 4. *If $\Omega\in\mathscr{A}$, then there exists a sequence of domains $\{\Omega_n\}$ with $\Omega_n\in\mathscr{A}_n$ such that $\Omega_n\to\Omega$ in the sense of kernel convergence.*

Proof. It obviously suffices to show that for each $\eta>0$ there exists an integer n and $\Omega_n\in\mathscr{A}_n$ such that $\partial\Omega_n$ is contained in an η neighborhood of $\partial\Omega$. Let $f\in S(\alpha,d,M)$ be such that $f(K)=\Omega$. For given r, $0<r<1$, we consider the function $f_r(z)=f(rz)/r$, $z\in K$. From (1.1) we see that f_r is an α starlike function. Moreover the maximum and minimum modulus principle guarantee the existence of a d_1 and M_1 such that

$$d<d_1<|f_r(z)/z|<M_1<M,\qquad z\in K. \tag{7.1}$$

We put $\Omega^*=f_r(K)$. Then since f is continuous on $K\cup\partial K$, we may choose r near enough 1, such that each point of $\partial\Omega^*$ is contained in an $\eta/2$ neighborhood of $\partial\Omega$. From Lemma 2 we see that Ω^* may be approximated by an α starlike domain G with the following properties:

(i) $G\subset\Omega^*$,
(ii) $\partial G\subset L(d_1,M_1)=\{z:d_1\leqq|z|\leqq M_1\}$,
(iii) ∂G is the union of a finite number of α arcs,
(iv) ∂G is contained in an $\eta/4$ neighborhood of $\partial\Omega^*$,
(v) if $\rho=\text{m.r. } G<1$, then $\dfrac{1}{\rho}\leqq\min\left\{M/M_1,1+\dfrac{\eta}{4M}\right\}$.

From (ii), (iii), and (v) we see that $1/\rho G\in\mathscr{A}_n$ for some n. Also, (iv) and (v) imply that $\partial(1/\rho G)$ is contained in an $\eta/2$ neighborhood of Ω^*. Hence if $\Omega_n=1/\rho G$, then $\partial\Omega_n$ is contained in an η neighborhood of Ω. This completes the proof of Lemma 4.

Lemma 4 and A Theorem of Carathéodory imply that if $f(K)=\Omega$, $f_n(K)=\Omega_n$, where $f,f_n\in S(\alpha,d,M)$, then

$$\lim_{n\to\infty}f_n=f \tag{7.2}$$

uniformly on compact subsets of K.

For a given positive integer n, let C_n denote the class of functions $f \in S(\alpha, d, M)$ with $f(K) \in \mathcal{A}_n$. As in the proof of Lemma 1, we see that if $f \in C_n$, then f may be written in the form

$$\alpha \operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}+(1-\alpha)\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}=\sum_{k=1}^{m} a_k \operatorname{Re}\left\{\frac{1+e^{i\theta_k}z}{1-e^{i\theta_k}z}\right\},$$

where $m \leqq n$, $a_k > 0$ $(1 \leqq k \leqq m)$, and $\Sigma_{k=1}^{m} a_k = 1$. From this formula it is easily seen that C_n is compact. Hence if $C_n \neq \{\phi\}$, then there exists an extremal function F_n for Problem 1 with $C = C_n$. Choose a subsequence $(n_j)_1^\infty$ of $(n)_1^\infty$ such that if $H_j = F_{n_j}$, $j = 1, 2, \cdots$, then $\lim_{j\to\infty} H_j = H \in S(\alpha, d, M)$ uniformly on compact subsets of K. Then from (7.2) we see that H is an extremal function for Problem 1 with $C = S(\alpha, d, M)$.

