PACIFIC JOURNAL OF MATHEMATICS
Vol. 58, No. 1, 1975

ON THE STRUCTURE OF THE FOURIER-STIELTJES ALGEBRA

MARTIN E. WALTER

If G is a locally compact group, denote its Fourier-Stieltjes algebra by $B(G)$ and its Fourier algebra by $A(G)$. If G is compact, then $B(G) = A(G)$ and $\sigma(B(G))$, the spectrum of $B(G)$, is G. If G is not compact then $\sigma(B(G))$ contains partial isometries and projections different from e, the identity of G. More generally, $\sigma(B(G))$ is closed under operations that commute with "representing" and the "taking of tensor products". It is shown that $\sigma(B(G))$ contains a smallest positive element, z_F; and that $g \in G \subset \sigma(B(G)) \mapsto z_F g \in \sigma(B(G))z_F$ is an epimorphism of G into $\bar{G}$, the almost periodic compactification of G.

A structure theorem is given for the closed, bi-translation, invariant subspaces of $B(G)$. In so doing we introduce the concepts of inverse Fourier transform localized at π, and the standardization of π, where π is a continuous, unitary representation of G.

Introduction. In this paper we establish some facts about the structure of the Fourier-Stieltjes algebra, $B(G)$, of a locally compact group G, which were inspired by **[7]**, **[11]** and **[14]**. In particular, we apply the characterization of the (nonzero) spectrum of $B(G)$, $\sigma(B(G))$, obtained in Theorem 1 (ii) of **[17]** to investigate further the structure of this spectrum. As one of several applications, we relate the smallest, positive element of $\sigma(B(G))$ to the almost periodic compactification of G. It soon becomes apparent that a deep understanding of closed, bi-translation invariant subspaces (and more specially, sub-algebras and ideals) of $B(G)$ is needed. It is to this end that we introduce a canonical or standard form for any continuous, unitary representation π of G on Hilbert space, and with it the notion of the inverse Fourier transform "localized at π".

We follow the notational conventions of **[17]** and define in the text any new notations introduced.

The spectrum of $B(G)$. If $s \in \sigma(B(G))$ there are naturally associated two (norm-decreasing, algebra) endomorphisms of $B(G)$, viz., $\gamma_s : b \in B(G) \mapsto s \,.\, b \in B(G)$ and $\delta_s : b \in B(G) \mapsto b \,.\, s \in B(G)$ where, for example, $\langle x, s \,.\, b \rangle = \langle xs, b \rangle$ for all $x \in W^*(G) = B(G)'$. Letting $s \,.\, B(G) = \{s \,.\, b \in B(G) : b \in B(G)\} = \gamma_s(B(G))$, similarly for $B(G) \,.\, s$,

we observe that these are right, respectively left, translation invariant subalgebras of $B(G)$; where we adopt the convention that the right translate of $b \in B(G)$ by $g \in G$ is $b \,.\, g$ and $\langle x, b \,.\, g\rangle = \langle gx, b\rangle$ for all $x \in W^*(G)$. We also observe that the kernels of γ_s and δ_s are right, respectively left, translation invariant ideals of $B(G)$. In case $s = s^2$ we write, for example, $(e - s) \,.\, B(G) = \ker \gamma_s$, where e is the unit in $W^*(G)$. We should also observe that $s^2 = s$ implies that $s \,.\, B(G)$ and $(e - s) \,.\, B(G)$ are norm-closed. We now have the following:

PROPOSITION 1. *If $s \in W^*(G)$ is an idempotent, i.e., $s^2 = s$, then the following are equivalent:*

(i) $s \in \sigma(B(G))$;

(ii) $s \,.\, B(G)$ *is an algebra and* $(e - s) \,.\, B(G)$ *is an ideal in* $B(G)$;

(ii)′ γ_s *is an endomorphism*;

(iii) $B(G) \,.\, s$ *is an algebra and* $B(G) \,.\, (e - s)$ *is an ideal in* $B(G)$;

(iii)′ δ_s *is an endomorphism.*

Proof. That (i) implies (ii) and (ii)′ is immediate. We now show that (ii) $\Rightarrow$ (ii)′ $\Rightarrow$ (i). Consider that for $b_1, b_2 \in B(G)$,

$$\begin{aligned} s \,.\, (b_1 b_2) &= s \,.\, ((s \,.\, b_1 + (e - s) \,.\, b_1)(s \,.\, b_2 + (e - s) \,.\, b_2)) \\ &= s \,.\, (s \,.\, b_1 \, s \,.\, b_2) + s \,.\, (((e - s) \,.\, b_1)(b_2)) \\ &\qquad + s \,.\, ((s \,.\, b_1)(e - s) \,.\, b_2) \\ &= s \,.\, (s \,.\, b_1 \, s \,.\, b_2) \text{ (since } (e - s) \,.\, B(G) \text{ is an ideal and } s^2 = s) \\ &= s \,.\, b_1 \, s \,.\, b_2 \text{ (since } s \,.\, B(G) \text{ is an algebra),} \end{aligned}$$

hence (ii)′. Evaluation at e shows that $s \in \sigma(B(G))$, thus (ii)′ $\Rightarrow$ (i). The remainder of the proposition follows immediately by symmetry.

If $s^2 = s \in \sigma(B(G))$, we call $(e - s) \,.\, B(G)$ a right-prime or δ-prime ideal; similarly, $B(G) \,.\, (e - s)$ is called a left-prime or γ-prime ideal, where our terminology here is influenced by **[13]**. Note that a δ-prime ideal $I \subset B(G)$ has the property that if $(b_1 \,.\, g_1)(b_2 \,.\, g_2) \in I$ for all $g_1, g_2 \in G$, then either $b_1 \in I$ or $b_2 \in I$.

