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THE SPECTRUM OF AN EQUATIONAL CLASS OF GROUPOIDS

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The *spectrum* of an equational class $\mathscr{K}$ is the set of positive integers $\mathrm{Spec}(\mathscr{K}) = \{n \mid \exists \mathfrak{A} \in \mathscr{K}, |\mathfrak{A}| = n\}$. It is obvious that $1 \in \mathrm{Spec}(\mathscr{K})$ and $x, y \in \mathrm{Spec}(\mathscr{K})$ implies $xy \in \mathrm{Spec}(\mathscr{K})$ for any equational class $\mathscr{K}$; i.e. $\mathrm{Spec}(\mathscr{K})$ is a multiplicative monoid of positive integers. Conversely, G. Grätzer showed that given any multiplicative monoid of positive integers $\mathscr{S}$ there is an equational class $\mathscr{K}$ such that $\mathscr{S} = \mathrm{Spec}(\mathscr{K})$. In this paper we show that $\mathscr{K}$ can be chosen to be an equational class of groupoids.

Our first step is to give a simplified proof of Grätzer's theorem. For $n \geqq 1$ let $A_n = \{0, 1, \cdots, n\}$. Define the function $p(x)$ on A_n by $p(x) = x + 1 \pmod{n+1}$. Let $t(x, y, z)$ be the ternary discriminator function (t.d.f.) on A_n; i.e. $t(x, y, z) = z$ if $x = y$ and $t(x, y, z) = x$ if $x \neq y$. If the reader is not familiar with the properties of $t(x, y, z)$ he should consult **[6]**; for the concepts and notations of universal algebra see **[2]**. Let $\mathfrak{A}_n = \langle A_n; t, p\rangle$.

THEOREM 1. (G. Grätzer **[1]**). *Let $\mathscr{S}$ be a multiplicative monoid of positive integers. There is an equational class $\mathscr{K}$ of type $\langle 3, 1\rangle$ such that* $\mathrm{Spec}(\mathscr{K}) = \mathscr{S}$.

Proof. Let $\mathscr{K}' = \{\mathfrak{A}_{n-1} \mid n \in \mathscr{S} - \{1\}\}$ and let $\mathscr{K} = HSP(\mathscr{K}')$. Because the t.d.f. is represented by $t(x, y, z)$ on each $\mathfrak{A}_i$, $\mathscr{K}$ has distributive congruences. Hence by the well known theorem of B. Jónsson **[3]** we have that $\mathscr{K} = P_S HSP_P(\mathscr{K}')$. In particular the subdirectly irreducible members of $\mathscr{K}$ are contained in $HSP_P(\mathscr{K}')$. Let $\mathfrak{A}'$ be a prime product of members of $\mathscr{K}'$, say $\{\mathfrak{A}_j \mid j \in J\}$ (the reader is referred to **[2]** for properties of prime products). If $\mathfrak{A}'$ is finite then it is isomorphic to some $\mathfrak{A}_j$. Thus let $\mathfrak{A}'$ be infinite. Since $t(x, y, z)$ represents the t.d.f. on $\mathfrak{A}'$, all subalgebras of $\mathfrak{A}'$ are simple. Using $p(x)$ we can form a sentence σ_n in the first order theory of $\mathscr{K}$ which implies the existence of at least n distinct elements and which is true in $\mathfrak{A}_m$ for $m \geqq n - 1$. Since $\mathfrak{A}'$ is infinite, σ_n is true in almost all members of $\{\mathfrak{A}_j \mid j \in J\}$ and so σ_n is true in $\mathfrak{A}'$ for all n. Hence every subalgebra of $\mathfrak{A}'$ is infinite. This means that the finite subdirectly irreducible members of $\mathscr{K}$ are contained in $HS(\mathscr{K}')$. But each $\mathfrak{A}_i \in \mathscr{K}'$ is simple and has no proper subalgebras. Hence up to isomorphism the finite subdirectly irreducible members of $\mathscr{K}$ are the members of $\mathscr{K}'$. Finally we note that because of $t(x,y,z)$, $\mathscr{K}$ has permutable congruences. Since each $\mathfrak{A}_i \in \mathscr{K}'$ is simple this means that every finite algebra in $\mathscr{K}$ is a direct product of algebras from $\mathscr{K}'$ so that $\mathscr{S} = \mathrm{Spec}(\mathscr{K})$ and the theorem is proved.

COROLLARY 1. *Let* $\{\mathfrak{A}_n \mid n \geqq 1\}$ *be algebras of type* τ *with* $|\mathfrak{A}_n| = n+1$. *Let* $t(x,y,z)$ *and* $p(x)$ *be polynomials of type* τ *such that* $t(x,y,z)$ *represents the* t.d.f. *on each* $\mathfrak{A}_n$ *and* $p(x) = x + 1 (\mathrm{mod}\ n+1)$ *in each* $\mathfrak{A}_n$. *Given any multiplicative monoid of positive integers* $\mathscr{S}$ *there is an equational class* $\mathscr{K}$ *of type* τ *such that* $\mathscr{S} = \mathrm{Spec}(\mathscr{K})$.