We note for given $f \in S(\alpha, d, M)$ and $t \in K$ that the function $f(tz)/t, z \in K$, is also in $S(\alpha, d, M)$. It follows from this fact and a result of Kirwan [9] that

$$\Phi'\left[\log\frac{H(z)}{z}\right] \neq 0. \tag{7.3}$$

Here Φ and z are as in Problem 1, and $H = \lim_{j\to\infty} H_j$ is as above. Hence we may choose n_0 large enough such that

$$\Phi'[\log(H_j(z)/z)] \neq 0,\ j \geqq n_0. \tag{7.4}$$

We use (7.4) to obtain a partial description of $\Omega_j = H_j(K), j \geqq n_0$. Indeed, we have

LEMMA 5. *Let H_j and $\Omega_j = H_j(K)$ be as above for $j \geqq n_0$. Then all but at most two of the α sides of $\partial\Omega_j$ are either α chords of ∂K_M or are tangent to ∂K_d.*

Proof. Assume for some $j \geqq n_0$ that Lemma 5 is false. Put $D(0) = \Omega_j$, $\Gamma(0) = \partial\Omega_j$, and $f_0 = H_j$. Then $\Gamma(0)$ has at least three α sides which are not α chords of ∂K_M and which are not tangent to ∂K_d. The preimage of these sides consists of three arcs of ∂K, disjoint except possibly for endpoints. As in §6 we choose two of these arcs, λ and τ such that (6.4) holds. Let $f_0(\lambda) = \mu$, $f_0(\tau) = \beta$. Then μ can be rotated inward as in variation I, and β can be rotated outward as in variation II (b) in such a way that we obtain $D(\epsilon)$ (see §4) for $\epsilon \in Y_1$. Also if $\epsilon_1 > 0$ is small enough, then $D(\epsilon)$ has the same number of vertices as $D(0)$. Hence $D(\epsilon) \in \mathcal{A}_{n_j}$ for $\epsilon \in Y_1$. It follows that the functions f_ϵ, corresponding to $D(\epsilon)$ are in C_{n_j}. Using this fact, (6.4), (7.4), and arguing in §6, we find that f_0 is not extremal for Problem 1 in C_{n_j}. Since $f_0 = H_j$, we have reached a contradiction. We conclude from this contradiction that Lemma 5 is true.

We recall that our goal is to show that a rotation of F defined by (i)-(iv) of §1 solves Problem 1. We shall need the following lemma.

LEMMA 6. *Let d_1 and M_1 be fixed positive numbers satisfying $d < d_1 < M_1 < M$. Let Ω_j be as in Lemma 5 for $j \geqq n_0$. Then there exists, independently of j, a maximum number N of α sides of $\partial\Omega_j$ that intersect the closed annulus $L(d_1, M_1)$.*

Proof. We first consider those α sides of $\partial\Omega_j$ (j fixed) that have their endpoints on ∂K_M. If an α chord of ∂K_M intersects $L(d, M)$, this chord subtends a minor arc of ∂K_M of arc length at least t_1. t_1 may be taken to be the arc length of the minor arc subtended by an α chord of ∂K_M which is tangent to ∂K_{M_1}. Again this statement is proved using (2.3b) and properties of convex domains. Choose an integer l_1 such that $l_1 t_1 > 2\pi M$. Then, independently of j, no more that l_1 sides of $\partial\Omega_j$ which are α chords of ∂K_M, can intersect $L(d_1, M_1)$.

Suppose next that β is an α side tangent to ∂K_d which intersects $L(d_1, M_1)$. Let P be a point of $\beta \cap L(d_1, M_1)$ and let P_1 be the radial projection of P on ∂K_d. Let P_2 be the point where β is tangent to ∂K_d. Then the length of the minor arc of ∂K_d with endpoints P_1, P_2, has length at least t_2, where t_2 depends only on d, d_1, and α. Since Ω_j is starlike, two arcs of ∂K_d, obtained from two different α sides tangent to ∂K_d in the above way, cannot overlap. Hence if l_2 is a positive integer satisfying $l_2 t_2 \geqq 2\pi d$, then $\partial\Omega_j$ has at most l_2, α sides intersecting $L(d_1, M_1)$ which are tangent to ∂K_d. Using Lemma 5 we conclude that $\partial\Omega_j$ has at most $N = l_1 + l_2 + 2$ sides which intersect $L(d_1, M_1)$.