The following results show that $\sigma(B(G))$ is closed under certain operations, and a basis for generalizing some of the results of **[14]**, **[15]** on the structure of the spectrum of convolution measure algebras is thus obtained. Recall first that any operator s on Hilbert space has a left and right polar decomposition, viz., $s = v_\gamma \,|\, s \,|_\gamma$ where $|\, s \,|_\gamma = (s^* s)^{1/2}$ and $s = |\, s \,|_\delta \, v_\delta$ where $|\, s \,|_\delta = (s s^*)^{1/2}$. Also for later notational convenience let $\sigma(B(G))_+$ and $\sigma(B(G))_p$ denote the positive, hermitian elements and

(self-adjoint) idempotents in $\sigma(B(G))$, respectively. Note that idempotents of norm one are self-adjoint and that $v_\delta = v_\gamma$. We then have:

THEOREM 1. *If $s \in \sigma(B(G))$, then $v_\gamma, v_\delta, |s|_\gamma$, and $|s|_\delta$ are also in $\sigma(B(G))$.*

LEMMA. *If $s \in \sigma(B(G))_+$, then the positive square root, $s^{1/2}$, is also in $\sigma(B(G))_+$.*

Proof of Lemma. Let π_1 and π_2 be any two continuous unitary representations of G on Hilbert space, and by the same letters denote their canonical extensions to $W^*(G)$. Abusing notation again, let $\pi_1 \otimes \pi_2$ denote both the usual tensor product group representation of G and its canonical extension to $W^*(G)$. Now in $W^*(G)$ $s^{1/2}s^{1/2} = s$, and $\pi_1 \otimes \pi_2(s^{1/2}s^{1/2}) = (\pi_1 \otimes \pi_2(s^{1/2}))^2$. But $\pi_1 \otimes \pi_2(s) = \pi_1(s) \otimes \pi_2(s)$ since $s \in \sigma(B(G))$, cf., [17], Theorem 1, (ii); and $\pi_1(s) \otimes \pi_2(s) = (\pi_1(s^{1/2}) \otimes \pi_2(s^{1/2}))^2$. Thus by uniqueness of the positive square root, we have $\pi_1(s^{1/2}) \otimes \pi_2(s^{1/2}) = \pi_1 \otimes \pi_2(s^{1/2})$, hence $s^{1/2} \in \sigma(B(G))$ by [17] Theorem 1, (ii) again.

Proof of Theorem 1. We prove v_γ and $|s|_\gamma$ are in $\sigma(B(G))$, the remainder of the theorem follows by symmetry. Note first that $s^*s \in \sigma(B(G))$, since $s^*s \neq 0$, cf., [17] Theorem 1 (iii). Thus by the lemma $|s|_\gamma \in \sigma(B(G))$. Now again let π_1, π_2 be representations as above. We have $\pi_1 \otimes \pi_2(s) = \pi_1 \otimes \pi_2(v_\gamma)\pi_1 \otimes \pi_2(|s|_\gamma)$ and $\pi_1(s) \otimes \pi_2(s) = (\pi_1(v_\gamma) \otimes \pi_2(v_\gamma))(\pi_1(|s|_\gamma) \otimes \pi_2(|s|_\gamma))$. Since $s, |s|_\gamma \in \sigma(B(G))$ we have

$$\pi_1 \otimes \pi_2(v_\gamma)\,\pi_1(|s|_\gamma) \otimes \pi_2(|s|_\gamma)$$
$$= (\pi_1(v_\gamma) \otimes \pi_2(v_\gamma))(\pi_1(|s|_\gamma) \otimes \pi_2(|s|_\gamma))$$

Now $v_\gamma^* v_\gamma$ is the support of $|s|_\gamma$, by the definition of the polar decomposition. But it is easy to see that $\pi_1 \otimes \pi_2(v_\gamma)$ and $\pi_1(v_\gamma) \otimes \pi_2(v_\gamma)$ are partial isometries, both with initial projections equal to the support of $\pi_1(|s|_\gamma) \otimes \pi_2(|s|_\gamma)$. Thus again by uniqueness of the polar decomposition and [17] Theorem 1 (ii), we have $v_\gamma \in \sigma(B(G))$.

As corollaries of the method of argument in the foregoing proofs we have:

COROLLARY. *If $s \in \sigma(B(G))_+$, then $s^z \in \sigma(B(G))$ for all complex z with* $\operatorname{Re} z > 0$.

REMARK. We understand by s^0 the support projection of s, which is in $\sigma(B(G))$; and the map $z \mapsto s^z$ is analytic for $\operatorname{Re} z > 0$.

REMARK. Speaking loosely, the weakly compact *-semigroup $\sigma(B(G))$, see first corollary of Theorem 2, is closed under any operation that commutes with representing and the taking of tensor products. To see that "raising to the z power" has these properties when defined, application of the spectral theorem for self-adjoint operators will suffice; or alternatively apply a standard analytic function proof.

COROLLARY. *Let s and t be in $\sigma(B(G))_+$ and let $s \leqq t$, then there exists a unique $a \in \sigma(B(G))$ satisfying $s^{1/2} = at^{1/2}$, with support of a majorized by that of t.*

Proof. This follows from **[4]** Chap. 1, §1.6, Lemma 2.

We now show that $\sigma(B(G))_+$ has a smallest element z_F, which is a central idempotent.

THEOREM 2. $z_F = \sup\{z[\pi]: z[\pi] = $ *support in* $W^*(G)$ *of finite dimensional* (unitary) *representation* $\pi\}$. *Then z_F is a central projection in $W^*(G)$, and $z_F \in \sigma(B(G))_+$. Moreover if $s \in \sigma(B(G))_+$, we have $z_F s = z_F$,* i.e., $z_F \leqq s$.

Proof. It is clear that $z_F \,.\, B(G)$ is an algebra, since the tensor product of two finite dimensional representations is itself finite dimensional. That $(e - z_F) \,.\, B(G)$ is an ideal in $B(G)$ follows from **[13]**. Briefly, if $b_1 \in (e - z_F) \,.\, B(G)$ and $b_2 \in B(G)$, let π_{p_1} and π_{p_2} be the cyclic representations arising from, say, the left absolute values p_1 and p_2 which arise from the left polar decompositions of b_1 and b_2, respectively. Then $z[\pi_{p_1}]z_F = 0$, and thus $z[\pi_{p_1} \otimes \pi_{p_2}]z_F = 0$, by **[13]**. But $z[\pi_{p_1}] \,.\, b_1 = b_1$; and $z[\pi_{p_2}] \,.\, b_2 = b_2$. Hence $z[\pi_{p_1} \otimes \pi_{p_2}] \,.\, b_1 b_2 = b_1 b_2$, and thus $z_F \,.\, b_1 b_2 = 0$. Thus by Proposition 1, $z_F \in \sigma(B(G))$.

