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MAPS WITH 0-DIMENSIONAL CRITICAL SET

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Let $f: M^n \to N^p$ be C^n with $n - p = 0$ or 1, let $p \geqq 2$, and let $R_{p-1}(f)$ be the critical set of f. If $\dim(R_{p-1}(f)) \leqq 0$, then (1.1) at each $x \in M^n$, f is locally topologically equivalent to one of the following maps:

(a) the projection map $\rho: R^n \to R^p$,

(b) $\sigma: C \to C$ defined by $\sigma(z) = z^d (d = 2, 3, \cdots)$, where C is the complex plane, or

(c) $\tau: C \times C \to C \times R$ defined by $\tau(z, w) = (2z \cdot \bar{w}, |w|^2 - |z|^2)$, where $\bar{w}$ is the complex conjugate of w.

Under the additional hypothesis that $\dim(f(R_{p-1}(f))) \leqq p-2$ this result was proved in an earlier paper of the authors. They show here that $\dim(R_{p-1}(f)) \leqq 0$ implies something like $\dim(f(R_{p-1}(f))) \leqq p - 2$.

For general background material, the reader is referred to that earlier paper [5]. The *branch set* B_f [5, p. 616, (1.5)] is the set of points at which f fails to be locally topologically equivalent to ρ. A map $g: J^{n-m} \times R^m \to L^{p-m} \times R^m$ is called a *layer map* if for each $t \in R^m$, $g(J^{n-m} \times \{t\}) \subset L^{p-m} \times \{t\}$.

1.2. Outline of the proof. We suppose that f is not an open map, and from some technical differential lemmas of §3 obtain in (3.4) by restriction and change of coordinates a layer map satisfying the hypotheses of (2.1). By that lemma $\dim(B_f) = p - 1$, so that $\dim(R_{p-1}(f)) = p - 1$, contradicting the hypothesis of (1.1). Thus f *is* open, and from the local structure for open maps given in [7] we conclude in (4.1) that $\dim(f(B_f)) \leqq p - 2$. This is (essentially) the additional hypothesis assumed in [5], and our conclusion results. A global structure theorem is also given (4.5).

2. A topological lemma.

In order to read the proof of (2.1) the reader will need to have at hand the definition and certain properties of spoke sets [7, (2.1), (2.2), (2.3)].

LEMMA 2.1. *Let $f: D^2 \times R^{p-1} \to R \times R^{p-1}$ be a layer map with $B_f \neq \emptyset$, $f(\partial D^2 \times \{t\})$ a single point not in $f(B_f)$, and $\dim(B_f \cap (D^2 \times \{t\})) = \dim(f(B_f \cap (D^2 \times \{t\}))) \leqq 0$ for each $t \in R^{p-1}$. Then $\dim B_f = p - 1$.*

Proof. The last hypothesis implies that $\dim f(B_f) \leqq p - 1$ [9, p.

44, Theorem IV 3], so that $\dim B_f \leqq p-1$ [9, p. 91, Theorem VI 7]. If $p=1$ and $B_f=\emptyset$, then f is open and a contradiction results from [7, (3.1)(b) or (d)]. Thus, for $p=1$ $\dim B_f=0$, i.e., $p-1$. Hence we may suppose that $p \geqq 2$, and will prove that $\dim B_f \geqq p-1$.

Let $I=[0,1]$, let $I^{p-1} \subset I^p$ be $\{(x_1, x_2, \cdots, x_p): x_p=0\}$, let $r=0,1,\cdots,p-1$, and, for $a \in I^{p-1}$, let $\Gamma_{a,r}=\{x \in I^{p-1}: x_i=a_i \text{ for } i \geqq r+1\}$. For

$$X \subset \Gamma_{a,r} \quad \text{and} \quad \alpha > 0 ,$$

let $X(r,\alpha)=\{x \in I^{p-1}: (x_1,\cdots,x_r,a_{r+1},\cdots,a_{p-1}) \in X \text{ and } |x_i-a_i|<\alpha \text{ for } i \geqq r+1\}$. Thus $\Gamma_{a,r}(r,\alpha)=\{x \in I^{p-1}: |x_i-a_i|<\alpha \text{ for } i \geqq r+1\}$.