Proof. Note that the proof of Theorem 1 only requires that $t(x,y,z)$ and $p(x)$ be polynomials, not that they be operations.

Thus we need to construct a set of groupoids $\{\mathfrak{B}_n\}$ satisfying the conditions of Corollary 1. First we will construct $\{\mathfrak{B}_n \mid n \geqq 3\}$ and later construct $\mathfrak{B}_1$ and $\mathfrak{B}_2$. The multiplication table for $\mathfrak{B}_n = \langle\{0, 1, \cdots, n\}; \omega\rangle$ for $n \geqq 3$ is given in Fig. 1.

MULTIPLICATION TABLE FOR $\mathfrak{B}_n$, $n \geqq 3$

	ω	0	1	2	3			$n-1$	n
	0	1	0	2	3	·····		n	$n-1$
	1	n	2	1					
	2	2	0	3	1				
	3	3		1	4				
		⋮			2				
		⋮						1	
	$n-1$	$n-1$						n	1
$	n	0	0	0			0	$n-2$	0

FIGURE 1

The multiplication table is filled in according to the following rules; the reader should check that for $n \geqq 3$ these rules are consistent (all addition is mod $n+1$):

(1) $\omega(x,x) = x + 1$ for all x.

(2) $\omega(x+1,x) = x - 1$ for all x.

(3) $\omega(0,1)=\omega(n,0)=0$; $\omega(x,x+1)=1$ for $x\neq 0,n$.
(4) $\omega(n,x)=0$ for $x\neq 0,n-1,n$.
(5) $\omega(0,n-1)=n$; $\omega(0,x)=x$ for $x\neq 0,\ 1,n-1,n$.
(6) $\omega(x,0)=x$ for $x\neq 0,1,n$.
(7) In all other cases, $\omega(x,y)\neq 1$.

LEMMA 1. *There are groupoid polynomials $t'(x,y,z)$ and $p(x)$ such that $t'(x,y,z)$ represents the* t.d.f. *on each $\mathfrak{A}_n, n\geqq 3$ and such that $p(x)=x+1 (\mathrm{mod}\ n+1)$ in $\mathfrak{A}_n, n\geqq 3$.*

Proof. The proof will consist of a list of definitions of polynomials together with their values on $\mathfrak{A}_n$. The reader should have no trouble verifying each member of the list.

(1) $\alpha(x)\equiv\omega(x,x)=x+1$. Thus $p(x)=\alpha(x)$.
(2) $\beta(x)\equiv\omega(\alpha(x),x)=x-1$.
(3) $\gamma(x)\equiv\omega(x,\alpha(x))=\begin{cases}0 \text{ if } x=0,n,\\ 1 \text{ otherwise.}\end{cases}$
(4) $C_1(x)\equiv\gamma(\alpha(\gamma(x)))=1$.
(5) $C_0(x)\equiv\beta(C_1(x))=0$; $C_n(x)\equiv\beta(C_0(x))=n$.
(6) $\delta_{n-1}(x)\equiv\gamma(\omega(C_n(x),x))=\begin{cases}1 \text{ if } x=n-1,\\ 0 \text{ otherwise.}\end{cases}$
(7) $\delta_n(x)\equiv\delta_{n-1}(\beta(x))=\begin{cases}1 \text{ if } x=n,\\ 0 \text{ otherwise;}\end{cases}$

$\delta_0(x)=\delta_n(\beta(x))=\begin{cases}1 \text{ if } x=0,\\ 0 \text{ otherwise.}\end{cases}$

