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# COEFFICIENT BOUNDS FOR SOME CLASSES OF STARLIKE FUNCTIONS

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**Let $t$ be given, $1/4 \leqq t \leqq \infty$, and let $S(t)$ denote the class of normalized starlike univalent functions $f$ in $|z| < 1$ satisfying (i) $|f(z)/z| \geqq t$, $|z| < 1$, if $1/4 \leqq t \leqq 1$, (ii) $|f(z)/z| \leqq t$, $|z| < 1$, if $1 < t \leqq \infty$. If $f(z) = z + \sum_{k=2}^{\infty} a_k z^k \in S(t)$ and $n$ is a fixed positive integer, then the authors obtain sharp coefficient bounds for $|a_n|$ when $t$ is sufficiently large or sufficiently near 1/4. In particular a sharp bound is found for $|a_3|$ when $1/4 \leqq t \leqq 1$ and $5 \leqq t \leqq \infty$. Also a sharp bound for $|a_4|$ is found when $1/4 \leqq t \leqq 1$ or $12.259 \leqq t \leqq \infty$.**

**1. Introduction.** Let $S$ denote the class of starlike univalent functions $f$ in $K = \{z : |z| < 1\}$ with the normalization, $f(0) = 0$, $f'(0) = 1$. Given $t$, $1/4 \leqq t \leqq \infty$, let $S(t)$ denote the subclass of functions $f \in S$ satisfying

$$|f(z)/z| \geqq t, \; z \in K, \text{ if } 1/4 \leqq t \leqq 1, \tag{1.1}$$

$$|f(z)/z| \leqq t, \; z \in K, \text{ if } 1 < t \leqq \infty. \tag{1.2}$$

If $1/4 < t \leqq 1$, we let $F = F(\cdot, t)$ be defined by

$$zF'(z)/F(z) = [1 + 2(2b^2 - 1)z + z^2]^{1/2}/(1 - z), \; z \in K, \tag{1.3}$$

where $0 \leqq b < 1$ and $t = [(1+b)^{1+b}(1-b)^{1-b}]^{-1}$. The function $F = F(\cdot, t)$ defined by (1.3) is in $S(t)$ for $1/4 < t \leqq 1$, as can be shown by a long but straightforward calculation (see Suffridge [9]). For fixed $t$, $1/4 < t \leqq 1$, this function maps $K$ onto the complex plane minus a set

$$\{w : |w| \geqq t, \qquad \pi b \leqq \arg w \leqq 2\pi - \pi b\}.$$

If $1 < t < \infty$, we let $F = F(\cdot, t) \in S(t)$ be defined by

$$\frac{F(z)}{[1 - t^{-1}F(z)]^2} = \frac{z}{(1-z)^2}, \; z \in K. \tag{1.4}$$

It is well known (see Nehari [4, p. 224, ex. 4]) that the function $F$ maps $K$ onto a domain whose boundary consists of $\{w : |w| = t\}$, and a slit along the negative real axis from $-t$ to $-\lambda$ where $4\lambda t^2 = (t + \lambda)^2$. If $t = 1/4$ or $t = \infty$, we let

$$F(z, 1/4) = F(z, \infty) = z/(1-z)^2, z \in K.$$

In [2] the authors proved a subordination theorem for some classes of univalent functions. For $S(t)$ this theorem may be stated as follows:

THEOREM A. *Let $t$ be given, $1/4 \leqq t \leqq \infty$. Let $F = F(\cdot, t)$ be as in (1.3) and (1.4). If $f \in S(t)$, then $\log f(z)/z, z \in K$, is subordinate to $\log F(z)/z$, $z \in K$.*

Theorem A implies for a given t, $1/4 \leqq t \leqq \infty$, that $F = F(\cdot, t)$ solves a number of extremal problems in $S(t)$. Some of these problems were pointed out in [2]. There, however, only general prperties of subordination were used. In this note, for certain values of $t$, we use our specific knowledge of $F$, together with Theorem A, to obfain coefficient bounds for functions $f \in S(t)$. More specifically, we prove