We now show that z_F is the smallest element in $\sigma(B(G))_+$. First consider the case where q is an idempotent in $\sigma(B(G))_+$. Now $z_F q$ is an idempotent in $\sigma(B(G))_+$ satisfying $z_F \geqq z_F q$, or else $z_F q = 0$. In the latter case $(e - q) \,.\, B(G)$ is an ideal of $B(G)$ that contains 1, the identity of $B(G)$; hence $q = 0$, which is impossible since $0 \notin \sigma(B(G))$. More generally, if $z_F q \neq z_F$, consider that $(e - z_F q) \,.\, B(G)$ is a closed, right translation invariant ideal in $B(G)$ which contains a positive definite function p which is a coefficient of a finite dimensional, irreducible representation π. If π is on Hilbert space H_π, there is an orthonormal

basis $\{\xi_i\}_{i=1}^{\dim \pi}$ of H_π so that, supposing $p = \omega_{\xi_1}, \omega_{\xi_1,\xi_i} \in (e - z_F q) \cdot B(G)$ $i = 1, 2, \cdots, \dim \pi$; but then

$$1 = \sum_{i=1}^{\dim \pi} \bar{\omega}_{\xi_1,\xi_i}\, \omega_{\xi_1,\xi_i} \in (e - z_F q) \cdot B(G)$$

where $\bar{\omega}_{\xi_1,\xi_i}(\pi(x)) = \overline{(\pi(x)\xi_1 \mid \xi_i)}$, the bar denoting complex conjugation, $i = 1, 2, \cdots \dim \pi$. Thus again we get $q = 0$, an impossibility. Thus z_F is the smallest idempotent in $\sigma(B(G))_+$. In general, let $s \in \sigma(B(G))_+$; and let $e_s = \text{weak-lim}_{n\to\infty} s^n$, the projection on the eigenspace of s corresponding to eigenvalue 1. Since $\sigma(B(G))$ is weakly compact, $e_s \in \sigma(B(G))_+$ (because $e_s \neq 0$). But then $z_F \leqq e_s \leqq s$, and we are done.

We can now refine [17] Theorem 1, (iii):

COROLLARY. *$\sigma(B(G))$ is a weakly compact *-semigroup.*

REMARK. The reader should be careful to note that $\sigma(B(G))$ is not a topological semigroup (in general) in the weak topology. However, $\sigma(B(G))$ is a topological semigroup in the strong topology (see discussion of topology following Proposition 3). Then $\sigma(B(G))$ is not (in general) compact in the strong topology, neither is * strongly continuous, though * is weakly continuous.

Proof. All that remains to be shown is that if $x, y \in \sigma(B(G))$, then $xy \neq 0$. But by Theorem 2 above, we have that z_F is smaller than either the support or range projections of x and y. Thus it is easy to see that $z_F xy \neq 0$, hence $xy \neq 0$.

The following corollary is stated to illustrate in the simplest case, a relationship between the topology of G and the idempotents in $\sigma(B(G))_+$.

COROLLARY. *G is compact if and only if the only central element in $\sigma(B(G))_p$ is e.*

Proof. If G is compact, $A(G) = B(G)$; and $\sigma(B(G))$ is G. Thus the only idempotent in $\sigma(B(G))$ is e. Conversely, let $s \in \sigma(B(G))$, and let $s = v_\gamma \mid s \mid_\gamma$. Then $z_F \leqq v_\gamma^* v_\gamma$, and $z_F \leqq v_\gamma v_\gamma^*$. But $z_F = e$ by hypothesis, hence v_γ is unitary, and $\mid s \mid_\gamma = e$. Thus $\sigma(B(G))$ is topologically isomorphic with G, e.g., G is compact, cf. [17] Theorem 1.

EXAMPLE. Consider the group $SL(2, \mathbf{R})$. In this case $z_F = z_0 =$ support of the trivial representation of G. We must always have $z_F \geqq z_0$, and this example shows that equality may be obtained.

Any analysis of the structure of a semigroup should include a discussion of its ideals, idempotents, and groups. To begin with we have:

PROPOSITION 2. *If* $s \in \sigma(B(G))$, *the principal left ideal* $\sigma(B(G))s = \sigma(B(G))|s|_\gamma = \{t \in \sigma(B(G)): t^*t \leqq s^*s\}$. *Similarly,*

$$s\sigma(B(G)) = |s|_\delta\sigma(B(G)) = \{t \in \sigma(B(G)): tt^* \leqq ss^*\}.$$

Proof. Clearly $\sigma(B(G))s \subset \{t \in \sigma(B(G)): t^*t \leqq s^*s\}$, since if $x \in \sigma(B(G))$, $\|x\|_{W^*(G)} = 1$, and $(xs)^*(xs) \leqq s^*s\,\|x\|^2_{W^*(G)}$. Now $s = v_\gamma|s|_\gamma$ implies $\sigma(B(G))s = \sigma(B(G))v_\gamma|s|_\gamma \subset \sigma(B(G))|s|_\gamma$. But $v_\gamma^*s = |s|_\gamma$ yields the opposite inclusion. Finally, suppose $t^*t \leqq s^*s$, $t \in \sigma(B(G))$. Then by the second corollary of Theorem 1, $|t|_\gamma = a|s|_\gamma$ for some $a \in \sigma(B(G))$. But then $t = v'_\gamma|t|_\gamma = v'_\gamma a|s|_\gamma$ is in $\sigma(B(G))|s|_\gamma = \sigma(B(G))s$. To get the corresponding "right-handed" proposition just observe that the * operation on $\sigma(B(G))$ induces a symmetry between right and left.

Letting $|s|^\infty_\gamma$ denote the projection on the eigenspace of $|s|_\gamma$ corresponding to eigenvalue 1, we have the following chain of inclusions:

COROLLARY. *If* $s \in \sigma(B(G))$, *then for* $1 < \alpha < \beta$,

$$\sigma(B(G))|s|^\infty_\gamma \subset \sigma(B(G))|s|^\beta_\gamma \subset \sigma(B(G))|s|^\alpha_\gamma \subset \sigma(B(G))s$$
$$\subset \sigma(B(G))|s|^{1/\alpha}_\gamma \subset \sigma(B(G))|s|^{1/\beta}_\gamma \subset \sigma(B(G))|s|^0_\gamma.$$

A similar statement holds for the corresponding principal, right ideals.

Proof. We have, e.g., $|s|^\beta_\gamma = |s|^{\beta-\alpha}_\gamma|s|^\alpha_\gamma$. The rest is clear.

PROPOSITION 3. $I \subset \sigma(B(G))$ *is a left-ideal if and only if* $s \in I$, $t \in \sigma(B(G))$, *and* $t^*t \leqq s^*s$ *imply* $t \in I$. *A corresponding statement holds for right ideals.*

Proof. If s and t satisfy the above conditions, then for some $a \in \sigma(B(G))$,

$$t = v_\gamma|t|_\gamma = v_\gamma a|s|_\gamma \in \sigma(B(G))|s|_\gamma = \sigma(B(G))s \subset I.$$

Conversely, given $s \in I \subset \sigma(B(G))$ where I satisfies the above condition, we must show that $xs \in I$ if $x \in \sigma(B(G))$. But $(xs)^*(xs) \leqq s^*s$, hence we are done.