Consider statement S_r: (1) for every $\varepsilon>0$ and $a \in I^{p-1}$, there are a triangulation $\mathfrak{T}$ of the r-cell $\Gamma_{a,r}$ and $\alpha>0$, and (2) for every closed r-simplex σ of $\mathfrak{T}$, there are spoke sets $L_{j,\sigma}(j=0,1,\cdots,q(\sigma))$ satisfying conclusions (i)–(vi) of [7, (2.1) and (2.2)] with W replaced by $\mathrm{Cl}[\sigma(r,\alpha)]$ and $E=B_f \cap (D^2 \times I^{p-1})$. Moreover, (3) let σ and τ be closed r-simplices of $\mathfrak{T}$, and let $D^2 \times \mathrm{Cl}[(\sigma \cap \tau)(r,\alpha)]$ be denoted by T. Then, for any $L_{i,\sigma}$ and $L_{j,\tau}$, one of the following statements is true: $L_{i,\sigma} \cap T=L_{j,\tau} \cap T$, $L_{i,\sigma} \cap T \subset (L_{j,\tau}-\Omega_{j,\tau}) \cap T$, $L_{j,\tau} \cap T \subset (L_{j,\sigma}-\Omega_{j,\sigma}) \cap T$, or $L_{i,\sigma} \cap (L_{j,\tau}-\Omega_{j,\tau}) \cap T=\emptyset$.

Since $\Gamma_{a,0}=\{a\}$ and $\{a\}$ is the only 0-simplex of T, statement S_0 follows immediately from [7, (2.2)]. We will suppose that S_r is true $(r<p-1)$ and deduce S_{r+1}.

Let $\varepsilon>0$ and $a \in I^{p-1}$ be given. For $[u,v] \subset R$ and $\eta>0$, let

$$\Psi(u,v,\eta)=\{x \in I^{p-1}: u<x_{r+1}<v \text{ and } |x_i-a_i|<\eta \text{ for } i>r+1\} .$$

If $c \in \Gamma_{a,r+1}$, then $\Gamma_{c,r} \subset \Gamma_{a,r+1}$ and $\Gamma_{c,r}(r,\eta)=\Psi(c_{r+1}-\eta, c_{r+1}+\eta, \eta)$. For $c \in \Gamma_{a,r+1}$, let $\alpha(c)>0$, $\mathfrak{T}(c)$, and $\{L_{c,j,\sigma}\}$ be as given by S_r for ε (and a replaced by c). There are $c(i)(i=1,2,\cdots,m)$ such that $\{\Gamma_{c(i),r}(r,\alpha(c_i))\}$ covers $\Gamma_{a,r+1}$. We may suppose that $\{c_{r+1}(i)\}$ are in increasing order and the cover is minimal. If the open interval $(c_{r+1}(i)-\alpha(c(i)), c_{r+1}(i)+\alpha(c(i))$ is denoted by A_i, then $0 \in A_1-\bigcup_{i\neq 1}A_i$, $1 \in A_m-\bigcup_{i\neq m}A_i$, and $A_i \cap A_j \neq \emptyset$ if and only if $j=i-1, i$, or $i+1$. Choose $b(i) \in \Gamma_{a,r+1}$, $0<b_{r+1}(i)<1$, and $\gamma>0$ so that the intervals $F_i=[b_{r+1}(i)-\gamma, b_{r+1}(i)+\gamma]$ are mutually disjoint and $F_i \subset A_i \cap A_{i+1} \cap (0,1)(i=1,2,\cdots,m-1)$.