THEOREM 1. *Let $t$ be given, $1/4 \leqq t \leqq \infty$. Let $F(z) = F(z,t) = z + \sum_{k=2}^{\infty} A_k(t) z^k$, $z \in K$, be as in (1.3) and (1.4). Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, $z \in K$, be in $S(t)$. If $n$ a positive integer is given $(n > 2)$, then there exist $\alpha_n$, $\beta_n$ satisfying $1/4 < \alpha_n \leqq 1$, $1 \leqq \beta_n < \infty$, with the property that*

$$|a_n| \leqq A_n(t), \tag{1.5}$$

*whenever $1/4 \leqq t < \alpha_n$ or $\beta_n < t \leqq \infty$. $\alpha_n$ and $\beta_n$ may be chosen in such a way that equality holds in (1.5) only if $f(z) = \eta^{-1} F(\eta z)$, $z \in K$, for some $\eta$, $|\eta| = 1$. In particular*

$$|a_3| \leqq A_3(t) \text{ if } 1/4 \leqq t \leqq 1 \text{ or } 5 < t \leqq \infty, \tag{1.6}$$

$$|a_4| \leqq A_4(t) \text{ if } 1/4 \leqq t \leqq 1 \text{ or } 12.259 \leqq t \leqq \infty. \tag{1.7}$$

*Equality holds in (1.6) and (1.7) only if $f(z) = \eta^{-1} F(\eta z)$, $z \in K$, for some $\eta$, $|\eta| = 1$.*

Let $f$ and $t$ be as in Theorem 1. We note that the inequality $|a_2| \leqq A_2(t)$, $1/4 \leqq t \leqq \infty$, is an easy consequence of Theorem A (see [2]). We also note for $1 \leqq t \leqq e$ that $|a_3| \leqq 1 - t^{-2}$, where equality holds for the function $f \in S(t)$ defined by $f(z) = F(z^2, t^2)^{1/2}$, $z \in K$. This inequality is due to Tammi [10]. The problem of finding a sharp upper bound for $|a_3|$ when $f \in S(t)$, $e < t < 5$, is still open. However, Barnard [1] has shown that the function which maximizes $|a_3|$ in $S(t)$ is either $F$ or a function which maps $K$ onto a domain whose boundary consists $\{w : |w| = t\}$ and two radial slits of equal length.

We remark that several authors have considered similar problems in the class $U(t)$ of normalized univalent functions $f$ (i.e., $f(0) = 0$,

$f'(0) = 1$) bounded above by $t$, $1 < t < \infty$. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $z \in K$, is in $U(t)$,then Schiffer and Tammi **[6]** showed that $|a_4| \leqq A_4(t)$, for $t \geqq 33\,1/3$. If in addition $f$ has real coefficients, then Singh **[8]** proved that $|a_4| \leqq A_4(t)$ for $t \geqq 11$. Moreover, Schiffer and Tammi **[7]** have proved for each positive integer $n \geqq 2$, that there exists $\delta_n$, $1 < \delta_n < \infty$, with the following property: If $f \in U(t)$ and $1 < t \leqq \delta_n$, then

$$|a_n| \leqq \frac{2}{n-1}(1 - t^{1-n}).$$

Here equality holds for $f(z) = F(z^{n-1}, t^{n-1})^{1/(n-1)}$, $z \in K$, which in fact is in $S(t)$. Hence the above inequality is also sharp for functions in $S(t)$ when $1 < t \leqq \delta_n$. Finally we remark that Schiffer and Tammi **[6]** have shown that if suffices to take $\delta_4 \leqq 34/19$.

**2.** *Proof of Theorem* 1. Let $G$, $\omega$, be analytic in $K$ and suppose that

$$\omega(0) = 0, \tag{2.1}$$

$$|\omega(z)| \leqq 1, z \in K. \tag{2.2}$$

Put $g(z) = G[\omega(z)]$, $z \in K$. Suppose that $G(z) = \sum_{k=1}^{\infty} c_k z^k$, and $g(z) = \sum_{k=1}^{\infty} b_k z^k$. Then Rogosinski **[5**, Thm. VI] proved

THEOREM B. *Let $n$ be a fixed positive integer. If $c_n > 0$ and if there exists an analytic function $P$ in $K$ with positive real part satisfying*

$$P(z) = \frac{c_n}{2} + c_{n-1}z + c_{n-2}z^2 + \cdots + c_1 z^{n-1} + \sum_{k=n}^{\infty} d_k z^k$$