We remark that there is a map from the (weakly) closed, right ideals in $\sigma(B(G))$ to the left translation invariant "radical" ideals in $B(G)$, where if I is such an ideal in $\sigma(B(G))$, the corresponding radical ideal is $\{b \in B(G)\colon b(s) = 0 \text{ for all } s \in I\}$.

Before going further we must discuss the strong and weak topologies on $\sigma(B(G))$, as is done in **[14]** for G abelian. We have that $\sigma(B(G))$ is compact, the involution * is continuous, and multiplication is separately continuous in the weak (or what is the same, weak operator) topology on $\sigma(B(G))$. Also, the weak topology is weaker than any of the following strong topologies. (Note that consequently principal ideals in $\sigma(B(G))$ are weakly hence strongly closed.) Due to the non-abelianess of G, there are four strong topologies on $\sigma(B(G))$: the strong operator topology; the left-strong topology, i.e., $s \to s_0$ in $\sigma(B(G))$ if and only if $\|\gamma_s(b) - \gamma_{s_0}(b)\| \to 0$ for each $b \in B(G)$; the right-strong topology, i.e., $s \to s_0$ in $\sigma(B(G))$ if and only if $\|\delta_s(b) - \delta_{s_0}(b)\| \to 0$ for each $b \in B(G)$; and the *-strong topology, i.e., $s \to s_0$ in $\sigma(B(G))$ provided both $s \to s_0$ and $s^* \to s_0^*$ in the strong operator topology. It is easy to verify that $s \to s_0$ strongly (as operators in $W^*(G)$) if and only if $s \to s_0$ left-strongly, and $s^* \to s_0^*$ strongly (as operators in $W^*(G)$) if and only if $s \to s_0$ right-strongly, and the involution * is a homeomorphism between the left and right strong topologies. Multiplication in $\sigma(B(G))$ is jointly continuous in all the strong topologies, whereas the involution is continuous in the *-strong topology. Finally, it is clear that the map $s \in \sigma(B(G)) \to s^*s \in \sigma(B(G))_+$ (resp., $ss^* \in \sigma(B(G))_+$) is continuous from the left-strong (resp., right-strong) topology to the weak topology.

It is well to note the following for later use.

PROPOSITION 4. (i) *If $\{s_\alpha\}$ is a net in $\sigma(B(G))$, $s \in \sigma(B(G))$, and $s_\alpha^* s_\alpha \leqq s^*s$* (resp., $s_\alpha s_\alpha^* \leqq ss^*$) *for all α, then $s_\alpha \to s$ left-strongly* (resp., right-strongly) *if and only if $s_\alpha \to s$ weakly*;

(ii) *the weak and left-strong* (resp., right-strong) *topologies agree on any set of the form $\{s \in \sigma(B(G))\colon s^*s = t,\ t \in \sigma(B(G))_+\}$* (resp., $\{s \in \sigma(B(G))\colon ss^* = t,\ t \in \sigma(B(G))_+\}$);

(iii) *The weak and left-*(or right) *strong topologies agree on any subset of $\sigma(B(G))_+$ which is totally ordered.*

Proof. (i) Suppose $s_\alpha^* s_\alpha \leqq s^*s$ and $s_\alpha \to s$ weakly. Then for a positive definite function $p \in B(G)$, and $x \in W^*(G)$,

$$\|s_\alpha \cdot p - s \cdot p\| = \sup_{\|x\| \leqq 1} |\langle p, x\ (s_\alpha - s)\rangle| \leqq$$

$$\sup_{\|x\| \leqq 1} p(xx^*)^{1/2}\, p((s_\alpha - s)^*(s_\alpha - s))^{1/2} \leqq \|p\|^{1/2}(2p(s^*s) - 2\operatorname{Re} p(s^* s_\alpha))^{1/2}$$

which converges to zero. Since any $b \in B(G)$ is a linear combination of four positive definite functions we are done. The rest of (i) is by symmetry.

(ii) Immediate from (i).

(iii) Let s and net $\{s_\alpha\}$ be in a totally ordered subset of $\sigma(B(G))_+$. Our claim follows from the inequality $p((s_\alpha - s)^2) \leqq 2|p(s_\alpha - s)|$, where p is positive definite, cf., [4] Appendice II.

Now any interval, $\{t \in \sigma(B(G))_+ : s_1 \leqq t \leqq s_2\}$ in $\sigma(B(G))_+$, determined by $s_1, s_2 \in \sigma(B(G))_+$ is closed in both the weak and strong topologies. On the other hand,

PROPOSITION 5. *If $S \subset \sigma(B(G))_+$ is strongly closed, then S contains minimal and maximal elements.*

Proof. All strong topologies coincide on the set of self-adjoint elements in $\sigma(B(G))_+$, now apply Proposition 4 (iii), weak compactness of $\sigma(B(G))_+$, and Zorn's lemma.

As in the abelian case, minimal elements of strongly open-closed subsets of $\sigma(B(G))_+$ are especially important in the theory. Before discussing these objects, however, let us have the following notations, $G_{p,\gamma} = \{s \in \sigma(B(G)) : s^*s = p\}$, $G_{p,\delta} = \{s \in \sigma(B(G)) : ss^* = p\}$, $G_p = G_{p,\gamma} \cap G_{p,\delta}$, where $p \in \sigma(B(G))_p$.

PROPOSITION 6. *(i) G_p is a topological group with * for inverse, p for identity, and the right or left-strong topology or the weak topology — all of which coincide on G_p. G_p is *-strongly closed in $\sigma(B(G))$.*

(ii) *$G_{p,\gamma} \subset \sigma(B(G))p$, $G_{p,\delta} \subset p\sigma(B(G))$ and the following inclusions hold:* $\sigma(B(G))p\sigma(B(G)) \supset p\sigma(B(G)) \cup \sigma(B(G))p \supset p\sigma(B(G)) \cap \sigma(B(G))p = p\sigma(B(G))p \supset G_p$.

Proof. The proof is rather easy and left to the reader.

Now consider the following conditions on $s \in \sigma(B(G))_+$.

Condition (A): There does not exist a net $\{s_\alpha\} \subset \sigma(B(G))_+$ satisfying $s_\alpha \lneqq s$ and $\lim_\alpha s_\alpha = s$.

