PACIFIC JOURNAL OF MATHEMATICS
Vol. 58, No. 1, 1975

A GENERALIZED JENSEN'S INEQUALITY

TING ON TO AND YIP KAI WING

A generalized Jensen's inequality for conditional expectations of Bochner-integrable functions which extends the results of Dubins and Scalora is proved using a different method.

1. Introduction. Let $(\Omega, \mathbf{F}, P)$ be a probability space, $(\mathbf{U}, \|\cdot\|)$ a complex (or real) Banach space and $(\mathbf{V}, \|\cdot\|, \geqq_v)$ an ordered Banach space over the complex (or real) field such that the positive cone $\{v \in \mathbf{V} : v \geqq_v \theta\}$ is closed. Let x be a Bochner-integrable function on $(\Omega, \mathbf{F}, P)$ to $\mathbf{U}$. Let $\mathbf{G}$ be a sub-σ-field of the σ-field $\mathbf{F}$ and let f be a function on $\Omega \times \mathbf{U}$ to $\mathbf{V}$ such that for each $u \in \mathbf{U}$ the function $f(\cdot, u)$ is strongly measurable with respect to $\mathbf{G}$ and such that for each $\omega \in \Omega$ the function $f(\omega, \cdot)$ is continuous and convex in the sense that $tf(\omega, u_1) + (1-t)\ f(\omega, u_2) \geqq_v f(\omega, tu_1 + (1-t)u_2)$ whenever $u_1,\ u_2 \in \mathbf{U}$ and $0 \leqq t \leqq 1$. For any Bochner-integrable function z on $(\Omega, \mathbf{F}, P)$ to any Banach space $\mathbf{W}$, we define $E[z \mid \mathbf{G}]$ "a conditional expectation of z relative to $\mathbf{G}$" as a Bochner-integrable function on $(\Omega, \mathbf{F}, P)$ to $\mathbf{W}$ such that $E(z \mid \mathbf{G}]$ is strongly measurable with respect to $\mathbf{G}$ and that

$$\int_A E[z \mid \mathbf{G}](\omega) dP = \int_A z(\omega) dP, \qquad A \in \mathbf{G},$$

where the integrals are Bochner-integrals.

The purpose of this note is to prove the following generalized Jensen's inequality:

THEOREM. *If $f(\cdot, x(\cdot))$ is Bochner-integrable, then*

$$\text{(J)} \qquad E[f(\cdot, x(\cdot)) \mid \mathbf{G}](\omega) \geqq_v f(\omega, E[x \mid \mathbf{G}](\omega)) \qquad \text{a.e.}$$

The above theorem extends the results of Dubins **[2]** (cf. Mayer **[5**, p. 79]) and Scalora **[6**, p. 360, Theorem 2.3]. It is proved in **[2]** that the theorem is true for the case in which the spaces $\mathbf{U}$ and $\mathbf{V}$ are both the real numbers $\mathbf{R}$, while in **[6]** Scalora uses the methods of Hille-Phillips **[4]** to prove the theorem when the function $f(\omega, u)$ is replaced by a continuous, subadditive positive-homogeneous function $g(u)$ on $\mathbf{U}$ to $\mathbf{V}$. It should be noted that the method of the pr∂of used here is different than those used previously, the previous methods appear to be ineffective for a proof of the extension.

2. Preliminaries. We refer to **[4]** and **[6]** for the definitions and basic properties of the concepts of Bochner-integrals and the conditional expectation of a Bochner-integrable function. Our proof of the theorem is based on the following lemmas. Unless otherwise specified, functions in Lemma 1–5 are defined on $(\Omega, \mathbf{F}, P)$ to $\mathbf{U}$.

LEMMA 1. (**[4**, p. 73, Corollary 1]). *A function is strongly measurable if and only if it is the uniform limit almost everywhere of a sequence of countably-valued functions.*

LEMMA 2. (Egoroff's theorem, **[4**, p. 72] or **[3**, p. 149]). *A sequence $\{z_j\}_{i=1}^{\infty}$ of strongly measurable functions is almost uniformly convergent to a function z if and only if*

$$\|z_i(\omega) - z(\omega)\| \to 0 \text{ a.e. } as\, i \to \infty.$$

The following lemma is an immediate consequence of Lemma 1 and Lemma 2.