Let $\Omega=\bigcup_{i,j,\sigma}\Omega_{c(i),j,\sigma}$. Since $B_f \cap \Omega=\emptyset$ (by S_r (2) (iii) and (iv)), there is a δ with $0<\delta<\min(\varepsilon, d(B_f,\Omega))$ (d is distance). Let $\alpha(b(\mathrm{i}))>0$, $\mathfrak{T}(b(i))$, and $\{L_{b(i),j,\sigma}\}$ be as given by S_r for ε replaced by δ and a replaced by $b(i)(i=1,2,\cdots,m-1)$; let $\beta=\min\{\alpha(b(i)), \alpha(c(i)), \gamma\}$. By S_r (2) (vi) each $\dim L_{b(i),j}<\delta<d(B_f,\Omega)$ and by S_r (2) (iv) $B_f \cap L_{b(i),j} \neq \emptyset$; thus (*) if

$$(D^2 \times \Gamma_{a,r+1}(r, \beta)) \cap L_{b(i),j,\sigma} \cap L_{c(h),k,\tau} \neq \emptyset ,$$

then $(D^2 \times \Gamma_{a,r}(r, \beta)) \cap L_{b(i),j,\sigma} \subset (D^2 \times \Gamma_{a,r}(r, \beta)) \cap (L_{c(h),k,\tau} - \Omega_{c(h),k,\tau})$.

Let $d(t)(t = 1, 2, \cdots, 2m)$ be the numbers $0, 1, b_{r+1}(i) - \beta$, and $b_{r+1}(i) + \beta(i = 1, \cdots, m - 1)$ in increasing order. Then $\Psi(d(2i - 1), d(2i), \beta)$ (resp., $\Psi(d(2i), d(2i + 1), \beta)$) is contained in $\Gamma_{c(i),r}(r, \alpha(c(i)))$ (resp., $\Gamma_{b(i),r}(r, \alpha(b(i)))$).

For each closed r-simplex σ of $\mathfrak{T}(c(i))$ (resp., $\mathfrak{T}(b(i))$), let $\Sigma \subset \Gamma_{a,r+1}$ be the closed $(r + 1)$-cell defined by $x \in \Sigma$ if and only if $(x_1, \cdots, x_r, a_{r+1}, \cdots, a_{p-1}) \in \sigma$, $d(2i - 1) \leqq x_{r+1} \leqq d(2i)$ (resp., $d(2_i) \leqq x_{r+1} \leqq d(2i + 1)$), and $x_i = a_i$ for $i > r + 1$. There is a triangulation $\mathfrak{T}$ of $\Gamma_{a,r+1}$ such that each such Σ is a subpolyhedron [13, Chapter 1, p. 5]. For each closed $(r + 1)$-simplex ρ of $\mathfrak{T}$, there is an r-simplex σ of $\mathfrak{T}(c(i))$ or $\mathfrak{T}(b(i))$ with $\rho \subset \Sigma$. Define $L_{j,\rho} = L_{j,\sigma} \cap (D^2 \times \rho(r + 1, \beta))$ $(j = 1, 2, \cdots, q(\rho) = q(\sigma))$. It follows that S_{r+1} is satisfied for ε and a, with $\beta > 0$, $\mathfrak{T}$, and $\{L_{j,\rho}\}$ (conclusion (3) follows from (*) and S_r (3)).

Thus S_{p-1} *is true* for(say) 0 and any $\varepsilon > 0$; note that $\Gamma_{0,p-1} = I^{p-1}$ itself, and α does not arise in this case.

Let $e = 1, 2, \cdots$. Let $\mathfrak{T}_e$ be the triangulation of I^{p-1} and let $\{L_{j,\sigma,e}\}$ be as given in S_{p-1} for $\varepsilon = 1/e$, let $L_e = \bigcup_{j,\sigma} L_{j,\sigma,e}$, and let $\Omega_e = \bigcup_{j,\sigma} \Omega_{j,\sigma,e}$. Each $\mathfrak{T}_e$ is rectilinear in I^{p-1}, so we may suppose that each $\mathfrak{T}_{e+1}$ is a subdivision of $\mathfrak{T}_e$.

Define an equivalence relation $\sim$ on L_e by: for every $a \in I^{p-1}$, σ, and j, and for every $u, v \in L_{j,\sigma,e} \cap (D^2 \times \{a\})$, $u \sim v$. Let Y_e be the resulting identification space, and let $\omega_e\colon L_e \to Y_e$ be the identification map. Let $L_e \cap (D^2 \times \partial I^{p-1})$ be denoted by G_e, and $\omega_e(G_e)$ by ∂Y_e. Then $\omega_e\colon (L_e, G_e) \to (Y_e, \partial Y_e)$ is a homotopy equivalence, Y_e is a $(p - 1)$-dimensional finite polyhedron, viewed as a cell complex [13, Chapter 1, p. 5], its closed $(p - 1)$-cells are $\omega_e(L_{j,\sigma,e})$, their interiors $\omega_e(L_{j,\sigma,e} \cap (D^2 \times \operatorname{int} \sigma)) = \gamma_{j,\sigma,e}$ are mutually disjoint for distinct pairs (j, σ).