Next we use Lemma 6 to characterize $\Omega = H(K)$. We shall need some notation. Given θ, $0 < \theta \leqq 2\pi$, and $j \geqq n_0$, let $w_j(\theta)$ denote the unique point of intersection of $\partial\Omega_j$ with the ray from $w = 0$ which has direction $e^{i\theta}$. The uniqueness of $w_j(\theta)$ is guaranteed by Lemma 2. $w(\theta)$ is defined relative to $\partial\Omega$ in a similar way. For given $\epsilon > 0$ and θ, $0 < \theta \leqq 2\pi$, we claim there exists a positive integer $n_1 = n_1(\epsilon, \theta) \geqq n_0$ such that

$$|w_j(\theta) - w(\theta)| < \epsilon \quad \text{for} \quad j \geqq n_1. \tag{7.5}$$

This claim is a direct consequence of the fact that $\Omega_j \to \Omega$ as $j \to \infty$ in the sense of kernel convergence. We use (7.5) and Lemma 6 to prove.

LEMMA 7. *Let d_1 and M_1 be as in Lemma 6. Let $\Omega = H(K)$. Then $\partial\Omega \cap \{w: d_1 < |w| < M_1\}$ consists of a finite number of α arcs.*

Proof. Suppose x_i, $1 \leqq i \leqq N+2$, are $N+2$ points of $\partial\Omega$ with $0 \leqq \operatorname{Arg}(x_i\bar{x}_1) < \operatorname{Arg}(x_{i+1}\bar{x}_1) < \pi\alpha$, $1 \leqq i \leqq N+1$, and $d_1 < |x_i| < M_1$, $1 \leqq i \leqq N+2$. Let V be a sector, whose boundary consists of two rays drawn from $w = 0$, which contains each x_i, $1 \leqq i \leqq N+2$, in its interior. We may assume V has angle opening less than $\pi\alpha$. Draw the α arcs β_i, $1 \leqq i \leqq N+1$, which are contained in V, and have endpoints x_i, x_{i+1}. Let ϕ_i denote the smallest angle between the tangents of β_i and β_{i+1} at x_{i+1} for $1 \leqq i \leqq N$. Then if Lemma 7 is false we clearly can choose x_i, $1 \leqq i \leqq N+2$, as above, and such that

$$0 < \phi_i < \pi, \quad 1 \leqq i \leqq N. \tag{7.6}$$

Furthermore, from (7.5) we can choose, for arbitrarily small $\epsilon > 0$ and j large enough, $N+2$ points of $\partial\Omega_j$, say $y_1, y_2, \cdots, y_{N+2}$, so that

$$|x_i - y_i| < \epsilon, \qquad 1 \leqq i \leqq N+2.$$

However if ϵ is small enough this inequality and (7.6) imply that $\partial\Omega_j$ has $N+1$, α sides which intersect $L(d_1, M_1)$. We have reached a contradiction to Lemma 6. Hence Lemma 7 is true.

8. *Proof of Theorem* 1. Finally we prove.

LEMMA 8. *For some real θ, $\Omega = e^{i\theta}F(K)$, where $F = F(\cdot, \alpha, d, M)$ is as in* (i)-(iv).

Proof. First we extend the definition of an α side. Let γ be an α curve and suppose that $\beta = \gamma \cap \partial\Omega$ is a set consisting of more than one point. Then we shall call β an *α side* of $\partial\Omega$. From Lemma 2 we see that β is a closed α arc. Hence if $\partial\Omega \neq \beta$, then β has endpoints A, B, with $A \neq B$. In this case we assume, as we may, that $d \leqq |A| \leqq |B| \leqq M$. We assert that

(a) the left and righthand tangents to $\partial\Omega$ at B do not coincide.