Condition (B): There does not exist a net $\{s_\alpha\} \subset \sigma(B(G))_+$ satisfying $s_\alpha^2 \lneqq s^2$ (which implies $s_\alpha \lneqq s$) and $\lim_\alpha s_\alpha = s$.

Note that in both conditions weak and strong limits are equivalent. Also Condition (A) implies Condition (B). Both conditions imply that $s \in \sigma(B(G))_p$, since if $s^2 \lneqq s$ then $s^\alpha \lneqq s$ (respectively, $s^{2\alpha} \lneqq s^2$) and $\lim_{\alpha \downarrow 1} s^\alpha = s$. We should also observe that if s is central, then s satisfies condition (A) if and only if it satisfies condition (B), since if s_α and s commute, $0 \leqq s_\alpha \leqq s$ is equivalent to $0 \leqq s_\alpha^2 \leqq s^2$. What is much more important for us, however, is that if s satisfies

$s^2 = s \geqq 0$, then we have that $0 \leqq s_\alpha \leqq s^2 = s$ holds if and only if $0 \leqq s_\alpha^2 \leqq s^2 = s$. Note that $0 \leqq s_\alpha \leqq s^2 = s$ implies that $(e - s)s_\alpha = s_\alpha(e - s) = 0$, hence $s_\alpha s = s s_\alpha = s_\alpha$, and we are done, since for positive operators $s_\alpha, s, s_\alpha^2 \leqq s^2$ always gives $s_\alpha \leqq s$. Thus we have the following generalization of the notion of critical point introduced in **[14]:**

DEFINITION. If $p \in \sigma(B(G))_+$ satisfies condition (A), or equivalently condition (B), then p is called a critical element of $\sigma(B(G))_+$.

Observe that p is critical if and only if p is weakly isolated in

$$(p\sigma(B(G)))_+ = \{t \in \sigma(B(G))_+ : t^2 \leqq p\} = \{t \in \sigma(B(G))_+ : t \leqq p\}$$
$$= p\sigma(B(G))_+ p = (p\sigma(B(G))p)_+.$$

We now have the following characterization of critical elements:

PROPOSITION 7. (i) $p \in \sigma(B(G))_+$ *is critical*;
(ii) $G_{p,\gamma}$ *is left-strongly* (weakly) *open in* $\sigma(B(G))p$;
(iii) $G_{p,\delta}$ *is right-strongly* (weakly) *open in* $p\sigma(B(G))$;
(iv) G_p *is strongly* (weakly) *open in* $p\sigma(B(G))p$;
(v) p *is a minimal element of a strongly open and closed subset of* $\sigma(B(G))_+$.

Proof. Consider the map $\theta: s \in \sigma(B(G)) \mapsto s^* s \in \sigma(B(G))_+$ is continuous from the left-strong to the weak topology. Now in $\{t \in \sigma(B(G))_+ : t^2 \leqq p\}$, if p is critical $\{p\}$ is weakly open; and $G_{p,\gamma} = \theta^{-1}(p)$ is thus left-strongly open in

$$\sigma(B(G))p = \theta^{-1}(\{t \in \sigma(B(G))_+ : t^2 \leqq p\}).$$

That $G_{p,\gamma}$ is weakly open in $\sigma(B(G))p$ follows from Proposition 4 (i), thus (i) implies (ii). Clearly, (i) also imples (iii). Conversely, since $\{p\} = \sigma(B(G))_+ \cap G_{p,\gamma}$, and $(\sigma(B(G))p)_+ = \sigma(B(G))_+ \cap \sigma(B(G))p$, we have (ii) implies (i). Clearly (iii) implies (i) also. It is now easy to see that (i) is equivalent to (iv). Now suppose p is critical, then p is a minimal element of the strongly (weakly) open-closed set $\{t \in \sigma(B(G))_+ : p \leqq t^2\} = \{t \in \sigma(B(G))_+ : pt = p\}$ (which is the inverse image of weakly isolated point $\{p\}$ under weakly continuous map $t \in \sigma(B(G))_+ \mapsto ptp \in p\sigma(B(G))_+ p$). Conversely, if p is a minimal element of some strongly open and closed set $S \subset \sigma(B(G))_+$, then $\{p\} = S \cap \{t \in \sigma(B(G))_+ : t \leqq p\}$ is strongly (weakly) isolated in $p\sigma(B(G))_+ p = \{t \in \sigma(B(G))_+ : t \leqq p\}$, and p is thus critical. Hence (i) is equivalent to (v), and we are done. Since $G_{p,\delta}$, $G_{p,\gamma}$, and G_p are

weakly open in weakly compact $p\sigma(B(G))$, $\sigma(B(G))p$, $p\sigma(B(G))p$ respectively, we have:

COROLLARY. *If p is critical $G_{p,\delta}$, $G_{p,\gamma}$ are locally compact spaces, and G_p is a locally compact topological group.*

We now investigate a special critical point, viz., z_F, which is critical by Theorem 2. Note that if z is a central critical element then a continuous homomorphism $\theta_z: g \in G \mapsto gz \in G_z$ results. In general, at the very least one has that ${}'\theta_2: b \in B(G_z) \mapsto b \circ \theta_z \in B(G)$ is a normdecreasing homomorphism between the corresponding Fourier-Stieltjes algebras. In the case of z_F we have:

PROPOSITION 8. *G_{z_F} is the almost periodic compactification of G, and ${}'\theta_{z_F}$ is an isometry of $B(G_F)$ onto $B(G) \cap AP(G) = z_F \cdot B(G)$, where $AP(G)$ denotes the almost periodic functions on G.*

Proof. Let $\bar{G}$ denote the almost periodic compactification of G, then $i: G \to \bar{G}$ the canonical inclusion is such that ${}'i: B(\bar{G}) \to AP(G) \cap B(G)$ (isometrically), where $AP(G) \cap B(G)$ is a bi-translation invariant, closed subalgebra of $B(G)$, i.e., $AP(G) \cap B(G) = z_0 \cdot B(G)$ where z_0 is a central projection in $W^*(G)$, cf., **[5]** 2.27 and **[17]**. Now $z_F \cdot B(G) \subseteq z_0 \cdot B(G)$ since any element in $z_F \cdot B(G)$ is almost periodic, **[3]**, 16.2.1. Now $z_F \in \sigma(B(G))$ implies $z_F \in \sigma(B(\bar{G}))$, where we identify $B(\bar{G})$ and $z_0 \cdot B(G)$. But $z_0 \in \sigma(B(\bar{G}))$, namely, the identity. But $\bar{G}$ is compact, hence by the second corollary of Theorem 2, $z_F = z_0$. Thus $B(\bar{G}) \cong z_F \cdot B(G)$. Now the dual group (in the sense of **[18]**) of $B(\bar{G})$ is uniquely determined, and is $\bar{G}$; while the dual group of $z_F \cdot B(G)$ is the compact group $\sigma(B(G))z_F = G_{z_F}$. Thus $\bar{G}$ is topologically isomorphic with G_{z_F}.