With the index ξ of [7, (2.1)] $\sum_{j,\sigma} \xi(L_{j,\sigma,e}) \cdot \gamma_{j,\sigma,e}$ is a $(p - 1)$-chain β_e of $(Y_e, \partial Y_e)$. From the index formula [7, (2.3)] and from (2) (v) and (3) in S_{p-1} (note that $\mathrm{Cl}\,[(\sigma \cap \tau)(\alpha)]$ is merely $\sigma \cap \tau$ in this case), it follows that β_e is a cycle of $(Y_e, \partial Y_e)$. Since $\xi(D^2 \times \{s\}) = 1$, it follows again from the index formula that $\sum_j \xi(L_{j,\sigma}) = 1$ for each σ, so that $\beta_e \neq 0$. Since $\dim Y_e = p - 1$, β_e defines a nonzero element of $H_{p-1}(Y_e, \partial Y_e; Z) \approx H_{p-1}(L_e, G_e, Z)$ (Z the ring of integers). Let $\eta_e = \omega_{e*}^{-1}(\{\beta_e\}) \in H_{p-1}(L_e, G_e; Z)$.

Since $\Omega_e \cap B_f = \emptyset$ (by S_{p-1} (2) (iv)), there exists $\delta(e)$ with $0 < \delta(e) < d(\Omega_e, B_f)$ $(e = 1, 2, \cdots)$, and there is a subsequence $\{e(k)\}$ such that $e(1) = 1$ and $1/e(k + 1) < \min \{\delta(e(i))\colon i \leqq k\}$ $(k = 1, 2, \cdots)$. For every $L_{j,\sigma,e(k+1)}$, there are a unique $\tau \in T_{e(k)}$ with $\sigma \subset \tau$ and $x \in B_f \cap L_{j,\sigma,e(k+1)}$ by S_{p-1} (2) (iv). For a unique i, $x \in L_{i,\tau,e(k)}$ by S_{p-2} (2) (iv) and (V), and from the size of $1/e(k + 1)$ and S_{p-1} (2) (vi), (†) $L_{j,\sigma,e(k+1)} \subset$

$L_{i,\tau,e(k)}$. Let $\lambda_{k+1}: (L_{e(k+1)}, G_{e(k+1)}) \to (L_{e(k)}, G_{e(k)})$ be inclusion. From (†) and the index formula [7, (2.3)] it follows that $\lambda^*_{k+1}(\eta_{e(k+1)}) = \eta_{e(k)}(\neq 0)$. Thus the inverse limit of $\{\eta_{e(k)}\}$ is nonzero, so that the Čech homology group $H_{p-1}(\bigcap_e L_e, \bigcap_e G_e; Z) \neq 0$ by the Continuity Theorem. Hence

$$\dim (\bigcap_e L_e) \geqq p - 1$$

[9, p. 152, Theorem VIII 4], and since $\bigcap_e L_e \subset B_f(S_{p-1}$ (2) (iv) and (vi)), $\dim B_f \geqq p - 1$.

3. Differential lemmas. The following two lemmas are generalizations of lemmas that have been used repeatedly, and these generalizations will also be used elsewhere.