If $|B| < M$, then (a) is a consequence of Lemma 7. If $|B| = M$, then (a) is easily proved using (2.3b) and geometric properties of convex domains. Hence our assertion is true.

Next, we assert that one of (b), (c), or (d) is valid for A,

(b) The left and right hand tangents to $\partial\Omega$ at A do not coincide and $d < |A|$,

(c) $|A| = d$ and there exists a set $\{Q_n\}_1^\infty$ of distinct points in $\partial K_d \cap \partial\Omega$ with $\lim_{n\to\infty} Q_n = A$,

(d) $|A| = d$ and there exists set $\{\rho_n\}_1^\infty$ of distinct α sides $\subset \partial\Omega$, with endpoints A_n, B_n, $n = 1, 2, \cdots$, for which $d < |A_n| \leqq |B_n| \leqq M$ and $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = A$.

The proof of (b) is the same as the proof of (a). If $|A| = d$, then from Remark 1 we see that γ is the unique α curve tangent to $\partial K_d \cap \partial \Omega$ at A. Using this fact, Lemma 3, and Lemma 7, it follows that either (c) or (d) is true.

We now use (a)-(d) to show that $\partial \Omega$ contains at most two α sides. Suppose to the contrary that there are at least three distinct α sides in $\partial \Omega$. To obtain a contradiction we consider two possibilities. First assume that one of the statements (a), (b), or (c) is valid for each endpoint of the α sides. Then the preimage of these sides consists of three arcs, disjoint, except possibly for endpoints. As in §6 we can choose two of the arcs λ and τ such that (6.4) holds with $f_0 = H$. Let λ_1 be a subarc of λ with the property that

(†) λ_1 has an endpoint in common with λ,
(††) $H(\lambda_1) \cap \partial K_d = \{\phi\}$.

Clearly there exists such an arc λ_1. We put $H(\lambda_1) = \mu$, $H(\tau) = \beta$. We also put $\Omega = D(0)$ and $f_0 = H$. Then μ satisfies the conditions of variation I, and β satisfies the conditions of either variation II or III. Hence we can perform these variations on $D(0)$ in such a way that we obtain $D(\epsilon)$ (see §4) for $\epsilon \in Y_1$. From the construction of $D(\epsilon)$, we have $D(\epsilon) \in \mathscr{A}$. Hence if f_ϵ is the function corresponding to $D(\epsilon)$, then $f_\epsilon \in S(\alpha, d, M)$. Using this fact, (6.4), (7.3), and arguing as in §6, we find that $f_0 = H$ is not extremal for Problem 1 in $S(\alpha, d, M)$. We have reached a contradiction. Thus if the above possibility occurs, then $\partial \Omega$ contains at most two α sides.

Next consider the possibility that all of the statements (a), (b), and (c) are false for an endpoint of one of the above α sides. Then from (d) we see that $\partial \Omega \cap \{z: d < |z| < M\}$ contains three other α sides. Furthermore, either statement (a) or statement (b) is valid for each endpoint of these α sides. Hence we may apply the argument of the first case to these α sides. Again we obtain a contradiction. We conclude from this contradiction that $\partial \Omega$ contains at most two α sides.

Since Ω is α starlike and we have Lemma 7, it follows from the above that $\partial \Omega$ consists of at most two α sides, at most two arcs of ∂K_M, and possibly one or two points or a proper arc of ∂K_d. Consider first the case when $\partial \Omega$ contains exactly one α side. Then from the discussion in §3 for $d = \delta(M, \alpha)$ we see that Lemma 8 is true. Second consider the case when $\partial \Omega$ contains two α sides. In this case we shall show that one endpoint of each α side must be on ∂K_d. It then follows from Remark 1 that these α sides are tangent to ∂K_d and there upon from the discussion in §3 that Lemma 8 is true.