A natural discussion now arises. Given a central critical element z, then the closure of $\theta_z(G)$, call it $G_{\theta z}$, in G_z is a locally compact group, and (with a slight abuse of notation) ${}'\theta_z: B(G_{\theta_z}) \to B(G)$ is an isometric isomorphism onto a closed, bi-translation invariant subalgebra of $B(G)$. Also, of course, the inclusion $i: G_{\theta_z} \to G_z$ induces a norm-continuous homomorphism ${}'i: B(G_z) \to B(G_{\theta_z})$ with the additional property that ${}'i(A(G_z)) = A(G_{\theta_z})$, cf., **[8]**. One question then is $G_{\theta_z} = G_z$? By Proposition 8 the answer is yes if $z = z_F$. In general, it is not hard to see that the complete analysis of a central critical point z in $\sigma(B(G))_+$ depends ultimately on the resolution of the following question: Does the algebra of functions $z \cdot B(G)$ contain an element of $A(G_z)$? The affirmative answer to this question in case G is abelian was furnished by Taylor, cf. **[14]**, **[15]** and references therein, with much machinery and

considerable work. A closely related question is: what types of commutative Banach algebras are dual to a locally compact group G (in the sense of [18])? Must such a dual algebra contain a copy of $A(G)$? A tool which we hope will help resolve these questions is considered in the next section.

Generalized inverse Fourier-transform. In [7], [8] C. S. Herz demonstrates that $A(G)$ is the quotient of $L^2(G)\hat{\otimes}L^2(G)$ (the projective tensor product of $L^2(G)$ with itself) by the kernel of the continuous surjection $P: L^2(G)\hat{\otimes}L^2(G)\to A(G)$ determined by $P(\xi\otimes\eta)=\eta*\check{\xi}$, where $\eta*\check{\xi}(g)=\int\xi(g^{-1}x)\eta(x)dx$ for $\xi,\eta\in L^2(G)$. With this norm $A(G)$ is a Banach algebra. Now we note that $L^2(G)$ is a Hilbert G-module, i.e., there is a continuous unitary representation of G on $L^2(G)$, viz., the left regular representation λ, and that $A(G)$ is just the collection of coefficients of λ. A natural question is: can this result be generalized to an arbitrary Hilbert G-module H_π, i.e., to the case where we have a continuous, unitary representation π of G in H_π? We give an affirmative answer to this question, and in so doing introduce the notion of the *generalized inverse Fourier transform localized at* π, as well as the notion of the *standardization of* π. These concepts have been motivated by our desire to better understand closed, bi-translation invariant subspaces, subalgebras, and ideals in $B(G)$. We present this section with the hope that it will be a useful tool which will bring to bear on any unitary group representation almost the entire calculus previously only used in association with the left-regular representation. Technically we have been motivated by [7], [8], [10], [11] as will become apparent, but the Tomita-Takesaki theory makes the dominant contribution.

We first note that $L^2(G)\hat{\otimes}L^2(G)$ may be identified with the nuclear (or trace class) operators, $\mathcal{T}(L^2(G))$, on $L^2(G)$ via the map $\tau: L^2(G)\hat{\otimes}\overline{L^2(G)}\to\mathcal{T}(L^2(G))$ determined by $\tau(\xi\otimes\eta)=\langle\ ,\eta\rangle\xi$, where although $L^2(G)$ and its dual $\overline{L^2(G)}$ are "the same" we prefer to retain the distinction. Note that $\langle\ ,\eta\rangle$ indicates we view η as in $\overline{L^2(G)}$, $(\ |\eta)$ indicates we view $\eta\in L^2(G)$.

REMARK. From an intuitive point of view we regard $\mathcal{T}(L^2(G))$ as a semi-abelianized, discretized version of another noncommutative L^1-measure algebra associated with a weight. The precise meaning of this statement will be made clear when we discuss the standardization of π. Suffice it to say that the map P of C. S. Herz behaves very much like an inverse Fourier-transform of an L^1-space onto $A(G)$.

A version of our next theorem, we have been informed by mail, was obtained independently by a student of P. Eymard, G. Arsac, in his Ph.D. thesis. The research of this paper was carried out independently

by the present author without knowledge of the work of Arsac. Our point of view and motivation are different, and our "concrete" transform and standardization concepts, as far as we know, have not been discussed by Arsac. Whereas our proof of Theorem 3 is based on an inverse transform of nuclear (i.e., trace class) operators, Arsac's proof is based on the more abstract projective tensor product representation of this object as a Banach space. Our proofs differ in that we look at a "concrete" transform of nuclear operators; also, we have a C^*-algebra of operators to deal with; and thus we obtain more detailed results. Our approach emphasizes the action of G and closely resembles the classical Fourier-transform theory.

DEFINITION. Given a continuous, unitary, representation π of G on H_π, we denote the nuclear operators on H_π by $\mathcal{T}(H_\pi)$. We define the inverse Fourier-transform of $t \in \mathcal{T}(H_\pi)$ to be that complex-valued function on G defined by $\hat{t}_\pi : g \in G \mapsto \mathrm{Tr}(\pi(g)t)$, where Tr is the normalized trace on $\mathcal{L}(H_\pi)$. We refer to this map as the *inverse Fourier transform (localized) at* π.

REMARK. This transform is obtained by considering $t \in \mathcal{T}(H_\pi)$ as an element in the predual of $\mathcal{L}(H_\pi)$ and then restricting to the von Neumann algebra $\{\pi(g): g \in G\}'' \subset \mathcal{L}(H_\pi)$. In this way we shall see that $\hat{t}_\pi \in z[\pi] \cdot B(G)$. If we define the transform by $g \in G \mapsto \mathrm{Tr}(\pi(g)^* t)$, then $\hat{t}_\pi \in z[\bar{\pi}] \cdot B(G)$, where $\bar{\pi}$ is the representation "conjugate" to π.