LEMMA 3.1. *Let $f: M^n \to N^p$ be C^m, let K^q be a C^m q-manifold ($m = 1, 2, \cdots$; or $m = \infty$; or $m = \omega$; $q = 0, 1, \cdots, p-1$), let ρ be a C^m diffeomorphism of a region in N^p onto $K^q \times R^{p-q}$, and let Ω be a nonempty compact subset of $f^{-1}(\rho^{-1}(K^q \times \{0\}))$. If $f \mid \Omega$ is transverse regular on $\rho^{-1}(K^q \times \{0\})$, then there are $\varepsilon > 0$, a $C^m(n - p + q)$-manifold L, and a C^m diffeomorphism σ of $L \times S(0, \varepsilon)$ onto a neighborhood of Ω in M^n such that $\rho \circ f \circ \sigma$ is a layer map.*

This is proved in [6, (4.1)] and is a generalization of [8, p. 80, (3.5)] and [3, p. 376, (2.7)]. The condition that "$f \mid \Omega$ is transverse regular" means that f is transverse regular at x for each $x \in \Omega$.

LEMMA 3.2. *Let $q = 1, 2, \cdots$, let $f: M^n \to N^p$ be a C^r map with $\max(n - q + 1, 1) \leqq r \leqq \infty$, let $\Omega \subset M^n$ be compact, and let $Y \subset N^p$ be closed, with $\dim Y \geqq q$. Then for some m ($m = 0, 1, \cdots, p - q$) there is a C^r embedding λ of $S^m \times R^{p-m}$ in N^p such that $f \mid \Omega$ is transverse regular on $\lambda(S^m \times \{t\})$ and $\lambda(S^m \times \{t\}) \cap Y \neq \emptyset$ for each $t \in R^{p-m}$.*

If Ω is omitted, "$f \mid \Omega$ is transverse regular" is replaced by "f is transverse regular", and f is assumed proper, this is [8, p. 80, (3.7)]. The proof is an immediate generalization of that proof. (Although we do not need it in this paper, the same comments apply to [8, p. 82, (3.8)], except that J need not be compact.)

DEFINITION 3.3. Let K^n and L^p be C^r-manifolds with nonempty boundary, and let $f: K^n \to L^p$ be a $C^r (r \geqq 1)$ proper map with $f^{-1}(\partial L^p) = \partial K^p$ and $f(R_{p-1}(f)) \subset \operatorname{int} L^p$. Let $D(K^n)$ and $D(L^p)$ be the doubles K^n and L^p, respectively [10, p. 52, (5.10) and p. 62, (6.3)]. We now define a C^r map $g: D(K^n) \to D(L^p)$, called a *double of* f, such that the restriction of g to each half is C^r equivalent to f [5, p. 616, (1.3)].

Let $K_i = K \times i$, let $L_i = L \times i$, and let $f_i: K_i \to L_i$ be defined by $f_i(x, i) = (f(x), i)(i = 0, 1)$. Let $J_0 = [0, 1)$ and $J_1 = (-1, 0]$. There is an open neighborhood U of ∂L in L disjoint from $f(R_{p-1}(f))$ and C^r diffeomorphisms $\psi_i: U_i = U \times i \to \partial L_i \times J_i$ [10, p. 51, (5.9)]. Let $\alpha_i: f_i^{-1}(U_i) \to U_i$ and $\beta_i: \partial K_i \to \partial L_i$ be the restrictions of f_i.

There exist manifolds $V_i = V_i^n$ with $\partial V_i = \varnothing$ and $f^{-1}(U_i) \subset V_i$ and $W_i = W_i^p$ with $\partial W_i = \varnothing$ and $U_i \subset W_i$, and a C^r extension $\gamma_i: V_i \to W_i$ of α_i. By restricting γ_i we may suppose that it is proper. Now γ_i is the projection map of a C^r bundle (e.g. from (3.1) with K a single point), so that α_i and β_i are also. Thus there are diffeomorphisms $\phi_i: f_i^{-1}(U_i) \to \partial K_i \times J_i$ such that $\psi_i \circ \alpha_i = (\beta_i \times \iota) \circ \phi_i$ (where ι is the identity map on J_i) [11, p. 53, (11.4)].