LEMMA 3. *If z is a strongly measurable function, then for any positive number M there exists a sequence $\{z_i\}_{i=1}^{\infty}$ of simple functions such that $\|z_i(\omega)\| \leqq \|z(\omega)\| + M$ a.e., $i = 1, 2, \cdots$, and $\|z_i(\omega) - z(\omega)\| \to 0$ a.e. as $i \to \infty$.*

LEMMA 4. (**[6**, p. 356, Theorem 2.2]).

(a) *If $z(\omega) = u$ on Ω then $E[z \mid \mathbf{G}](\omega) = u$ a.e.*

(b) *If z and z_i, $i = 1, 2, \cdots$, are Bochner-integrable functions such that $z(\omega) = \sum_{i=1}^{n} t_i z_i(\omega)$ a.e. where t_i are scalars then $E[z \mid \mathbf{G}](\omega) = \sum_{i=1}^{n} t_i E[z_i \mid \mathbf{G}](\omega)$ a.e.*

(c) $\|E[z \mid \mathbf{G}](\omega)\| \leqq E[\|z\| \mid \mathbf{G}](\omega)$ a.e., *for any Boxhner-integrable function z.*

(d) *If z is a Bochner-integrable function and $z_i, i = 1, 2, \cdots$, are strongly measurable functions such that $\|z_i(\omega) - z(\omega)\| \to 0$ a.e. as $i \to \infty$, and if there is a real-valued integrable function $y(\omega) \geqq 0$ such that $\|z_i(\omega)\| \leqq y(\omega)$ a.e., $i = 1, 2, \cdots$, then z_i's are Bochner-integrable and $\|E[z_i \mid \mathbf{G}](\omega) - E[z \mid \mathbf{G}](\omega)\| \to 0$ a.e. as $i \to \infty$.*

LEMMA 5. *If z is a Bochner-integrable function and $z_i, i = 1, 2, \cdots$, are strongly measurable functions such that $\|z_i(\omega) - z(\omega)\| \to 0$ uniformly a.e. as $i \to \infty$, then there exists an integer N such that $z_i, i = N, N+1, \cdots$, are Bochner-integrable functions, and*

$$\|E[z_i \mid \mathbf{G}](\omega) - E[z \mid \mathbf{G}](\omega)\| \to 0 \text{ } uniformly$$

a.e. *as* $i \to \infty$.

Proof. An immediate consequence of Lemma 4 and the fact that $E[\cdot|\mathbf{G}]$ is a positive operator on the space of all real-valued integrable functions.

LEMMA 6. *If z is a strongly measure function on $(\Omega, \mathbf{G}, P)$ to a Banach space $\mathbf{W}$, and if on $(\Omega, \mathbf{F}, P)$, y is a numerically-valued integrable function such that zy is a Bochner-integrable function with values in $\mathbf{W}$, then*

$$E[zy|\mathbf{G}](\omega) = zE[y|\mathbf{G}](\omega) \text{ a.e..}$$

Proof. By using Lemma 3 and Lemma 4, the proof when $\mathbf{W}$ is the real numbers $\mathbf{R}$ as given by Billingsley [**1**, p. 110, Theorem 10.1] can be applied to obtain the general result.

LEMMA 7. *Let g be a convex function on $\mathbf{U}$ to $\mathbf{V}$. If $u_i \in \mathbf{U}$ and $t_i \in \mathbf{R}$, $t_i \geqq 0$, $i = 1,2,\cdots n$, such that*

$$\sum_{i=1}^{n} t_i = 1, \text{ then } \sum_{i=1}^{n} t_i g(u_i) \geqq_v g\left(\sum_{i=1}^{n} t_i u_i\right).$$

Proof. By induction.