*for $z \in K$, then $|b_n| \leqq |c_n|$. Equality can occur only if $g(z) = G(\eta z)$ for some $\eta$, $|\eta| = 1$, or if $n > 1$ and $P$ has the form,*

$$P(z) = \sum_{i=1}^{J} \lambda_i \left(\frac{1 + \varepsilon_i z}{1 - \varepsilon_i z}\right), z \in K, \tag{2.3}$$

*where $\lambda_i > 0$, $|\varepsilon_i| = 1$, $1 \leqq i \leqq J$, and $J \leqq n - 1$.*

Furthermore, Carathéodory (see Tsuji **[11**, Ch. 4 §7]) proved

THEOREM C. *The function $P$ in Theorem B exists if and only if the $n$ by $n$ matrix*

$$\begin{pmatrix} c_n & c_{n-1} & c_{n-2} & \cdots & c_1 \\ c_{n-1} & c_n & c_{n-1} & \cdots & c_2 \\ c_{n-2} & c_{n-1} & c_n & \cdots & c_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_1 & c_2 & c_3 & \cdots & c_n \end{pmatrix}$$

*is positive semi definite. If $P$ exists, then $P$ has the form* (2.3) *only if the above matrix has determinant zero.*

We now use Theorems A, B, and C to prove Theorem 1. Let $t$ be fixed, $1/4 \leqq t \leqq \infty$, and $f \in S(t)$. Then Theorem A implies there exists a function $\omega$ satisfying (2.1) and (2.2) for which $f(z)/z = F[\omega(z)]/\omega(z)$, $z \in K$. Hence we may use Theorems B and C with $g(z) = f(z)/z) - 1$, $G(z) = (F(z)/z) - 1$, $z \in K$, and $c_i = A_{i+1}(t)$, $1 \leqq i \leqq n-1$, to prove Theorem 1. To do so we shall want some notation.

Let $n$ and $k$ be fixed positive integers satisfying $2 \leqq k \leqq n$. Let $\delta(k,n,t)$ be the $k-1$ by $k-1$ matrix

$$\delta(k,n,t) = \begin{pmatrix} A_n(t) & A_{n-1}(t) & \cdots & A_{n-k+2}(t) \\ A_{n-1}(t) & A_n(t) & \cdots & A_{n-k+3}(t) \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ A_{n-k+2}(t) & A_{n-k+3}(t) & \cdots & A_n(t) \end{pmatrix} \tag{2.4}$$

Let $|\delta(k,n,t)|$ denote the determinant of $\delta(k,n,t)$. Then it is well known (see Hohn [3, Thm. 9, 17.3]) that $\delta(n,n,t)$ is positive definite if and only if $|\delta(k,n,t)| > 0$ for $2 \leqq k \leqq n$.

We note that $A_n(\infty) = A_n(1/4) = n$ for $n \geqq 2$. Using this fact we obtain that $|\delta(k,n,\infty)| = |\delta(k,n,1/4)| = (2n+2-k)\ 2^{k-3}$ for $2 \leqq k \leqq n$ and $n > 2$. Since (1.3) and (1.4) imply $A_n$ is continuous as a function of $t$, $1/4 \leqq t \leqq \infty$, it follows that

$$\lim_{t\to\infty} |\delta(k,n,t)| = |\delta(k,n,\infty)| = \lim_{t \to 1/4} |\delta(k,n,t)| > 0$$

for each positive integer $n > 2$ and $2 \leqq k \leqq n$. From this inequality and our previous remark we see that $\delta(n,n,t)$ is positive definite for sufficiently large $t$ and $t$ near 1/4, say $1/4 \leqq t < \alpha_n$, $\beta_n < t \leqq \infty$. Using Theorems A, B, and C, it follows that (1.5) is true.