(8) $\delta_k(x)\equiv\delta_0(\beta^k(x))=\begin{cases}1 \text{ if } x=k,\\ 0 \text{ otherwise.}\end{cases}$
(9) $\bar{\delta}_j(x)\equiv\delta_0(\delta_j(x))=\begin{cases}0 \text{ if } x=j,\\ 1 \text{ otherwise.}\end{cases}$
(10) $\Delta_{j,k}(x,y)\equiv\delta_0(\omega(\bar{\delta}_j(x),\delta_k(y)))=\begin{cases}1 \text{ if } (x,y)=(j,k),\\ 0 \text{ otherwise.}\end{cases}$
(11) $\bar{\Delta}_{j,k}(x,y)\equiv\delta_0(\Delta_{j,k}(x,y))=\begin{cases}0 \text{ if } (x,y)=(j,k),\\ 1 \text{ otherwise.}\end{cases}$
(12) $\omega(0,\omega(0,y))=y$ and $\omega(n,\omega(n,y))=0$.
(13) $x\cdot y\equiv\alpha(\omega(\beta(\delta_0(x)),\ \omega(\beta(\delta_0(x)),\ \beta(y))))$.
(14) $1\cdot y=1,\ 0\cdot y=y$.
(15) $\sigma(x)\equiv\omega(C_0(x),x)$; $\sigma^2(x)=x$.
(16) $\tau(x)\equiv\omega(x,C_0(x))$; $\tau^3(x)=x$.
(17) $x+y\equiv\omega(\tau^2(x),\sigma(y))$; $1\dot{+}y=y+1=y$.
(18) $\omega(x,\alpha(y))=1$ iff $(x=y$ and $x\neq 0,n)$ or $((x,y)=(0,n))$ or $((x,y)=(3,1))$.
(19) $\bar{\varepsilon}(x,y)\equiv\bar{\Delta}_{0,0}(x,y)$
$+(\bar{\Delta}_{n,n}(x,y)+(\Delta_{0,n}(x,y)\cdot(\Delta_{3,1}(x,y)\cdot\bar{\delta}_1(\omega(x,\alpha(y))))))$.

$$(20)\quad \bar{\varepsilon}(x,y)=\begin{cases}0 \text{ if } x=y,\\ 1 \text{ otherwise.}\end{cases}$$

$$(21)\quad \varepsilon(x,y)\equiv\delta_0(\bar{\varepsilon}(x,y))=\begin{cases}1 \text{ if } x=y,\\ 0 \text{ otherwise.}\end{cases}$$

$$(22)\quad t'(x,y,z)\equiv(\bar{\varepsilon}(x,y)\cdot z)+(\varepsilon(x,y)\cdot x)=\begin{cases}z \text{ if } x=y,\\ x \text{ otherwise.}\end{cases}$$

This concludes the proof of Lemma 1.

Each of the $\mathfrak{B}_n$, $n \geqq 3$, is a primal algebra (i.e. a finite nontrivial algebra such that every function on the algebra is representable by a polynomial). A theorem of E. S. O'Keefe **[4]** asserts that a set of pairwise nonisomorphic primal algebras of a type consisting of just one operation is independent. In particular this guarantees that for any finite subset of $\{\mathfrak{B}_n\}$ there is a polynomial representing the t.d.f. However, this does not guarantee that there is a polynomial representing the t.d.f. on all $\mathfrak{B}_n$.

Now consider $\mathfrak{B}_1$ and $\mathfrak{B}_2$ as given in Fig. 2. It is well known that $\mathfrak{B}_1$ is primal. To see that $\mathfrak{B}_2$ is primal we invoke a theorem of G. Rousseau **[7]** which states that if $\mathfrak{A}$ is a finite nontrivial algebra of type $\langle n\rangle$ with $n \geqq 2$ then $\mathfrak{A}$ is primal iff $\mathfrak{A}$ has no proper subalgebras, has no proper automorphisms, and is simple. It will be shown shortly that every element of $\mathfrak{B}_2$ is the value of a constant polynomial. Hence the first two conditions hold. To see that $\mathfrak{B}_2$ is simple note that if $0\equiv 1$ then $0=\omega(0,1)\equiv\omega(1,1)=2$; if $0\equiv 2$ then $1=\omega(0,0)\equiv\omega(2,0)=2$, and if $1\equiv 2$ then $0=\omega(0,1)\equiv\omega(0,2)=1$. Hence $\mathfrak{B}_2$ is primal. Thus by the above mentioned theorem of O'Keefe there is a polynomial $t''(x,y,z)$ representing the t.d.f. on $\mathfrak{B}_1$ and $\mathfrak{B}_2$.

ω	0	1
0	1	1
1	1	0

$\mathfrak{B}_1$

ω	0	1	2
0	1	0	1
1	1	2	0
2	2	1	0

$\mathfrak{B}_2$

FIGURE 2

LEMMA 2. *There is a polynomial $\phi(x,y)$ such that $\phi(x,y)=x$ in $\mathfrak{B}_1,\mathfrak{B}_2$ while $\phi(x,y)=y$ in $\mathfrak{A}_n$, $n \geqq 3$.*

Proof. Again we make a series of definitions and statements each of which is easily verifiable.

(1) $\alpha(x) \equiv \omega(x,x) = x + 1$ in $\mathfrak{B}_n, n \geqq 1$.