The proof is again by contradiction. Assume that $\partial \Omega$ contains two α sides with at least one of the sides having both its endpoints off of

∂K_d. Observe from Lemma 1 that the other side then must also have both its endpoints off of ∂K_d. Let ξ_1 and ξ_2 denote these two α sides. Let $\psi_1, \psi_2, \subset \partial K$ be such that $H(\psi_1) = \xi_1$, $H(\psi_2) = \xi_2$. Put $f_0 = H$ and suppose that σ defined by (6.3) obtains its minimum at $\zeta_0 \in \partial K$. We first assume that ζ_0 is an interior point of either ψ_1 or ψ_2. We may assume that ζ_0 is in the interior of ψ_1 since otherwise we renumber. Then by the monotonic property of σ (see §6), there is a subarc $\psi_3 \subset \psi_1$ possessing an endpoint in common with ψ_1 and satisfying $\max_{\zeta \in \psi_3} \sigma(\zeta) \leqq \min_{\zeta \in \psi_2} \sigma(\zeta)$. Choose a subarc λ of ψ_3 possessing a common endpoint with ψ_1 and for which $H(\lambda) \cap \partial K_d = \{\phi\}$. This choice is possible since ξ_1 has both endpoints off of ∂K_d. We note that λ and $\tau = \psi_2$ satisfy (6.4). Also if $H(\lambda) = \mu$, $H(\tau) = \beta$, $f_0 = H$, $\Omega = D(0)$, then μ and β satisfy the requirements of variations I and II respectively. Using this fact and arguing as previously in §8, we obtain a contradiction to the fact that H is extremal for Problem 1 in $S(\alpha, d, M)$. Hence ζ_0 is not an interior point of either ψ_1 or ψ_2.

Now consider the case when ζ_0 is not an interior point of either ψ_1 or ψ_2. In this case σ clearly varies in a strictly monotonic manner on one of the sides, say ψ_1. Let ζ_1, ζ_2, denote the endpoints of ψ_1. Let the labelling of these points be such that

$$\sigma(\zeta_1) < \sigma(\zeta_2). \tag{8.1}$$

Choose a subarc $\nu \subset \psi_1$ with the property that $\zeta_1 \in \nu$ and $H(\nu) \cap \partial K_d = \{\phi\}$.

Again we let $H = f_0$, $D(0) = \Omega$, and use the notation introduced in §4. Let μ be a subarc of $H(\nu)$, with $H(\zeta_1) \in \mu$, $\mu \cap \partial K_d = \{\phi\}$, and such that if $\beta = \xi_1 - \mu$, then q defined as in (5.2) satisfies

$$q > 1. \tag{8.2}$$

This choice is possible since from (5.2) and (5.3) we have $q \to \infty$ as the arc length of $\mu \to 0$. Let $A = E$ denote the common endpoint of β and μ. Then μ satisfies the hypotheses of variation I, but β does not satisfy the hypotheses of either variation II or III. However from the remark after (4.6) we see that we still can apply variations I and II to obtain a starlike domain $D(\epsilon)$ for $\epsilon \in Y_1$ with m.r. $D(\epsilon) = 1$ and $\partial D(\epsilon) \subset L(d, M)$. We assert that in fact $D(\epsilon)$ is α starlike for $\epsilon \in Y_1 \cap [0, \epsilon_2]$ when $\epsilon_2 > 0$ is small enough.

To prove this assertion we introduce a new domain $\hat{D}(\epsilon)$, $\epsilon \in Y_1$. We obtain $\hat{D}(\epsilon)$ by applying variations I and II to $D(0)$. More specifically, put $\hat{D}(\epsilon) = D(\epsilon, \epsilon)$ (see §4 for the definition of $D(\epsilon, \delta)$). Then $\partial \hat{D}(\epsilon)$ contains α arcs $\hat{\mu}$, $\hat{\beta}$, with ϵ the smallest angle between μ, $\hat{\mu}$, and β, $\hat{\beta}$, at $E = A$. Hence, $\hat{\mu} \cup \hat{\beta}$ is an α arc. Using this fact and Lemma 1, we find that $\hat{D}(\epsilon)$ is α starlike.