We may define the (C^r structures on the) doubles $D(K^n)$ and $D(L^p)$ using the maps ϕ_i and ψ_i (identify $(x, 0)$ in ∂K_0 with $(y, 1)$ in ∂K_1 if $\phi_0(x, 0)$ and $\phi_1(y, 1)$ have the same first coordinate), and let λ_i: $K_i \to D(K^n)$ and $\mu_i: L_i \to D(L^p)$ be the natural (C^r) embeddings. Define g by $g(x) = f_i(x)$ for $x \in K_i$. Clearly g is C^r except possibly on ∂K. If $U' = U_0 \cup U_1$ and $\psi: U' \to \partial L \times (-1, 1)$ and $\phi: g^{-1}(U) \to \partial K \times (-1.1)$ are defined by the ψ_i and ϕ_i, respectively, then $\psi \circ g | g^{-1}(U') = (\beta \times \iota) \circ \phi$ (where ι is the identity map on $(-1, 1)$ and $\beta = \beta_1 = \beta_2$), so that g is C^r everywhere.

LEMMA 3.4. *Let $f: M^n \to N^p$ be a C^n map with $n - p = 0$ or 1, $\dim B_f \leqq p - 2$, and $\dim (B_f \cap f^{-1}(y)) \leqq 0$ for each $y \in N^p$. Then f is open.*

Proof. In case $n = p$, f is light and the conclusion is given by [2, p. 94, (2.3)], so we may suppose that $n = p + 1$. Suppose that f is not open. Let E_f be the set of points at which f fails to be open, and let $x \in E_f$. According to [5, p. 622, (2.6)] there is a connected (not necessarily compact) manifold $K^{p+1} \subset M^{p+1}$ with boundary such that $x \in \operatorname{int} K^{p+1} (= K^{p+1} - \partial K^{p+1})$ and the closure $\bar{K}^{p+1}$ of K^{p+1} in M^{p+1} is compact; there is an open p-cell $D^p \subset N^p$ with $f(K^{p+1}) \subset D^p$; and the restriction map $g: K^{p+1} \to D^p$ is proper with $B_g \cap \partial K^{p+1} = \varnothing$. Let $\psi = g | \operatorname{int} K^{p+1}$, and let $\Omega \subset \operatorname{int} K^{p+1}$ be the compact set E_ψ. Since f is not open, $\dim \psi(E_\psi) \geqq p - 1$ [5, p. 623, (3.4)], and by (3.2) there is a C^{p+1} embedding $\lambda: S^m \times R^{p-m} \to D^p$ such that $\psi | \Omega$ is transverse regular on $\lambda(S^m \times \{t\})$ and $\lambda(S^m \times \{t\}) \cap \psi(E_\psi) \neq \varnothing$ for each $t \in R^{p-m}$ and $m = 0$ or 1. From (3.1) $m \neq 0$ and, for some $\varepsilon > 0$, the restriction of ψ to some neighborhood of E_ψ is C^{p+1} equivalent to the C^{p+1} layer map $\alpha: Q^2 \times R^{p-1} \to S^1 \times R^{p-1}$ with $E_\alpha \cap (Q^2 \times \{t\}) \neq \varnothing$ for every $t \in R^{p-1}$.

Since $B_\alpha \subset R_{p-1}(\alpha)$ (the Rank Theorem [5, p. 617, (1.6)], $\dim (\alpha(B_\alpha) \cap$

$(S^1 \times \{t\})) \leqq 0$ for each $t \in R^{p-1}$ (by Sard's theorem); and since

$$\dim (B_\alpha \cap \alpha^{-1}(u, t)) \leqq 0$$

for each $(u, t) \in S^1 \times R^{p-1}$ by hypothesis, $\dim (B_\alpha \cap (Q^2 \times \{t\})) \leqq 0$ [9, p. 91, Theorem VI 7].

Let $(q, s) \in E_\alpha \subset B_\alpha$ (we may suppose that $s = 0$), and let $T \subset Q^2 \times R^{p-1}$ be a closed $(p+1)$-cell neighborhood of $(q, 0)$. Since $\{(q, 0)\}$ is the component of $\alpha^{-1}(\alpha(q, 0))$ containing $(q, 0)$ [5, p. 622, (3.2)], there is an interval $I \subset S^1$ with $\alpha_0(q) \in \operatorname{int} I$ and $\delta > 0$ such that the component F of $\alpha^{-1}(I \times S(s, \delta))$ is contained in int T. We may suppose that the endpoints of I are regular values of α_0, and thus, for δ sufficiently small, of α_t for every $t \in S(0, \delta)$. Thus F is an n-manifold with boundary, and each $F_t = F \cap (Q^2 \times \{t\})$ is compact. Let G be the double of F, and let $\beta\colon G \to S^1 \times S(0, \delta)$ be the double of the proper map $\alpha | F\colon F \to I \times S(0, \delta)$ (3.3).