(2) $\beta(x) \equiv \omega(\alpha(x),x) = \begin{cases} 1 & \text{in } \mathfrak{B}_1, \mathfrak{B}_2, \\ x-1 & \text{in } \mathfrak{B}_n, n \geqq 3. \end{cases}$

(3) $\rho(x) \equiv \beta(\alpha(x)) = \begin{cases} 1 & \text{in } \mathfrak{B}_1, \mathfrak{B}_2, \\ x & \text{in } \mathfrak{B}_n, n \geqq 3. \end{cases}$

(4) $\alpha'(x) \equiv \omega(\rho(x),x) = \begin{cases} \text{a permutation in } \mathfrak{B}_1, \mathfrak{B}_2, \\ \alpha(x) \text{ in } \mathfrak{B}_n,\ n \geqq 3. \end{cases}$

(5) $\beta'(x) \equiv \omega(\alpha(x),\rho(x)) = \begin{cases} \text{a permutation in } \mathfrak{B}_1, \mathfrak{B}_2, \\ \beta(x) \text{ in } \mathfrak{B}_n,\ n \geqq 3. \end{cases}$

(6) $\gamma'(x) \equiv \omega(\rho(x),\alpha'(x)) = \begin{cases} \text{a permutation in } \mathfrak{B}_1, \mathfrak{B}_2, \\ \gamma(x) \text{ in } \mathfrak{B}_n,\ n \geqq 3. \end{cases}$

(7) $C_1'(x) \equiv \gamma'(\alpha(\gamma'(x))) = \begin{cases} \text{a permutation in } \mathfrak{B}_1, \mathfrak{B}_2, \\ 1 \text{ in } \mathfrak{B}_n,\ n \geqq 3. \end{cases}$

(8) $C_0'(x) \equiv \beta'(C_1'(x)) = \begin{cases} \text{a permutation in } \mathfrak{B}_1, \mathfrak{B}_2, \\ 0 \text{ in } \mathfrak{B}_n,\ n \geqq 3. \end{cases}$

(9) Compose $C_0'(x)$ with itself sufficiently many times to get

$$\eta(x) = \begin{cases} x \text{ in } \mathfrak{B}_1, \mathfrak{B}_2, \\ 0 \text{ in } \mathfrak{B}_n, n \geqq 3. \end{cases}$$

(10) $\mu(x,y) \equiv \omega(\eta(x),\rho(y))$

$$= \begin{cases} \text{a permutation in } x \text{ of order 2 in } \mathfrak{B}_1, \mathfrak{B}_2, \\ \text{a permutation in } y \text{ of order 2 in } \mathfrak{B}_n, n \geqq 3. \end{cases}$$

(11) $\phi(x,y) \equiv \mu(\mu(x,y),\mu(x,y)) = \begin{cases} x \text{ in } \mathfrak{B}_1, \mathfrak{B}_2, \\ y \text{ in } \mathfrak{B}_n,\ n \geqq 3. \end{cases}$

This concludes the proof of Lemma 2.

THEOREM 2. *Given any multiplicative monoid of positive integers $\mathcal{S}$ there is an equational class of groupoids $\mathcal{K}$ such that $\mathcal{S} =$ Spec($\mathcal{K}$). If $\mathcal{S} \neq \{1\}$ then there are uncountably many such equational classes of groupoids and each is generated by its finite members.*

Proof. Let $\{\mathfrak{B}_n \mid n \geqq 1\}$ be as defined in Fig. 1 and 2. Let $\mathcal{K}' = \{\mathfrak{B}_{n-1} \mid n \in \mathcal{S} - \{1\}\}$ and let $\mathcal{K} = HSP(\mathcal{K}')$. Then taking $p(x) = \alpha(x)$ and $t(x,y,z) = \phi(t''(x,y,z), t'(x,y,z))$ we see that by Corollary 1, $\mathcal{S} =$ Spec($\mathcal{K}$). If $\mathcal{S} \neq 1$ let $m \in \mathcal{S}$ with $m > 1$. Then for $n > 1$ we can include or exclude $\mathfrak{B}m^n$ from $\mathcal{K}'$ without changing the spectrum of $HSP(\mathcal{K}')$.

Problem 1. For which equational subclasses of groupoids does Theorem 2 hold? It is known to be false for semigroups. If we consider idempotent groupoids, note that there are up to isomorphism, only three two element idempotent groupoids and any equational class containing one of them has a complete spectrum: all positive integers. For $2 \not\in \mathcal{S}$ it is likely that there is an equational class of idempotent groups whose spectrum is $\mathcal{S}$.

Problem 2. If $\mathscr{S}$ is finitely generated then we may take $\mathscr{K}'$ to consist only of those $\mathfrak{B}_{n-1}$ for n in a given finite generating set of $\mathscr{S}$. Thus $\mathscr{K}$ will be generated by a finite algebra (the product of the $\mathfrak{B}_{n-1}$). Hence by a result of Kirby Baker, $\mathscr{K}$ is finitely based and so by [5] 1-based. On the other hand, if $\mathscr{K}$ is finitely based then necessarily $\mathscr{S}$ is recursive. Is the converse true; namely if $\mathscr{S}$ is recursive is the corresponding $\mathscr{K}$ finitely based?

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