THEOREM 3. (i) *The function* $\hat{t}_\pi(g) = \mathrm{Tr}(\pi(g)t)$ *on* G *is in* $z[\pi] \cdot B(G)$, *where* $z[\pi]$ *is the support of* π *in* $W^*(G)$, *i.e.,* $z[\pi]B(G)$ *is the closed, bi-translation invariant subspace of* $B(G)$ *determined by the coefficients* $\{(\pi(\cdot)\xi \mid \eta): \xi, \eta \in H_\pi\}$ *of* π.

(ii) *If* t *is a positive operator in* $\mathcal{T}(H_\pi)$, *then* $\hat{t}_\pi \in z[\pi] \cdot P(G)$, *and* $\|\hat{t}_\pi\|_{B(G)} = \hat{t}_\pi(e) = \|t\|_{\mathcal{T}(H_\pi)} = \mathrm{Tr}(t)$. *If* $t = v \mid t \mid \in \mathcal{T}(H_\pi)$ (left polar decomposition in $\mathcal{L}(H_\pi)$), *then* $\hat{t}_\pi = v \cdot |t|^\wedge_\pi$ (left polar decomposition with respect to $\mathcal{L}(H_\pi)$), *and* $\|\hat{t}_\pi\|_{B(G)} \leqq \|t\|_{\mathcal{T}(H_\pi)}$.

(iii) *For each* $b \in z[\pi] \cdot B(G)$, *there is a* $t \in \mathcal{T}(H_\pi)$ *such that* $b = \hat{t}_\pi$.

(iv) *The map* $t \in \mathcal{T}(H_\pi) \mapsto \hat{t}_\pi \in B(G)$ *is one-to-one if and only if* π *is irreducible.*

Proof. Given $t \in \mathcal{T}(H_\pi)$, let $t = v \mid t \mid$ be its polar decomposition, with $|t| = \sum_{i=1}^\infty \lambda_i (\cdot \mid \xi_i)\xi_i$ where $\xi_i \in H_\pi$, $\|\xi_i\| = 1$, and $\lambda_i \geqq 0$ for all i, $\sum_{i=1}^\infty \lambda_i = \mathrm{Tr}(|t|)$. Thus $\hat{t}_\pi(g) = \mathrm{Tr}(\pi(g)t) = \mathrm{Tr}(\sum_{j=1}^\infty \lambda_i (\cdot \mid \xi_i)\pi(g)v\xi_i))$

$= \Sigma_{i=1}^{\infty} \lambda_i \operatorname{Tr}((\cdot \mid \xi_i)\pi(g)v\xi_i)$, where the last equality follows from the Hölder inequality

$$\left| \operatorname{Tr}\left(\pi(g)v \sum_{i=n}^{\infty} \lambda_i(\cdot \mid \xi_i)\xi_i\right)\right| \leqq \|\pi(g)v\|_{\mathscr{L}(H_\pi)} \left\| \sum_{i=n}^{\infty} \lambda_i(\cdot \mid \xi_i)\xi_i \right\|_{\mathscr{T}(H_\pi)}.$$

Thus $f_\pi(g) = \Sigma_{i=1}^{\infty} \lambda_i(\pi(g)v\xi_i \mid \xi_i) = \Sigma_{i=1}^{\infty} \lambda_i(v \cdot \omega_{\xi_i})(g)$ is in $B(G)$; since (restricted to G) $\Sigma_{i=1}^{\infty} \lambda_i \omega_{\xi_i} \in P(G)$ norm — converges in $B(G)$ to an element in $P(G)$; and since $v \cdot (\Sigma_{i=1}^{\infty} \lambda_i \omega_{\xi_i}) = \Sigma_{i=1}^{\infty} \lambda_i v \cdot \omega_{\xi_i}$, when restricted to $\{\pi(g): g \in G\}''$ is in $z[\pi] \cdot B(G)$. Note that $\|\Sigma_{i=1}^{\infty} \lambda_i v \cdot \omega_{\xi_i}\|_{B(G)} \leqq \Sigma_{i=1}^{\infty} \lambda_i = \operatorname{Tr}(|t|)$. Thus (i) and (ii) have been demonstrated. As for (iii) and (iv) they are almost obvious from the remark immediately above, since $z[\pi] \cdot B(G)$ is the predual of the von Neumann algebra $\{\pi(g): g \in G\}''$, $\mathscr{T}(H_\pi)$ is the predual of $\mathscr{L}(H_\pi)$, cf., **[4]** Chap. 1, §3 Théorème 1, and $\{\pi(g): g \in G\}'' = \mathscr{L}(H_\pi)$ if and only if π is irreducible.

REMARK. Note that the partial isometry v in part (ii) of Theorem 3 is in general not in $\{\pi(g): g \in G\}''$ but only in $\mathscr{L}(H_\pi)$. Thus $\|f_\pi\|_{B(G)}$ can be a zero even if t is not zero, and $z[\pi] \cdot B(G)$ is the Banach space coimage of $\hat{}$.

REMARK. Theorem 3 can immediately and obviously be applied to any group representation π such that, for example, $\pi(L^1(G)) \cap \mathscr{T}(H_\pi)$ is large; and there are many groups whose irreducible representations, for example, have this property. Thus one might say that $B(G)$ is "sufficient" for the Fourier analysis of such groups. We contend, however, in a forthcoming paper that $B(G)$ is "sufficient" for the Fourier analysis of any locally compact group, cf., the final remark of this paper.

We now introduce the concept of the standardization of a continuous, unitary group representation π. This procedure amounts basically to translation of the Tomita-Takesaki theory into the special context of group theory. This standardization process gains added significance when one realizes that with the machinery of this theory any continuous unitary, representation π of group G becomes a "modified left-regular" representation accompanied by the calculus thereof. As an application we will apply Theorem 3 in this setting.

Given any π, as above, let $M(\pi)$ (or M_π, whichever notation is more convenient) be the von Neumann algebra $\{\pi(g): g \in G\}'' \subset \mathscr{L}(H_\pi)$. On $M(\pi)$ there exists a normal, faithful, semi-finite weight denoted by $\varphi(\pi)$, or φ_π; we can thus put the pair $\{M(\pi), \varphi(\pi)\}$ into standard form, cf., **[6], [11], [12], [16].** Very briefly, we take left-ideal $n_{\varphi(\pi)} = \{x \in M(\pi): \varphi_\pi(x^*x) < +\infty\}$; $H_{\varphi(\pi)}$, the completion of $n_{\varphi(\pi)}$ with

respect to the nondegenerate inner product induced by $\varphi(\pi)$, and $\eta: x \in n_\varphi \mapsto \eta(x) \in H_\pi$ the usual inclusion. We then denote by $\lambda(\varphi_\pi)$ the faithful *-representation of $M(\pi)$ on $H_{\varphi(\pi)}$ determined by $\lambda(\varphi_\pi)(x)$ $\eta(y) = \lambda(\varphi_\pi)\eta(xy)$ for all $x \in M_\pi$, $y \in n_{\varphi(\pi)}$. But $\lambda(\varphi_\pi) \circ \pi$ is thus also a continuous, unitary representation of G; and it is *quasi-equivalent to* π. Thus in particular, $z[\lambda(\varphi_\pi) \circ \pi] = z[\pi]$, and both representations determine the same subspace of $B(G)$.