Choose an open 2-cell U with $q \in U$ and $U \times \{0\} \subset \operatorname{int} F_0 \subset G_0$, and choose η, $0 < \eta < \delta$, with $U \times S(0, \eta) \subset \operatorname{int} F \subset G$. There exists ξ, $0 < \xi < \eta$, and an interval $J \subset \operatorname{int} I \subset S^1$ such that $\beta_0(q) \in \operatorname{int} J$, the component X of $\beta^{-1}(J \times S(0, \xi))$ containing $(q, 0)$ is contained in $U \times S(0, \xi)$, and the end points of J are regular values of β_t for each $t \in S(0, \xi)$. Thus $X \cap (U \times \{0\})$, call it A^2, is a 2-disk with holes, and $\alpha_0(\partial A^2) \subset \partial J$.

We now apply [1, p. 196, (3.4)] to β, $K_0 = S^1 \times \{0\}$, $\Gamma_1 = J \times \{0\}$, $K_1 = \partial \Gamma_1$, and ρ the identity map. There exists ζ, $0 < \zeta < \xi$, and a C^{p+1} (layer) diffeomorphism ω of $\beta^{-1}(S^1 \times \{0\}) \times S(0, \zeta)$ onto $\beta^{-1}(S^1 \times S(0, \zeta))$ with $\omega(A^2 \times S(0, \zeta)) = X$. Let D be the closed 2-cell with $A^2 \subset D \subset U$ and $\partial D \subset \partial A^2$, and let $\gamma\colon D \times S(0, \zeta) \to \operatorname{int} I \times S(0, \zeta)$ be the restriction of $\beta \circ \omega$. Now $(0, q) \in E_\gamma \subset B_\gamma$ and by (2.1) $\dim B_\gamma = p - 1$, so that $\dim B_f \geqq p - 1$, and a contradiction results.

4. Conclusions.

PROPOSITION 4.1. *Let $f\colon M^{p+1} \to N^p$ be C^{p+1} with $B_f \neq \emptyset$,* $\dim B_f \leqq p - 2$, *and* $\dim (f^{-1}(y) \cap B_f) \leqq 0$ *for each $y \in N^p$. Then* $\dim B_f = p - 3$ *and there is a closed set $Y \subset B_f$ such that* $\dim Y < p - 3$ *and, for every $x \in B_f - Y$, f at x is locally topologically equivalent to*

$$\tau \times \mathrm{id}\colon R^4 \times R^{p-3} \longrightarrow R^3 \times R^{p-3} .$$

According to the Rank Theorem [5, p. 617, (1.6)] $B_f \subset R_{p-1}(f)$ and the following corollary results.

COROLLARY 4.2. *Let $f\colon M^{p+1} \to N^p$ be C^{p+1} with critical set $R_{p-1}(f)$, let* $\dim R_{p-1}(f) \leqq p - 2$, *and let* $\dim (f^{-1}(y) \cap R_{p-1}(f)) \leqq 0$ *for each $y \in N^p$. Then there is a closed set $Y \subset M^{p+1}$ such that* $\dim Y < p - 3$

and, for each $x \in M^{p+1} - Y$, f *at* x *is locally topologically equivalent to either the projection map* $\rho: R^{p+1} \to R^p$ *or to*

$$\tau \times \mathrm{id}: R^4 \times R^{p-3} \longrightarrow R^3 \times R^{p-3} .$$

Proof of (4.1). By (3.4) f is open, and $p \geqq 2$ since $B_f \neq \emptyset$ and $\dim B_f \leqq p - 2$. According to [7, (4.1) and (1.1)], if $f: M^{p+1} \to N^p$ is a C^3 open map with $\dim (B_f \cap f^{-1}(y)) \leqq 0$ for each $y \in N^p$, then there is a closed set $X \subset M^{p+1}$ such that $\dim f(X) \leqq p - 2$ and, for every $x \in M^{p+1} - X$, there is a natural number $d(x)$ with f at x locally topologically equivalent to the map

$$\phi_{d(x)}: C \times R^{p-1} \longrightarrow R \times R^{p-1}$$

defined by $\phi_{d(x)}(z, t) = (\mathscr{R}(z^{d(x)}), t)(\mathscr{R}(z^{d(x)})$ is the real part of the complex number).