To prove (1.6) and (1.7) we make some explicit calculations. The case $t = 1$ is trivial since then $S(t)$ consists only of the identity function. First from (1.4) we find for $x = t^{-1}$, and $1 < t \leqq \infty$, that

$$\begin{aligned} A_2(t) &= 2(1-x), \\ A_3(t) &= (3-5x)(1-x), \\ A_4(t) &= (4+14x^2-16x)(1-x). \end{aligned} \tag{2.5}$$

Second if $1/4 \leqq t < 1$ and $a = 2b^2 - 1$ [$b$ as in (1.3)], then from (1.3) we get

$$\begin{aligned} A_2(t) &= 1+a, \\ A_3(t) &= (1+a)(5+a)/4, \\ A_4(t) &= (1+a)(17+6a+a^2)/12. \end{aligned} \tag{2.6}$$

Here $-1 < a \leqq 1$.

To prove (1.6) it suffices, by the previous argument, to show that $A_2(t) > 0$ and

$$|\delta(3,3,t)| = A_3(t)^2 - A_2(t)^2 > 0$$

for $5 < t < \infty$ or $1/4 \leqq t < 1$. From (2.5) and (2.6) we see that these inequalities are valid for the above values of $t$. To prove (1.7), we need to show that $\delta(3,4,t) > 0$, $\delta(4,4,t) > 0$, for the stipulated values of $t$ in Theorem 1. To do this we consider two cases. If $1 < t \leqq \infty$, and $x = 1/t$, then from (2.4) and (2.5) we have

$$|\delta(4,4,t)| = (1-x)^3 \begin{vmatrix} 4+14x^2-16x & 3-5x & 2 \\ 3-5x & 4+14x^2-16x & 3-5x \\ 2 & 3-5x & 4+14x^2-16x \end{vmatrix}$$

Adding the second row to the first and third rows we get

$$|\delta(4,4,t)| = (1-x)^5 \begin{vmatrix} 7(1-2x) & 1(1 \times 2x) & 5 \\ 3-5x & 4+14x^2-16x & 3-5x \\ 5 & 7(1-2x) & 7(1-2x) \end{vmatrix}$$

Evaluating this determinant we obtain

$$|\delta(4,4,t)| = 4(1-x)^5(1-7x)[3-47x+126x^2-98x^3] > 0$$

for $12.259 \leqq t \leqq \infty$. It is easily checked that $|\delta(3,4,t)| = A_4^2(t) - A_3^2(t) > 0$ for $12.259 \leqq t \leqq \infty$. Hence (1.7) is true for $12.259 \leqq t \leqq \infty$.

If $1/4 \leqq t < 1$, then from (2.4), (2.6), we obtain

$$(12)^3|\delta(4,4,t)| = (1+a)^3 \begin{vmatrix} 17+6a+a^2 & 3(5+a) & 12 \\ 3(5+a) & 17+6a+a^2 & 3(5+a) \\ 12 & 3(5+a) & 17+6a+a^2 \end{vmatrix}$$

Subtracting the second row from the first and third rows, we get

$$(12)^3|\delta(4,4,t)| = (1+a)^5 \begin{vmatrix} a+2 & -a-2 & -3 \\ 3(5+a) & 17+6a+a^2 & 3(5+a) \\ -3 & -a-2 & a+2 \end{vmatrix}$$

Adding six times the first and third rows to the second of this determinant, we find that

$$(12)^3|\delta(4,4,t)| = (1+a)^6 \begin{vmatrix} a+2 & -a-2 & -3 \\ 9 & a-7 & 9 \\ -3 & -a-2 & a+2 \end{vmatrix}$$

Evaluating this determinant we obtain

$$(12)^3|\delta(4,4,t)| = (1+a)^6(a^3+15a^2+93a+215) > 0$$

for $-1 < a \leqq 1$. Hence $|\delta(4,4,t)| > 0$ for $1/4 \leqq t < 1$. It is easily checked that $|\delta(3,4,t)| > 0$ for $1/4 \leqq t < 1$. We conclude that (1.7) is true for $1/4 \leqq t < 1$. The proof of Theorem 1 is now complete.

Finally we remark for $1/4 \leqq t < 1$ that

$$48A_5(t) = (1+a)(74+38a+10a^2-2a^3) < 48A_4(t)$$

for $t$ near 1, $t < 1$. It follows that $|\delta(3,5,t)| < 0$ for $t$ near 1, $t < 1$. Hence our method does not imply for all $t$, $1/4 \leqq t \leqq 1$, that

$|a_5| \leqq A_5(t)$. However, it is still possible our method implies that $\alpha_n$ in Theorem 1 can be chosen independent of $n$.

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