3. Proof of the theorem. We first note that if $F \in \mathbf{F}$ with $P(F) > 0$ and z is a simple function on $(\Omega, \mathbf{F}, P)$ to $\mathbf{U}$ such that $\chi_F f(\cdot, z(\cdot))$ is Bochner-integrable, then

$$(1) \quad E[\chi_F f\Big(\cdot, z(\cdot)|\mathbf{G}](\omega) \geqq_v E[\chi_F|\mathbf{G}](\omega) f\Big(\omega, \frac{E[\chi_F z|\mathbf{G}](\omega)}{E[\chi_F|\mathbf{G}](\omega)}\Big) \text{ a.e. on } F.$$

To see this, let $z = \sum_{i=1}^{n} u_i \chi_{A_i}$, where $u_i \in \mathbf{U}$ and A_i's are disjoint sets of $\mathbf{F}$ such that $\sum_{i=1}^{n} \chi_{A_i} = 1$. It is clear that $F \subset \{\omega: E[\chi_F|\mathbf{G}](\omega) > 0\}$ a.e.. Since $f(\cdot, u_i)$ is strongly measurable with respect to $\mathbf{G}$ and $f(\omega, \cdot)$ is convex, by using Lemma 4, (b), Lemma 6 and Lemma 7, we then have

$$\frac{1}{E[\chi_F|\mathbf{G}](\omega)} E[\chi_F f(\cdot, z(\cdot))|\mathbf{G}](\omega)$$

$$= \frac{1}{E[\chi_F|\mathbf{G}](\omega)} \sum_{i=1}^{n} f(\omega, u_i)\, E[\chi_F \chi_{A_i}|\mathbf{G}](\omega) \text{ a.e. on } F.$$

$$\geqq_v f\left(\omega, \frac{1}{E[\chi_F \mid \mathbf{G}](\omega)} \sum_{i=1}^{n} u_i E[\chi_F \chi_{A_i} \mid \mathbf{G}](\omega)\right) \text{a.e. on } F$$

$$= f\left(\omega, \frac{E[\chi_F z \mid \mathbf{G}](\omega)}{E[\chi_F \mid \mathbf{G}](\omega)}\right) \text{ a.e. on } F.$$

Nextly, since x is assumed to be a Bochner-integrable function on $(\Omega, \mathbf{F}, P)$ to $\mathbf{U}$, x is strongly measurable, and hence by the definition of strong measurability (or by Lemma 3) there exists a sequence $\{x_i\}_{i=1}^{\infty}$ of simple functions on $(\Omega, \mathbf{F}, p)$ to $\mathbf{U}$ such that $\|x_i(\omega) - x(\omega)\| \to 0$ a.e.. Furthermore, since $f(\omega, \cdot)$ is continuous on $\mathbf{U}$ it follows that $\|f(\omega, x_i(\omega)) - f(\omega, x(\omega))\| \to 0$ a.e..

Therefore, by Lemma 2 we can find an increasing sequence, $\Omega_1 \subset \Omega_2 \subset \cdots$, of sets of $\mathbf{F}$ with $P(\Omega - \Omega_k) < 1/k$, $k = 1, 2, \cdots$, such that

(2) $\|\chi_{\Omega_{k_0}}(\omega) x_i(\omega) - \chi_{\Omega_{k_0}}(\omega) x(\omega)\| \to 0$ uniformly a.e. and

(3) $\|\chi_{\Omega_k}(\omega) f(\omega, x_i(\omega)) - \chi_{\Omega_k}(\omega) f(\omega, x(\omega))\| \to 0$ uniformly a.e., as $i \to \infty$, for each $k = 1, 2, \cdots$.