Since $\dim B_f \leqq p - 2$ by hypothesis, $B_f \subset X$, so that $\dim f(B_f) \leqq p - 2$. Thus f satisfies the hypothesis of [5, p. 626, (4.7)]. (For $n = p + 1$ that proposition is identical with the present one except that the hypothesis $\dim B_f \leqq p - 2$ is replaced by $\dim f(B_f) \leqq p - 2$.)

COROLLARY 4.3. *If* $f: M^{p+1} \to N^p$ *is a* C^{p+1} *map with* $\dim B_f = 0$ *and* $p \geqq 2$, *then* $p = 3$ *and at each* $x \in B_f$, f *is locally topologically equivalent to* τ.

4.4. *Proof of* (1.1). From the Rank Theorem [5, p. 617, (1.6)] $B_f \subset R_{p-1}(f)$, and the conclusion for $n - p = 1$ results from (4.3). For $n = p \geqq 3$ $\dim (R_{p-1}(f)) \leqq 0$ implies $B_f = \emptyset$ [2, p. 94, (2.2)]; for $n = p = 2$, f is light open [2, p. 94, (2.3)], and so has the desired structure (e.g. by [2, p. 90, (1.10)]).

Let G be a compact, connected Lie group, and let M be a closed, connected, oriented G-manifold with orbit space a manifold. The action is called *almost free* if it is free except for the fixed point set F, and F is discrete nonempty set. In [4] Church and Lamotke classified such actions globally, up to equivariant homeomorphism (they also treated the smooth case): invariants are the oriented homeomorphism type of the orbit space and the number (which is even) of fixed points. This classification gives significance to the following corollary of (1.1), a global classification of maps with 0-dimensional critical set.

COROLLARY 4.5. *Let* M^{p+1} *and* N^p *be closed, connected, oriented manifolds, and let* $f: M^{p+1} \to N^p$ *be a* C^{p+1} *map with critical set* $R_{p-1}(f)$ *of dimension at most* 0. *Then there is a unique factorization* $f = h \circ g$, *where* $g: M^{p+1} \to K^p$ *is the orbit map of a topological* S^1 *free or almost free action on* M^{p+1} (*and thus is classified by* [4]), *and* $h: K^p \to N^p$ *is an* r*-to-*1 *covering map* ($r = 1, 2, \cdots$).

Proof. By (1.1) either the branch set $B_f = \emptyset$, or $p = 3$ and at each point of B_f f is locally topologically equivalent to τ, i.e., to the cone map of the Hopf fibration $\psi: S^3 \to S^2$ [5, p. 618, (1.10)]. According to [12, p. 64, (2.5)] there is a natural number k such that $f^{-1}(y)$ has exactly k components for each $y \in N^p - f(B_f)$, and at most k components for each $y \in f(B_f)$. From the local structure, $f^{-1}(y)$ has exactly k components for every $y \in N^p$, and thus according to [12, p. 63, (2.1)] there is a (unique) factorization $f = h \circ g$, where $g: M^{p+1} \to K^p$ is a C^{p+1} monotone map and $h: K^p \to N^p$ is an r-to-1 covering map.

In case $B_f = \emptyset$, $B_g = \emptyset$ also, so that g is a bundle map [5, p. 618, (1.9)] with fiber S^1. The structure group can be reduced to $S^1 = SO(2)$ [12, pp. 64–65], and thus g is the orbit map of a free S^1 action. In case $B_f \neq \emptyset$, the map $\alpha: M^{p+1} - B_g \to K^p - g(B_g)$ defined by restriction of g is also a free S^1 action; since B_g is discrete, g itself is the orbit map of an almost free action.

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