We claim that (5.3) is valid, where now $g_\epsilon(\cdot, w_1)$ is Green's function for $\hat{D}(\epsilon)$ with pole at $w_1 \in \hat{D}(\epsilon)$. Indeed, it is easily checked that (5.3) holds under the weaker assumption, $\lim_{\epsilon\to 0}$ m.r. $\hat{D}(\epsilon) = 1$. Using (5.3) we deduce that (5.13)–(5.16) still hold for g_ϵ when $\delta = \epsilon$. It follows from these equalities with $w_1 = 0$, $\delta = \epsilon$, and (8.2) that

$$\lim_{\epsilon\to 0} \frac{1 - \text{m.r.}\hat{D}(\epsilon)}{\epsilon} = \frac{1}{2\pi}[I_1 - I_2] = \frac{I_1}{2\pi}[1-q] < 0.$$

Hence, m.r. $\hat{D}(\epsilon) > 1$ for $\epsilon > 0$ and small.

Since m.r. $\hat{D}(\epsilon) > 1$ for $\epsilon > 0$ and small, we may now apply variation I (with $\hat{D}(\epsilon)$, $\hat{\mu}$, replacing $D(0)$, μ, in I) to obtain an α starlike domain $\tilde{D}(\epsilon)$ with m.r. $\tilde{D}(\epsilon) = 1$. We note that $\tilde{\beta} \subset \partial\tilde{D}(\epsilon) \cap \partial D(\epsilon)$. Using this fact and the monoticity of the mapping radius, we conclude that $\tilde{D}(\epsilon) = D(\epsilon)$. Hence our assertion is true.

Let $\lambda \subset \nu$ and $\tau = \psi_1 - \lambda$ be the preimages of μ, β, respectively under f_0. We observe that $\zeta_1 \in \lambda$. Using this observation, (8.1), and the monoticity of σ on ψ_1 we deduce that (6.4) holds for λ and τ. Using (6.4), (7.3), and arguing as in §6, we find that H is not extremal for Problem 1 in $S(\alpha, d, M)$. We have reached a contradiction. Therefore ζ_0 must be an interior point of either ψ_1 or ψ_2. However, we have already shown this case cannot occur. Hence the assumption that ξ_1 does not have an endpoint on ∂K_d is false. We conclude that Lemma 8 is true.

Next we use Lemma 8 to prove Theorem 1. We note that F defined by (i)-(iv) of §1 is circularly symmetric. Using this fact and Theorem 2 of Jenkins **[6]** we see that

$$(8.3) \qquad |F(re^{i\theta_1})| > |F(re^{i\theta_2})|, \qquad 0 \leqq \theta_1 < \theta_2 \leqq \pi,$$

whenever $0 < r < 1$. From (8.3) and Theorem 3 of Kaplan **[8]**, we deduce that the function $g(z) = \log F(z)/z$, $z \in K$, is univalent and convex in the direction of the imaginary axis. Suppose now for some $f \in S(\alpha, d, M)$ that the function $h(z) = \log f(z)/z$, $z \in K$, is not subordinate to $g(z)$. Then for some $z_0 \in K$-$\{0\}$ we would have $w_0 = h(z_0) \notin g(K)$. It would then follow from Runge's Theorem (see Rudin **[14**, Thm. 13.9**]**) that there exists a polynomial P with

(i) $|P(w)| \leqq \frac{1}{4}$ for $w \in g(K_{|z_0|})$,
(ii) $|P(w_0)| \geqq \frac{1}{2}$.