DEFINITION. Given any quasi-equivalence class $\{\pi\}$ of continuous, unitary representations of locally compact group G, then for $\pi \in \{\pi\}$ construct $\pi_s = \lambda(\varphi_\pi) \circ \pi \in \{\pi\}$, and call π_s the *standardization of* π.

REMARK. With abuse of notation we will often drop the subscript s and use π to denote both π and π_s, also H_π will henceforth refer only to $H_{\pi_s} = H_{\varphi(\pi)}$, etc.

We thus have the following corollary of Theorem 3:

COROLLARY. *Let π be the representation of G in standard form. Then the inverse Fourier-transform localized at π has, in addition to properties* (i), (ii), (iii), *of Theorem* 3,

(v) *If $b \in z[\pi] . B(G)$, there exists an operator of rank one, $t = (\cdot \mid \eta)\xi \in \mathcal{T}(H_\pi)$, such that $b = \check{t}_\pi$. Furthermore, $\xi, \eta \in H_\pi$ can be so selected that $\| b \|_{B(G)} = \| \xi \|_{H_\pi} \| \eta \|_{H_\pi}$.*

REMARK. This corollary is obvious if one is familiar with the Tomita-Takesaki theory. A quick proof is as follows: Observe that if π is standard, i.e., M_π on H_π, with unitary involution J_π, and self-dual cone $P_\pi \subset H_\pi$, then any sigma-finite projection in M_π has a cyclic vector ξ (which can be chosen from P_π). But now we are done, cf., **[4]** Chap. II, §1 cor. of Thm. 4 and the discussion of standard forms following this corollary. Each positive, weakly continuous functional on M_π, i.e., in $z[\pi] . P(G)$ is of the form ω_ξ, with $\xi \in P_\pi$. (In fact the map $\xi \in P_\pi \subset H_\pi \mapsto \omega_\xi \in (M_\pi)_+ = z[\pi] . P(G)$ is a norm, homeomorphism, cf. **[1]**, **[2]**, **[6]**.) Thus given $b \in z[\pi] . B(G)$, let $v . p = b$ be the (left) polar decomposition of b with respect to M_π, i.e., $v \in M_\pi$, $p \in z[\pi] . P(G)$. Then $\| p \|_{B(G)} = \omega_\xi(e) = \| \xi \|^2_{H_\pi}$, and $p = \check{t}_1$, where $t_1 = (\ \mid \xi)\xi$, and $b = \check{t}_2$, where $t_2 = (\ \mid \xi)v\xi$, where $\| b \|_{B(G)} = \| \xi \| \| v\xi \|$. Thus every element in $z[\pi] . B(G)$ is a transform of a rank-one operator of "minimal cross-norm". (We have in fact shown more, since we can select $\xi \in P_\pi$.)

REMARK. As a corollary of the above discussion we get a more detailed version of **[5]**, Thm. p. 218. Thus we may think of $b \in$

$z[\pi] \cdot B(G)$ as a generalized convolution (with a "twist") of two elements from H_π, cf., $L^2(G) * L^2(G)^\sim = A(G)$.

REMARK. We mentioned earlier that $\mathcal{T}(H_\pi)$ was a semi-abelianized, discretized version of another noncommutative L^1-measure "algebra". The measure "algebra" we have in mind is the L^1-space of weight φ_π. We have a definition and embryonic theory for this space analogous to the work done in [9] and [10] for the unimodular (trace) case. This L^1-space is the "proper" domain for the inverse Fourier transform; however, to go into details here would take us beyond the scope of this paper. We intend to go into this subject in depth in an upcoming paper.

REFERENCES

1. H. Araki, *A Radon-Nikodym theorem for W*-algebras with chain rule*, Queen's—Math. preprint, 1972.
2. A. Connes, C. R. Acad. Sc. Paris, **274** (1972), 1923–1926.
3. J. Dixmier, *Les C*-algèbres et leurs Representations*, Gauthier-Villars, Paris, 1964.
4. ———, *Les Algèbres d'Operateurs dans l'Espace Hilbertien*, Gauthier-Villars, deuxième édition, Gauthier-Villars, Paris, 1969.
5. P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France, **92** (1964), 181–236.
6. U. Haagerup, *The standard form of von Neumann algebras*, Kobenhavns Universitet Math. preprint No. 15, 1973.
7. C. S. Herz, *Remarques sur la note précédente de M. Varopolus*, C. R. Acad. Sc. Paris, **260** (1965), 6001–6004.
8. ———, *La rapport entre l'algèbre A d'un groupe et d'un sous-groupe*, C. R. Acad. Sc. Paris, **271** (1970), 244–246.
9. I. E. Segal, *A non commutative extension of abstract integration*, Ann. of Math., **57** (1953), 401–457.
10. W. F. Stinespring, *Integration theorems for gages and duality for unimodular groups*, Trans. Amer. Math. soc., **90** (1959), 15–56.
11. M. Takesaki, *Tomita's Theory of Modular Hilbert Algebras and Its Applications*, Springer-Verlag, Berlin, 1970.
12. ———, *Lectures on Operator Algebras*, U.C.L.A. Lecture Notes, 1970.
13. N. Tatsuuma, *Prime ideals in the dual objects of locally compact groups*, Proc. Japan Acad., **47**, No. 3 (1971), 249–251.
14. J. L. Taylor, *Inverses, logarithms, and idempotents, in M(G)*, Rocky Mountain J. Math., **2** (1972), 183–206.
15. ———, *Measure Algebras*, notes from N.S.F. regional conference on measure algebras, Univ. of Montana, 1972.
16. M. Tomita, *Standard forms of von Neumann algebras*, the Vth functional analysis symposium of the Mathematical Society of Japan, Sendai, (1967).
17. M. E. Walter, *W*-algebras and non-abelian harmonic analysis*, J. Functional Analysis, **11** (1972), 17–38.
18. ———, *A duality between locally compact groups and certain Banach algebras*, J. Functional analysis, (to appear).

Received March 5, 1974.

UNIVERSITY OF COLORADO