According to Lemma 5, (2) implies

(2′) $\|E[\chi_{\Omega_k} x_i \mid \mathbf{G}](\omega) - E[\chi_{\Omega_k} x \mid \mathbf{G}](\omega)\| \to 0$ uniformly a.e. as $i \to \infty$, for each $k = 1, 2, \cdots$, and (3) implies

(3′) $\|E[\chi_{\Omega_k} f(\cdot, x_i(\cdot)) \mid \mathbf{G}](\omega) - E[\chi_{\Omega_k} f(\cdot, x(\cdot)) \mid \mathbf{G}](\omega)\| \to 0$ uniformly a.e. as $i \to \infty$, for each $k = 1, 2, \cdots$.

Now by using the continuity of $f(\omega, \cdot)$ again, it follows from (2′) that

$$\left\| f\left(\omega, \frac{E[\chi_{\Omega_k} x_i \mid \mathbf{G}](\omega)}{E[\chi_{\Omega_k} \mid \mathbf{G}](\omega)}\right) - f\left(\omega, \frac{E[\chi_{\Omega_k} x \mid \mathbf{G}](\omega)}{E[\chi_{\Omega_k} \mid \mathbf{G}](\omega)}\right) \right\| \to 0 \tag{4}$$

a.e. on Ω_k as $i \to \infty$.

On the other hand, from (1) we obtain

$$E(\chi_{\Omega_k} f(\cdot, x_i(\cdot)) \mid \mathbf{G}](\omega) \geqq_v E[\chi_{\Omega_k} \mid \mathbf{G}](\omega) f\left(\omega, \frac{E[\chi_{\Omega_k} x_i \mid \mathbf{G}](\omega)}{E[\chi_{\Omega_k} \mid \mathbf{G}](\omega)}\right) \tag{1'}$$

a.e. on Ω_k, for each $k = 1, 2, \cdots$, and each $i = 1, 2, 3 \cdots$.

Letting $i \to \infty$ in (1′) and using (3′) and (4), we obtain

$$E[\chi_{\Omega_k} f(\cdot, x(\cdot)) \mid \mathbf{G}](\omega) \geqq_v E[\chi_{\Omega_k} \mid \mathbf{G}](\omega) f\left(\omega, \frac{E[\chi_{\Omega_k} x \mid \mathbf{G}](\omega)}{E[\chi_{\Omega_k} \mid \mathbf{G}](\omega)}\right), \tag{1''}$$

a.e. on Ω_k, since the positive cone of $(\mathbf{V}; \geqq_v)$ is closed.

Finally, since $|\chi_{\Omega_k}(\omega)| \leqq 1$ and $\chi_{\Omega_k}(\omega) \to 1$ a.e., by using Lemma 4, (a) and (d), and the continuity of $f(\omega, \cdot)$, when $k \to \infty$ we have

$$\text{(J)} \qquad E[f(\cdot, x(\cdot))|\mathbf{G}](\omega) \geqq_v f(\omega, E[x|\mathbf{G}](\omega)) \quad \text{a.e.}.$$

4. Remark. In particular, when $\mathbf{G}$ is the trivial sub-σ-field $\mathbf{Z} = \{\Omega, \phi\}$, inequality (J) reduces to

$$\text{(J')} \qquad \int_\Omega f(\omega, x(\omega))dP \geqq_v f\Big(\omega, \int_\Omega x(\omega)dP\Big).$$

When the function $f(\omega, u)$ is replaced by a continuous and convex function g on $\mathbf{U}$ to $\mathbf{V}$, inequalilties (J) and (J') become

$$\text{(K)} \qquad E[g(x(\cdot))|\mathbf{G}](\omega) \geqq_v g(E[x|\mathbf{G}](\omega)) \quad \text{a.e. and}$$

$$\text{(K')} \qquad \int_\Omega g(x(\omega))dP \geqq_v g\Big(\int_\Omega x(\omega)dP\Big).$$

As we have mentioned in the introduction, this result extends a theorem of Scalora **[6]** in which the stronger condition that g is subadditive and positive-homogeneous is assumed.

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Received January 22, 1974.

UNIVERSITY OF CANTERBURY, CHRISTCHURCH, NEW ZEALAND
AND
UNITED COLLEGE, CHINESE UNIVERSITY OF HONG KONG