We choose γ such that $\operatorname{Re}\{e^{i\gamma}P(w_0)\} = |P(w_0)|$. Then the function $\Phi(w) = e^{i\gamma}P(w)$ is entire and from (i), (ii), we have

$$\max_{0\leqq\theta\leqq 2\pi} \operatorname{Re}\left\{\Phi\left[\log\frac{F(e^{i\theta}z_0)}{e^{i\theta}z_0}\right]\right\} \leqq \frac{1}{4} < \operatorname{Re}\left\{\Phi\left[\log\frac{f(z_0)}{z_0}\right]\right\}.$$

This inequality contradicts Lemma 8. We conclude that Theorem 1 is true, for fixed α, d, and M satisfying $0<\alpha<\infty$, $1<M<\infty$, and $0\leqq d<1$.

The case $0<\alpha<\infty$, $M=\infty$, $0\leqq d<1$, can be handled by treating it as a limiting case as $M\to\infty$ of the above cases. We omit the details.

9. *Proof of Theorem* 2. Let M and d be fixed numbers satisfying $1<M<\infty$, $0\leqq d<1$. Let $S^*(d,M)$ be as in §1. Given $f\in S^*(d,M)$ and r, $0<r<1$, let $f_r(z)=f(rz)/r$, $z\in K$. From the maximum principle for harmonic functions and (1.1) we see that $f_r\in S(\alpha,d,M)$ for $0<\alpha\leqq\alpha_0$, provided α_0 is small enough. Hence from Theorem 1, $\log f_r(z)/z$, $z\in K$, is subordinate, to the function $\log[F(z,\alpha,d,M)/z]$, $z\in K$, for $0<\alpha\leqq\alpha_0$. Using this fact and simple properties of subordination, it follows that if $F^*(\cdot,d,M)=\lim_{\alpha\to 0}F(\cdot,\alpha,d,M)$ exists, then $\log f_r(z)/z$, $z\in K$, is subordinate to $\log[F^*(z,d,M)/z]$, $z\in K$. Since $F(K,\alpha,d,M)$ converges as $\alpha\to 0$ in the sense of kernel convergence, we see that the above limit exists. Furthermore, $\partial F^*(K,d,M)$ consists of either

(i) An arc of ∂K_d, passing through $-d$, with endpoints $de^{i\theta}$, $de^{-i\theta}$, $0<0<\pi$,

(ii) The arc of ∂K_M, passing through M, with endpoints $Me^{i\theta}$, $Me^{-i\theta}$,

(iii) The radial line segments connecting $de^{i\theta}$, $Me^{i\theta}$, and $de^{-i\theta}$, $Me^{-i\theta}$, respectively, or

(iv) A line segment on the negative real axis with one endpoint $-M$, and ∂K_M.

Since $f=\lim_{r\to 1}f_r$ we conclude that (B) of Theorem 1 is valid.

Finally we show that $g(z)=\log[F^*(z,d,M)/z]$, $z\in K$, is convex univalent. Since g is the limit of univalent functions, it is clearly univalent. Let z_1, z_2, be fixed points in K-$\{0\}$ with $z_1/|z_1|=e^{i\theta_1}$, $z_2/|z_2|=e^{i\theta_2}$, and $r_1=|z_1|\leqq r_2=|z_2|$. Then for given t, $0<t<1$, and $r=r_1/r_2\leqq 1$, the function

$$h(z)=z\left[\frac{F^*(e^{i\theta_2}z,d,M)}{e^{i\theta_2}z}\right]^t\left[\frac{F^*(re^{i\theta_1}z,d,M)}{re^{i\theta_1}z}\right]^{1-t},$$

$z\in K$, is in $S^*(d,M)$. The above fact follows from a property of starlike functions stated in §1, and the maximum principle for harmonic functions. Since $\log[h(z)/z]$, $z\in K$, is subordinate to g, we see that $\log[h(r_2)/r_2]=tg(z_2)+(1-t)g(z_1)$ is in $g(K)$. Hence g is convex univalent. The proof of Theorem 2 is now complete for $1 < M < \infty$ and $0 \leqq d < 1$. The case $M = \infty$, $0 \leqq d < 1$, can be handled by treating it as a limiting case as $M \to \infty$ of the above cases. We omit the details.

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